Suppose R = commutative ring

Universal things I can do :

given $x \in R$ — square, add 1, combinations, ⋮

Let V = free abelian gp generated by x

$\rightsquigarrow$ map $V \longrightarrow R$ extends to (comm. ring) $\mathrm{Sym}(V) \longrightarrow R$

$$y \in \mathrm{Sym}(V) \longmapsto \text{universal expression in } x$$

X = Spectrum

$\mathrm{Sym}^k X = X \wedge \ldots \wedge X / \Sigma_k$ — k-sym power of X

this doesn't work as well as it does in algebra, eg.

$\mathrm{Sym}^k(\mathbb{Z}) = \mathbb{Z}$ b/c it's the unit in the tensor structure

but

$\mathrm{Sym}^k(S^0) = \Sigma^\infty (B\Sigma_k)_+$

b/c acts freely, acts trivially

($B\Sigma_k$ = classifying space for the symmetric gp.
Σ^∞ : look at all the suspensions
$+$: add base pt.)

We'll work over the ground ring spectrum K-theory, + p-complete everything.

Call it K_p.

Suppose we have $S^0 \xrightarrow{x} R$ comm. K_p-algebra, $x \in \pi_0 R$

look at

$$\begin{array}{ccc} \mathrm{Sym}^p(S^0) & \longrightarrow & \mathrm{Sym}^p_{K_p}(R) \\ \big\downarrow & & \big\downarrow \\ (\ldots)^\wedge & & R. \end{array}$$

so, given a map $S^0 \longrightarrow (K \wedge (B\Sigma_p)_+)^\wedge_p$

find

$$\begin{tikzcd} S^0 \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha(x)"'] & (K \wedge (B\Sigma_p)_+)^\wedge_p \arrow[r] & R \end{tikzcd}$$

Computations:

$$\pi_0 (K \wedge (B\Sigma_p)_+)^\wedge_p \underset{\substack{\uparrow \\ \text{univ.} \\ \text{coeff.} \\ \text{thm}}}{=} \mathrm{Hom}\Big(\underbrace{K_p^0(B\Sigma_p)}_{\substack{\text{we have to} \\ \text{calculate this} \\ \text{(want f.g.)}}}, \mathbb{Z}_p\Big)$$

thm (Atiyah)

G cpt Lie gp, $K^0(BG) = R[G]^\wedge_I$

representation ring of G completed at augmentation ideal.

$$\Rightarrow K_p^0(B\Sigma_p) = \underset{\substack{\uparrow \\ \text{trivial} \\ \text{representation} \\ \mathbb{1}}}{\mathbb{Z}_p} \oplus \underset{\substack{\uparrow \\ \text{permutation} \\ \text{repr. of order } p \\ \text{(permuting set of } p \text{ elts)} \\ \rho}}{\mathbb{Z}_p}.$$

Let $\psi, \vartheta \in \pi_0 (K \wedge (B\Sigma_p)_+) = \mathbb{Z}_p \oplus \mathbb{Z}_p$ generated by ψ, ϑ

$\vartheta(\rho) = 1 \qquad \psi(\rho) = 0$

$\vartheta(\mathbb{1}) = 0 \qquad \psi(\mathbb{1}) = 1$

R a K_p-algebra commutative

$x \in \pi_0 R \rightsquigarrow \vartheta(x), \psi(x) \in \pi_0 R$

universal construction

(want rel. between these and a poly of deg p)

$$\begin{tikzcd} \overbrace{S^0 \wedge \dots \wedge S^0}^{p} = S^0 = B\Sigma_{1+} \arrow[rr, "x \wedge \dots \wedge x"] \arrow[d, "\text{corresponds to the inclusion of the units}"] \arrow[d, dashed] & & R \wedge \dots \wedge R \arrow[d] \\ B\Sigma_{p+} \arrow[rr] & & \mathrm{Sym}_K(R) \arrow[d] \\ S^0 \wedge \dots \wedge S^0 / \Sigma_p & & R \end{tikzcd}$$

the operation

$x \mapsto x^p \longleftrightarrow$ unique elt α in $\pi_0(\Sigma \wedge B\Sigma_{p+})$
s.t.
$\alpha(\chi) = \chi(1)$ for χ character, so
$\alpha(\mathbb{1}) = 1$
$\alpha(\rho) = p$
$\Rightarrow \alpha = \psi + p\vartheta$

$\psi(x) = x^p - p\vartheta(x)$

claim: ψ is a ring homomorphism
(multiplicativity clear, check additivity)
but there is an extra structure: Frobenius

Cor: R commutative K_p-algebra $\Rightarrow \pi_0 R$ has a Frobenius ψ

Ex: define a functor $X \longrightarrow$ p-adic K-theory of X
adjoint to pth root of unit, ie
$K_p^*(X) \otimes \mathcal{O}$ (is a comm. ring)
where $\mathcal{O} = \mathbb{Z}_p(\zeta)/1+\dots+\zeta^{p-1}=0$

If this were a commutative ring spectrum $\Rightarrow$ Frobenius
But $\nexists$ ring homomorphism $\mathcal{O} \xrightarrow{\psi} \mathcal{O}$ s.t. $\psi(x) = x^p(p)$
$\Rightarrow \mathcal{O}$ not an E_∞ ring spectrum.

Suppose X a K_p-module
$\mathrm{Sym}_{K_p}(X)$ = free comm. algebra over K_p on X
look at $\pi_0 \mathrm{Sym}_{K_p}(K_p \wedge S^0)$, it contains $\mathbb{Z}_p$ and x, but also
$\vartheta(x), \vartheta(\vartheta(x)), \dots$
$\Rightarrow \mathbb{Z}[x, \vartheta(x), \vartheta(\vartheta(x)), \dots]$
then complete everything

$x_i = \vartheta(x_{i-1})$ then: $\mathbb{Z}[x_0, x_1, \dots] \longrightarrow \pi_0 \mathrm{Sym}_{K_p}(K_p \wedge S^0)$

Oct. 6

Height of a formal group in char $p > 0$.

(factors through twisted Frobenius uniquely)

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "F"'] & \varphi^* G \arrow[d, "g"] & \cdots\cdots (\varphi^h)^* G \arrow[dl, "\sim"] \\ & G & \end{tikzcd}$$

in terms of coordinates $\overbrace{x +_F \cdots +_F x}^{p} = a\, x^{p^h} + \cdots\cdots$

$\mathrm{ht} \geq h$

Def. An elliptic curve over a field k of char $p > 0$ is

- ordinary if its formal gp has height 1
- supersingular " " " " " 2

Prop. If E is a Weierstrass cubic over a field k (of char $p > 0$) and either $c_4(E)$ or $\Delta(E)$ is a unit, the E is supersingular or ordinary.

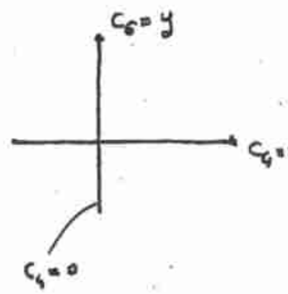
Pf not today.

Want to eliminate the bad "point" case $y^2 = x^3$. Consider

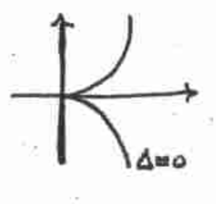
$\mathcal{M}_{Ell} \subset \mathcal{M}_{Weiers}$
open "sub stack" where either c_4 or Δ is a unit

ex: Work over $\mathbb{Z}[\frac{1}{6}]$

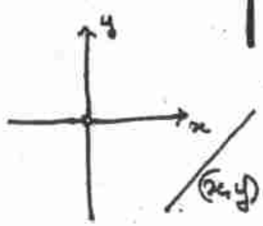
$\mathcal{M}_{Weier} = \mathbb{A}^2 / (x,y) \sim (\lambda^4 x, \lambda^6 y) =$



$$\Delta = \frac{c_4^3 - c_6^2}{1728}$$



$\mathcal{M}_{Ell} =$



$/ (x,y) \sim (\lambda^4 x, \lambda^6 y)$

removing origin get something like proj. space

How to tell whether an elliptic curve is supersingular or ordinary?

Prop. Suppose $F(x,y)=0$ is a Weierstrass cubic, and let $F(x,y,z)$ be the corresp. homog. equation
$\overset{\parallel}{z^3 F\left(\frac{x}{z},\frac{y}{z}\right)}$

$\boxed{\text{char } p}$

then the curve is supersingular iff coeff. of $(xyz)^{p-1}$ in $F(x,y,z)^{p-1}$ is zero.

No pf., but idea: (the proof uses that:)
supersingular $\Leftrightarrow$ absolute frobenius $N'(O) \to N'(O)$ is zero
(cf. Hartshorne)

ex. char 2
$y^2+y=x^3$ is supersingular (only one up to iso over char 2 alg closed field)

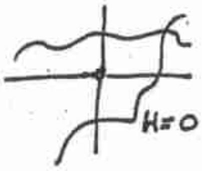
ex. char 3
$y^2=x^3-x$ is supersingular (only one in char 3 over alg. closed field)

char $p>3$
can always assume $y^2=x^3+\frac{c_4}{48}x+\frac{c_6}{864}$
let $F(x,y)=y^2-\left(x^3+\frac{c_4}{48}x+\frac{c_6}{864}\right)$
coeff of $(xyz)^{p-1}$ in $F(x,y,z)^{p-1}$ $=: H(c_4,c_6)$
$\parallel$
modular form of weight $p-1$
(normalized up to scaling)
$=E_{p-1}$ (Eisenstein series)

this is called
Hasse invariant
$=H(c_4,c_6)$

$H(c_4,c_6)=0 \Leftrightarrow$ supersingular.

$H(c_4, c_6)$ polynomial in c_4, c_6



$\mathcal{M}_{Ell}$

$H(\lambda^4 x, \lambda^6 y) = \lambda^{p-1} H(x,y)$

$H=0$ $\quad \underset{\dim 0}{\mathcal{M}^{ss}_{Ell}} \subset \mathcal{M}_{Ell}$

(ss = supersingular)

moduli stack of supersingular elliptic curve has dim 0

$\Rightarrow \mathcal{M}^{ss}_{Ell} = \overset{\circlearrowleft}{\bullet} \quad \overset{\circlearrowleft}{\bullet} \quad \overset{\circlearrowleft}{\bullet} \quad \overset{\circlearrowleft}{\bullet}$

How many "points" are there?

$= \dim_{\mathbb{F}_p} H^0(\mathcal{M}^{ss}_{Ell}; \mathcal{O})$ (# of iso. classes)

(compute by resolutions of the structure sheaf)

Fact: $\mathcal{M}^{ss}_{Ell}$ is reduced, i.e. H has distinct roots.

Moreover, $H^i(\mathcal{M}^{ss}_{Ell}; \mathcal{O}) = 0$, $i > 0$

b/c Aut gp. only involves 2's and 3's

On $\mathcal{M}_{Ell}$

$$\omega^{1-p} \xrightarrow{\cdot H} \mathcal{O}_{\mathcal{M}_{Ell}} \twoheadrightarrow \mathcal{O}^{ss}$$

$$\# \text{pts} = \underset{\text{Euler characteristic}}{\chi(\mathcal{O}^{ss})} = \chi(\mathcal{O}_{\mathcal{M}_{Ell}}) - \chi(\omega^{1-p})$$

$$H^*(\mathcal{M}_{Ell}; \omega^n) \to H^*(U_{c_4 \neq 0}; \omega^n) \oplus H^*(U_{c_6 \neq 0}; \omega^n) \to H^*(U_{c_4 \neq 0} \cap U_{c_6 \neq 0}) \to H^*(\mathcal{M}_{Ell})$$

take $\bigoplus_n$

$$\cdots \to \underset{\substack{\parallel \\ H^*(\mathcal{M}_{Ell})}}{\mathbb{Z}[\tfrac{1}{6}][c_4, c_6]} \to c_4^{-1}\mathbb{Z}[\tfrac{1}{6}][c_4, c_6] \oplus c_6^{-1}\mathbb{Z}[\tfrac{1}{6}][c_4, c_6] \to (c_4 c_6)^{-1}\mathbb{Z}[\tfrac{1}{6}][c_4, c_6] \to H^1(\mathcal{M}_{Ell}) \to 0$$

$$\begin{array}{l|cccc}
 & n=-10 & n=0 & n=4 & n=6 \\
\hline
H^1(\omega^n) & \cdots\ \frac{u}{c_6}\ \frac{u}{c_4}\ \cdots\ (c_4c_6)^{-1}=u & & & \\
H^0(\omega^n) & & \mathbb{Z}[\frac{1}{6}] & c_4 & c_6
\end{array}$$

$H^0(\omega^n) = 0,\ n<0$

$H^0(\omega^n) =$ dual of $H^1(\omega^{-10-n})$

(Serre duality ?)

$$\Rightarrow \chi(\mathcal{O}^{ss}) = \chi(\mathcal{O}_{\mathcal{M}_{Ell}}) - \chi(\omega^{1-p})$$
$$= 1 + \dim H^1(\omega^{1-p})$$
$$= 1 + \dim H^0(\omega^{p-11})$$

<u>Cor</u>: $\# \mathcal{M}_{Ell}^{ss} = \dim \deg(p-11)$ - part of $\mathbb{Z}[\underset{4}{c_4}, \underset{6}{c_6}]$

$$= \begin{cases} 1 + \left[\frac{p+1}{12}\right] & \text{if } p \not\equiv 1 \ (12) \\ \left[\frac{p+1}{12}\right] & \text{if } p \equiv 1 \ (12) \end{cases}$$

$\Rightarrow$ 11 is the first p s.t. 2 ss elliptic curves

<u>Ex</u>: $p = 11$, $H = c_4c_6$ b/c it's the only mod. form of weight 10

$y = x^3 + x$, $y = x^3 + 1$ the two ss elliptic curves

BOUSFIELD
~~Bousfield~~ LOCALIZATION (generalization of Serre's mod $\mathcal{C}$ theory)

E - homology theory

X - spectrum

<u>Def</u>. X is <u>E-acyclic</u> if $E_*X = 0$ i.e. $E \wedge X \simeq *$

Def. $X \to Y$ is an *E-equivalence* if $E_*X \xrightarrow{\simeq} E_*Y$

$E\wedge X \xrightarrow{\simeq} E\wedge Y$

Def. X is *E-local* if

$$E_*Z = 0 \implies [Z, X]_* = 0$$

Def. An *E-localization* of X is a map $X \xrightarrow{f} W$ s.t.

- f is an E-equivalence
- W is E-local

E-localizations exist, and they exist functorially, i.e. For every E, Bousfield constructed a functor L_E and a natural transformation

$$X \longrightarrow L_E X \qquad id \longrightarrow L_E \text{ s.t. } \forall X$$

which is an E-localization

Ex. (algebraic example)

Pretend Spectra = Chain complexes + ab. gps

- $E = \mathbb{Z}_{(p)}$ $\quad L_E X_0 = X \otimes \mathbb{Z}_p$

  i.e. $X \xrightarrow{f} X\otimes \mathbb{Z}_p$ is an E-localization

  - f is obviously an E-equivalence b.c.

    $X\otimes \mathbb{Z}_{(p)} \longrightarrow X \otimes \mathbb{Z}_{(p)} \otimes \mathbb{Z}_{(p)} = \mathbb{Z}\otimes\mathbb{Z}_{(p)}$

  - $Z \otimes \mathbb{Z}_p \simeq *$ $\qquad$ ($\Rightarrow$ E-local)

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & X\otimes \mathbb{Z}_{(p)} \arrow[d] \\ & Z\otimes \mathbb{Z}_{(p)} \sim * \end{tikzcd}$$

- **Ex.** $E = \mathbb{Z}/p$ $\quad$ the previous argument doesn't work anymore b/c

  $X\otimes \mathbb{Z}/p \otimes \mathbb{Z}/p \neq X \otimes \mathbb{Z}/p$

  need to replace by flat abelian gp

  before was [illegible] here replace with $\overset{X}{\downarrow p}$

tensorizing

$$\begin{pmatrix} X \\ \downarrow p \\ X \end{pmatrix} \otimes \mathbb{Z}/p = \begin{pmatrix} X \otimes \mathbb{Z}/p \\ \oplus \\ X \otimes \mathbb{Z}/p \end{pmatrix}$$

<u>Claim</u>: $X_p^\wedge = \varprojlim_m X \otimes \mathbb{Z}/p^m$ is $L_E X$.

doesn't commute with colimits (unlike the previous case)

want to show that

① $X_p^\wedge$ is E-local.

suppose $Z \otimes \mathbb{Z}/p \sim *$ and assume Z_n is free abelian

$\Rightarrow$ we have SES's

$$\begin{array}{ccc} Z_\cdot \otimes \mathbb{Z}/p & \rightarrowtail & Z_\cdot \otimes \mathbb{Z}/p^{n+1} \\ & & \downarrow \\ & & Z_\cdot \otimes \mathbb{Z}/p^n \end{array}$$

get LES in homology, and by induction on n

$$Z \otimes \mathbb{Z}/p^n \sim *$$

$\Rightarrow$ $[Z, X \otimes \mathbb{Z}/p^n] \sim *$

then we use Milnor's sequence

$$\begin{array}{ccccc} \varprojlim{}^1 [Z, X \otimes \mathbb{Z}/p^n] & \longrightarrow & [Z, X_p^\wedge] & \longrightarrow & \varprojlim [Z, X \otimes \mathbb{Z}/p^n] \\ \| & & \| & & \| \\ 0 & \therefore & 0 & & 0 \end{array}$$

② $X \longrightarrow X_p^\wedge$ is an E-equivalence

$X \otimes \mathbb{Z}/p \longrightarrow X_p^\wedge \otimes \mathbb{Z}/p$ is iso if X is dimensionwise free.

Lubin-Tate Formal Groups.

(see Lubin-Tate on formal grps or Serre in Cassels and Fröhlich.)

Wish to construct a formal group law F over $\mathbb{Z}_p$ with height n. So we must have

$$[p](x) = px + \cdots$$

$$[p](x) \equiv c \cdot x^{p^n} + \cdots \quad (\text{mod } p)$$

$$c \neq 0$$

Since $[p]$ is a homomorphism

$$[p](x +_F y) = [p](x) +_F [p](y).$$

Lemma (Lubin-Tate) Suppose $g(x) \in \mathbb{Z}_p[[x]]$ satisfies the following.

i) $g(x) = px + \cdots$

ii) $g(x) \equiv x^{p^n}$ (mod p).

Then, given any linear form $l(x_1, \_, x_n) = \sum a_i x_i$, $a_i \in \mathbb{Z}_p$, there exists a unique power series $F(x_1, -, x_n)$ s.t.

a) $F(x_1, --, x_n) = \ell(x_1, --, x_n) + --$

b) $g(F(x_1, -, x_n)) = F(g(x_1), --, g(x_n))$.

Assuming this, ~~let~~ $\ell(x,y) = x+y$ and consider the assoc. $F(x,y)$. Then

$$F(F(x,y), z) = F(x, F(y,z))$$

since ~~both~~ commute with g and have the same linear term $x+y+z$. So $F(x,y)$ is a formal grp. law over $\mathbb{Z}_p$. Since $[p](x)$ commutes with g,

$$[p](x) = px + --,$$

and since g commutes with g,

$$g(x) = px + ---,$$

so $[p](x) = g(x)$. Hence, F has height n.

Proof (of lemma) Suppose we have constructed $F_k(x_1, -, x_n)$ of degr. $\leq k$ s.t.

$$F_k(x_1, --, x_n) \equiv \ell(x_1, --, x_n) + --$$

and

$$g(F_k(x_1,\dots,x_n)) \equiv F_k(g(x_1),\dots,g(x_n))$$

modulo degr. $k+1$, and suppose that we have shown that this F_k is unique. Then, modulo degr. $k+2$,

$$g(F_k(x_1,\dots,x_n)) = F_k(g(x_1),\dots,g(x_n)) + \varepsilon_{k+1},$$

with ε_{k+1} homogeneous of degr. $k+1$. Modify F_k by φ_{k+1} homogeneous degr. $k+1$:

$$F_k + \varphi_{k+1} = F_{k+1}$$

Then

$$g(F_k + \varphi_{k+1}) = g(F_k) + p\varphi_{k+1} + \text{degr. } k+1$$

$$F_{k+1}(g(x_1),\dots,g(x_n)) = F_k(g(x_1),\dots,g(x_n))$$

$$+ \varphi_{k+1}(g(x_1),\dots,g(x_n))$$

$$= F_k(g(x_1),\dots,g(x_n)) + p^{k+1}\varphi_{k+1}(x_1,\dots,x_n))$$

$$= g(F_k(x_1,\dots,x_n)) - \varepsilon_{k+1} + p^{k+1}\varphi_{k+1}(x_1,\dots,x_n)$$

So we will have

$$g(F_{k+1}(x_1,\dots,x_n)) \equiv F_{k+1}(g(x_1),\dots,g(x_n))$$

modulo degr. $k+2$ if and only if

$$(p^{k+1}-p)\,\varphi_{k+1} = \varepsilon_{k+1} \in \mathbb{Z}_p[[x_1,\dots,x_n]] .$$

Hence, φ_{k+1} exists and is unique if and only if $\varepsilon_{k+1} \equiv 0$ modulo p. Since $g(x) = x^{p^n}$ (mod p),

$$g(F_k(\underline{x})) \equiv F_k(\underline{x})^{p^n} \equiv F_k(\underline{x}^{p^n}) \equiv F_k(g(\underline{x}))$$

so $\varepsilon_{k+1} \equiv 0$ as desired. //

Morava K-theories:

$K(n)$ even periodic cohomology,

$\pi_0 K(n) = \mathbb{Z}/p\mathbb{Z}$,

formal grp. = Lubin-Tate grp. of ht. n. reduced mod. p

ex $g(x) = 1-(1-x)^p = px + \dots \pm \binom{p}{i} x^i + \dots \pm x^p$,

$F(x,y) = 1-(1-x)(1-y) = x+y-xy$, $K(1)$ = mod p K-theory.

Bousfield Localization:

We assoc. to every homology th. E a functorial localization

$$X \longrightarrow L_E(X).$$

Lemma Every map in the stable category from X to an E-local spectrum Y factors uniquely as

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & L_E(X) \arrow[ur] & \end{tikzcd}$$

//

Lemma Every E-equivalence $X \to Y$ in the stable category factors uniquely

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & L_E(X) \arrow[ur] & \end{tikzcd}$$

//

Rem $L_E(-)$ depends only on

$$\{C \mid E_* C = 0\}.$$

Def Two spectra E and E' are Bousfield equivalent $E_* C = 0 \iff E'_* C = 0$. //

We write $\langle E \rangle$ for the equivalence class of E.

Thm (Okawa) The collection of Bousfield equivalence classes of spectra forms a set (cf. Dwyer-Palmieri). //

Will discuss:

1) Every p-local elliptic coh. th. E is local w.r.t. $K(0) \vee K(1) \vee K(2)$.

2) $L_{K(2)} E \longleftrightarrow$ completion at the super singular points

$L_{K(1)} E \longleftrightarrow v_1^{-1} \varprojlim E/p^n$

$\longleftrightarrow \mathcal{M}_{Ell} \otimes \mathbb{Z}/p^n\mathbb{Z} \setminus$ super-sing. points

Introduce process to recover a spectrum E local w.r.t. $K(0) \vee K(1) \vee K(2)$ from $L_{K(0)}E$, $L_{K(1)}E$, and $L_{K(2)}E$.