Simplicial objects:

Δ = (skeleton) category of finite ordered sets

Objects: $[n] = \{0 < 1 < \cdots < n\}$, $n \geq 0$.

$$\begin{array}{ccccl} \Delta & \longrightarrow & \text{Spaces} & & \\ [n] & \longmapsto & \Delta^n & = & \text{standard } n\text{-simplex} \\ \downarrow \theta & & \downarrow \theta_* & = & \text{affine linear} \\ [m] & \longmapsto & \Delta^m & & \end{array}$$

A simplicial object in cat. $\mathcal{C}$ is a functor

$$\Delta^{op} \longrightarrow \mathcal{C}$$

ex X space, $[n] \longmapsto \mathrm{Map}(\Delta^n, X) = \mathrm{Sin}_n(X)$ is a simplicial set.

Cosimplicial obj. are functors

$$\Delta \longrightarrow \mathcal{C}$$

ex $[n] \longmapsto \Delta^n$ cosimpl. space $\Delta^{\cdot}$.

If $P_\cdot$ is a simpl. A_∞-ring spectrum then $A_\infty(P_\cdot, F)$ is a cosimplicial space.

Def If $X^\cdot$ is a cosimpl. space, then

$$\mathrm{Tot}(X^\cdot) = \text{Space of maps } \Delta^\cdot \text{ to } X^\cdot$$

$$\subset \prod_{n\geq 0} \mathrm{Map}(\Delta^n, X^n).$$

An elem. of $\mathrm{Tot}(X^\cdot)$ consists of a pt. $x_0 \in X^0$, a path in X^1 from $d^0 x_0$ to $d^1 x_1$, ---

Cannot necessarily form

$$X^\cdot \rightsquigarrow \pi_k X^\cdot \quad \text{cosimpl. ab. grp}$$

— base-point problem.

Suppose we start with $x_0 \in X^0$ and a path in X^1 from $d^0 x_0$ to $d^1 x_1$. Get a map of cosimpl. spaces

$$sk_1 \Delta^\cdot \longrightarrow X^\cdot$$

Since $sk_1 \Delta^n$ is connected, we get

$\pi_k(X^n, x)$. Assuming $\pi_1(X^n, x)$ acts trivially on $\pi_k(X^n, x)$, $k \geq 0$, we can form

$$\pi_k(X^\cdot, x) \text{ — a cosimpl. ab. grp}$$

10/29/03

Discuss algebra from last time in more detail.

$A \xrightarrow{\alpha} B$ map of comm. rings.

$P_\cdot \longrightarrow B$ simpl. resolution with each P_n free associative A-algebra.

M B-module.

Then get a cosimpl. A-module

$\mathrm{Der}_A(P^\cdot, M)$

with assoc. cohomology groups the derived functors of $\mathrm{Der}_A(-, M)$.

Derivations: let A be a comm. ring and B a (not necessarily comm.) A-alg., i.e. a ring homomorphism

$$A \xrightarrow{\alpha} B$$

s.t. $\alpha(A)$ is contained in the center of B. Let M be a B-bimodule s.t. for all $a \in A$, $am = ma$.

An A-linear derivation of B into M is an A-linear map

$$B \xrightarrow{D} M$$

s.t. $D(b_1 b_2) = D(b_1) b_2 + b_1 D(b_2)$. There is a universal A-linear derivation of B into a B-bimodule:

$$0 \to I \to B \otimes_A B \xrightarrow{\mu} B \to 0$$

$$B \to I \quad \text{a derivation}$$

$$b \mapsto b\otimes 1 - 1 \otimes b$$

Universal:

$$\mathrm{Der}_A(B, M) \approx \mathrm{Hom}_{B-B}(I, M).$$

Given $P_\cdot \to B$ a simpl. res., then

$$I_\cdot = \ker(P_\cdot \otimes_A P_\cdot \xrightarrow{\mu} P_\cdot)$$

is a $P_\cdot$-bimodule, and as cosimpl. A-modules

$$\mathrm{Der}_A(P_\cdot, M) \approx \mathrm{Hom}_{P_\cdot - P_\cdot}(I_\cdot, M).$$

Since M is a $P_\bullet$ - bimodule via the augmentation $P_\bullet \to B$, we have

$$\mathrm{Hom}_{P_\bullet - P_\bullet}(I_\bullet, M)$$

$$\approx \mathrm{Hom}_{B-B}(B \otimes_{P_\bullet} I_\bullet \otimes_{P_\bullet} B, M)$$

where

$$D_{B/A} := B \otimes_{P_\bullet} I_\bullet \otimes_{P_\bullet} B$$

is Quillen's assoc. alg. homology object of B/A.

$$B^e := B \otimes_A B^{op} \quad \text{enveloping algebra}$$

$$B\text{-bimodule} = \text{left } B^e\text{-module}.$$

Suppose B is commutative and $bm = mb$. Then further

$$\mathrm{Hom}_{P_\bullet}(I_\bullet, M) \approx \mathrm{Hom}_{B^e}(D_{B/A}, M)$$

$$\approx \mathrm{Hom}_B(\underbrace{B \otimes_{B^e} D_{B/A}}_{\Lambda_{B/A}}, M)$$

Commutative algebra analog:

$A \longrightarrow B$, M B-module

$$\mathrm{Der}_A(B, M) := \left\{ B \xrightarrow{D} M \;\middle|\; \begin{array}{l} D(b_1 b_2) = D(b_1) b_2 + b_1 D(b_2) \\ D(a) = 0 \end{array} \right\}$$

Universal derivation

$$B \xrightarrow{D} B \underset{B^e}{\otimes} I = I/I^2 =: \Omega_{B/A}$$

~~Start~~ with B/A, pick simpl. res. $Q. \to B$ by free comm. A-algebras. Obtain as before the André-Quillen homology object

$$L_{B/A} = \Omega_{Q./A} \underset{Q.}{\otimes} B$$

s.t. the derived functors of the derivations for comm. algebras is

$$\mathrm{Der}^s_A(B, M) = H^s(\mathrm{Hom}_B(L_{B/A}, M)).$$

Thm Suppose B is flat over A. Then if $L_{B/A}$ is acyclic, $\Lambda_{B/A}$ is acyclic.

Before we give the proof ...

<u>ex</u> Suppose $A = \mathbb{F}_p$, $\varphi : B \xrightarrow{\sim} B$, $\varphi(x) = x^p$.
Then $L_{B/A}$ is acyclic, so by the thm. $\Lambda_{B/A}$ is acyclic, so derived functors of assoc. alg. derivations vanish:

$$\mathrm{Der}_A^s(B, M) = 0, \quad s \geq 0.$$

To see that $L_{B/A}$ is acyclic:

$$\begin{array}{ccc} Q. & \xrightarrow{\tilde{\varphi}} & Q. \\ \downarrow & & \downarrow \\ B & \xrightarrow[\sim]{\varphi} & B \end{array} \qquad Q_n = A[x_i] \xrightarrow{\tilde{\varphi}} A[x_i], \quad x_i \longmapsto x_i^p$$

Since $d(x_i^p) = p x_i^{p-1} dx_i = 0$, the map on $L_{B/A}$ induced by $\tilde{\varphi}$ is zero. But it is also chain-homotopic to an iso.
Therefore, $L_{B/A}$ is acyclic. //

<u>Prop</u> (Quillen) If B is flat over A,

$$H_k(\Lambda_{B/A}) = \mathrm{Tor}_{k+1}^{B^e}(B, B).$$

<u>proof</u> Since $P. \to B$ is a r.s. of B

by proj. A-modules, and since B/A is flat, $P_\cdot \otimes_A P_\cdot \longrightarrow B^e$ is again a res. by proj. $P_\cdot$-modules.

$$
\begin{array}{ccccccccc}
0 \to & I_\cdot & \to & P_\cdot \otimes_A P_\cdot & \xrightarrow{\mu} & P_\cdot & \to 0 \\
& \downarrow & & \downarrow \wr & & \downarrow \wr & \\
0 \to & I^B_\cdot & \to & B \otimes_A B & \xrightarrow{\mu} & B & \to 0
\end{array}
$$

so $I_\cdot \to I^B$ is a res. by proj. $P_\cdot^e$-mod. So get res. by proj. B^e-mod

$$B^e \otimes_{P_\cdot^e} I_\cdot \xrightarrow{\sim} I_B$$

Now

$$\Lambda_{B/A} := B \otimes_{B^e} (B^e \otimes_{P_\cdot^e} I_\cdot)$$

so

$$H_i(\Lambda_{B/A}) = \mathrm{Tor}^i_{B^e}(I_B, B)$$

The prop. follows from the long-exact seq. assoc. w. the s.e.s. of B^e-mod.

$$0 \to I_B \to B^e \xrightarrow{\mu} B \to 0.$$

//

ex $B = A[x_1, \ldots, x_n]$

$$\begin{aligned} B^e &= A[x_1, \ldots, x_n, y_1, \ldots, y_n] \\ &= A[x_1, \ldots, x_n, s_1, \ldots, s_n] \qquad s_i = y_i - x_i \\ &= B[s_1, \ldots, s_n] \end{aligned}$$

so

$$\begin{aligned} \mathrm{Tor}_*^{B^e}(B, B) &= \mathrm{Tor}_*^{B[s_1, \ldots, s_n]}(B, B) \\ &= B \otimes_A \mathrm{Tor}_*^{A[s_1, \ldots, s_n]}(A, A) \\ &= \Lambda_B^*\{ds_1, \ldots, ds_n\} \qquad \text{(Koszul res.)} \end{aligned}$$

Similarly, if $B = S_A(V)$ is the symmetric alg. on a proj. A-mod. V, then

$$\mathrm{Tor}_*^{B^e}(B, B) \approx \Lambda_B^*(V)$$

More generally, if B/A is smooth, then

$$\mathrm{Tor}_*^{B^e}(B, B) \approx \Omega^*_{B/A} = \Lambda_B^* \Omega_{B/A}. \quad //$$

<u>Prop</u> There is a spectral sequence

$$H_s(\Lambda_B^{t+1} L_{B/A}) \Rightarrow H_{s+t}(\Lambda_{B/A}).$$

<u>Rem</u> Here we take $\Lambda_B^{t+1} L_{B/A}$ as simpl. ab. groups and not as chain cx. This is necessary since Λ^{t+1} is not additive. Note for example

$$\begin{array}{ccc} \mathbb{Z} & & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\Lambda^2} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & & 0 \\ \text{acyclic} & & \text{not acyclic} \end{array}$$

//

<u>proof</u> (of prop) Consider bi-simpl. object

$$\begin{array}{ccccccccc} Q_2 & \leftarrow & P_{0,2} & \leftleftarrows & P_{1,2} & \Lleftarrow & P_{2,2} & \cdots \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & \\ Q_1 & \leftarrow & P_{0,1} & \leftleftarrows & P_{1,1} & \Lleftarrow & P_{2,1} & \cdots \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & \\ Q_0 & \leftarrow & P_{0,0} & \leftleftarrows & P_{1,0} & \Lleftarrow & P_{2,0} & \cdots \\ \downarrow & & & & & & & \\ B & & & & & & & \end{array}$$

Q_i free comm. A-alg.

$P_{i,j}$ free ass. A-alg.

$$\Lambda_{\cdot,\cdot} = B \underset{P^e_{\cdot,\cdot}}{\otimes} I_{\cdot,\cdot} \quad \text{bi-simpl. } B\text{-mod.}$$

$$I_{\cdot,\cdot} \to P^e_{\cdot,\cdot} \to P_{\cdot,\cdot}$$

Then $H_* \operatorname{Tot}(\Lambda_{\cdot,\cdot}) = H_*(\Lambda_{B/A})$. Consider spectral sequence

$$H_s^{vert}(H_t^{horiz}(\Lambda_{\cdot,\cdot})) \Rightarrow H_{s+t}(\Lambda_{B/A})$$

Since Q_i is a free comm. A-alg.,

$$H_t^{horiz}(\Lambda_{\cdot,\cdot}) = \Lambda_B^{t+1} L_{B/A}. \qquad //$$

The theorem is an immediate corollary.

3 Nov TMF

Quillen Model Category

$\mathcal{C}$ category, equipped with 3 classes of maps:

- cofibrations $\rightarrowtail$
- fibrations $\twoheadrightarrow$
- weak equivalences $\xrightarrow{\sim}$

There are ~~several~~ axioms:

M1 $\mathcal{C}$ has all limits and colimits

M2 2 of 3 in a composition of w.eq $\Rightarrow$ 3rd is

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

M3 Retract of w.eq., cof, fib is the same

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left, "id"] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[rr, bend right, "id"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

M4 (lifting properties).

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\exists$ if one of i or p is a weak eq.

5 (factorization)

$$\begin{tikzcd} & \bullet \arrow[dr, "\sim"] & \\ \bullet \arrow[ur, tail] \arrow[rr] \arrow[dr, tail, "\sim"'] & & \bullet \\ & \bullet \arrow[ur, two heads] & \end{tikzcd}$$

$\mathcal{C}$ - model category

$\cdot\,\mathcal{C}$ - homotopy category of $\mathcal{C}$ is the cat. obtained from $\mathcal{C}$ by inverting the w.eq.

References:

- Hovey, Model Category
- Hirschhorn (math.mit.edu/~psh)
- Dwyer - Spalinski in Handbook of alg. Topology

Rmk: 2 of the 3 classes (cof, fib, w eq.) determine the 3rd by M4 and M5

Examples

= top. spaces

w eq. = $f : X \to Y$ s.t. $\pi_* f : \pi_* X \xrightarrow{\sim} \pi_* Y$ $\forall$ base pt

fib = Serre fibration

this is a model category and $\text{Ho}\, \mathcal{C}$ = CW cxes / homotopy

= chain complexes of R-modules (R ring)

$$\begin{array}{c} \vdots \\ A_2 \\ \downarrow \\ A_1 \\ \downarrow \\ A_0 \end{array} = A_\bullet \qquad \pi_m A_\bullet = m^{\text{th}} \text{ homology gp.}$$

graded below in ≥ 0

* weak eq = iso's of homology gps.

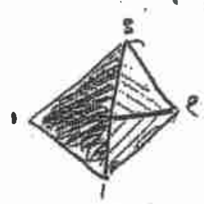
* $A_\bullet \to B_\bullet$ fibration if $A_n \to B_n$ onto for $n > 0$

Recall
Serre fibration

$$\begin{tikzcd} V_{n,k} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \quad \forall$$

the map doesn't need to be surjective

$V_{n,k}$ = k-horn of the standard n-simplex $= \bigcup_{i \neq k} \partial_i \Delta^n$



$V_{3,2}$

* cofibrations = mono whose kernel is complex of projectives

Chain complexes

$$S^n = \begin{array}{c} 0 \\ \downarrow \\ R \\ \downarrow \\ 0 \end{array} \qquad n \geq 0$$

$$S^n \subset D^{n+1} \longrightarrow S^{n+1}$$

cokernel of the inclusion

$$D^{n+1}: \quad \begin{array}{cc} R & {\scriptstyle n+1} \\ \downarrow 1 & \\ R & {\scriptstyle n} \end{array} \qquad n \geq 0$$

$$\begin{tikzcd} & R \arrow[r] \arrow[d] & R \\ R \arrow[r, hook] & R & \end{tikzcd}$$

A map $S^n \longrightarrow X_\bullet \quad \leftrightarrow \quad$ a cycle in X_n

A map $D^{n+1} \longrightarrow X_\bullet \quad \leftrightarrow \quad$ an elt. of X_n

$$S^n \hookrightarrow D^{n+1} \longrightarrow X_\bullet \qquad \begin{array}{c} X_{n+1} \\ \partial \downarrow \\ Z_n(X) \end{array}$$

$$\pi_n X = C(S^n; X) \big/ C(D^{n+1}; X)$$

Remark: $X \to Y$ is a fibration iff

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ D^{n+1} \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \qquad n \geq 0$$

(surjective $\forall n \geq 0$ but no way to test in dim. below zero)

$X \to Y$ is both a <u>fibration and a weak eq.</u> iff

"acyclic fibration"

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ S^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

(for the case $n = 0$ this need to be added)

(diagrams to detect fibrations and acyclic fibrations) to compute $\mathrm{ho}\,\mathcal{C}(X,Y)$ (homotopy classes from X to Y)

1) apply factorization

$$\begin{tikzcd} & P \arrow[d, "\wr"] \\ 0 \arrow[ur] \arrow[r] & X \end{tikzcd}$$

projective resolution of X

2) apply factorization

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\wr"'] & 0 \\ \tilde{Y} \arrow[ur] & \end{tikzcd}$$

and then $\mathrm{ho}\,\mathcal{C}(X,Y) = \mathcal{C}(P,Y) / \mathcal{C}(P \otimes D', Y)$ = chain homotopy classes of maps

maps to Y modulo
maps which extend to the disk

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we were considering
$P_\cdot \longrightarrow E$

$\pi_* P_\cdot \to \pi_* E$ weak eq. of chain complexes

$$\begin{array}{ccc} \pi_* P_n & & 0 \\ \big\downarrow \Sigma(-1)^i d_i & & \\ \pi_* P_{n-1} & & 0 \\ \vdots & & 0 \\ \pi_* P_0 & \longrightarrow & \pi_* E \end{array}$$

E_2 - model category. (due to Dwyer - Kan - Stover
Bousfield: Cosimplicial resolutions and homotopy spectral sequences in model category.

Hopf Topology archive: hopf.math.purdue.edu