3) Find a map

$$P_1 \longrightarrow K(\Omega^2 M, 3)$$

inducing an iso. on $\pi_* \pi_3$.

Is there a map?

$$\begin{tikzcd} K(\Omega M, 1) \arrow[r] & P_1 \arrow[r, dashed, "\exists ?"] \arrow[d] & K(\Omega^2 M, 3) \\ & P_0 & \end{tikzcd}$$

This, in general, might be obstructed; can describe obstruction in terms of homological algebra. However, by the formula for $\operatorname{Ho} C(X, K(N, s, t))$, there is no obstruction.

How many maps are there?

-- formula $\Rightarrow$ described in terms of

$$\prod_k \operatorname{Ext}^3_R (M_k, M_{k+2})$$

Form

$$\begin{array}{ccc} P_2 & & \\ \downarrow & & \\ P_1 & \longrightarrow & K(\Omega^2 M, 3). \end{array}$$

Then

$$\begin{array}{cccccccc} \pi_* \pi_* P_2 & = & M_* & 0 & 0 & 0 & \Omega^3 M & \cdots \\ & & 0 & 1 & 2 & 3 & 4 & \end{array}$$

Continue.

11/12/03

Reference for obstruction theory that we use: Blanc, Dwyer, Goerss: The realization space of Π-algebra: a moduli problem in algebraic topology.

Space of A_∞-structures on X:

$\mathcal{S}^{A_\infty}$ = cat. of A_∞-ring spectra.

$\mathcal{S}$ = cat. of spectra.

An A_∞-str. on X is a pair of R in $\mathcal{S}^{A_\infty}$ and $R \xrightarrow{\sim} X$ in $\mathcal{S}$.

Consider cat. A_∞/X with obj.

R in $\mathcal{S}^{A_\infty}$ $\qquad$ $R \xrightarrow{\sim} X$ in $\mathcal{S}$

and morphisms

$$\begin{matrix} R \\ \downarrow \\ R' \end{matrix} \text{ in } \mathcal{S}^{A_\infty} \qquad \begin{tikzcd} R \arrow[d] \arrow[dr, "\sim"] & \\ R' \arrow[r, "\sim"'] & X \end{tikzcd}$$

Then $B(A_\infty/X)$ is the moduli space of moduli space of A_∞-structures on X. A variation is the cat. $A_\infty\{X\}$

of R in $\mathcal{S}^{A_\infty}$ that are w.e. to X (but we do not record a particular eq.) and where morphisms are w.e. $R \xrightarrow{\sim} R'$. The two moduli spaces are related by

$$E\,haut(X) \times_{haut(X)} B(A_\infty/X)$$

$$\downarrow \sim$$

$$B A_\infty\{X\}$$

or equivalently, by a fiber seq

$$B(A_\infty/X) \to BA_\infty\{X\} \to B\,haut(X).$$

The right hand map forgets the A_∞-structure. (A model for $B\,haut(X)$ is the cat. of spectra w.e. to X.)

Suppose X has a htpy. assoc. mult. Can we find R in $\mathcal{S}^{A_\infty}$ such that $R \xrightarrow{\sim} X$ (or, more generally, s.t. there exists an E_*-homology eq. $R \xrightarrow{\sim} X$).

1) Find $V \to X$ s.t.

E_*V is free over E_*

$E_*V \twoheadrightarrow E_*X$

$T(V) = \bigvee_{n\geq 0} V^{\wedge n}$ = free A_∞-ring spectrum generated by V

We can extend $V \to X$ to a htpy multiplicative map

$$T(V) \to X$$

$$E_*T(V) = T_{E_*}E_*(V) \twoheadrightarrow E_*X.$$

2) $$F \to E_*T(V) \to E_*X$$

E_*-mono $\qquad$ E_*-epi

Choose $W \to F$

E_*W proj. (or free)

$E_*W \twoheadrightarrow E_*F$

2)
$$\begin{tikzcd} T(W) \arrow[r] \arrow[d] & T(*) = S^0 \arrow[d] \\ T(V) \arrow[r] & R^{(1)} \end{tikzcd}$$
push-out in A_∞-ring spectra.

However, it is difficult to get a map $R^{(1)} \to X$, since X does not have an A_∞-ring structure. So this process usually gets stuck and it is not easy to see what the obstructions to continuing it are ...

Further variation:

E_*X as an E_*E-comodule

Moduli space of A_∞-ring spectra R s.t. E_*R is a given associative alg. in E_*E-comodules.

Adams-Atiyah condition:

$\mathcal{P}$ = smallest full subcategory of fin. CW-spectra X s.t.

E_*X is projective, and for all E-modules M,

$$[X, M]_* \xrightarrow{\cong} \mathrm{Hom}_{E_*}(E_*X, M_*).$$

Then the Adams-Atiyah cond. is that E is a filtered colimit of E_α s.t. the dual DE_α is in $\mathcal{P}$.

ex MU satisfies the cond., and this implies that every Landweber exact theory satisfies the cond. //

Suppose E satisfies the A.-A. cond

$$\begin{tikzcd} S^0 \arrow[r, "x"] \arrow[dr, dashed] & E \wedge X \\ & E_\alpha \wedge X \arrow[u] \end{tikzcd}$$

$$\updownarrow$$

$$DE_\alpha \xrightarrow{f} X$$

$$E_* DE_\alpha \xrightarrow{f_*} E_*X$$

$$i_\alpha \longmapsto x$$

so the Adams-Atiyah condition implies

that there exists $V \to X$ s.t.

E_*V -- proj. E_*-module.

$E_*V \twoheadrightarrow E_*X$. //

Prop (Dwyer-Kan-Stover, Goerss-H., Bousfield)
The cat. $s\mathcal{S}$ of simplicial spectra is a simplicial model category with the following structure.

1) weak equivalences:
Let $\pi_k(X;P) = \mathrm{ho}\,\mathcal{S}(\Sigma^k P, X)$, $P \in \mathcal{P}$.
Then $X \to Y$ is a w.e. if for all $P \in \mathcal{P}$ and integers k,

$$\pi_k(X_\cdot;P) \to \pi_k(Y_\cdot, P)$$

is a w.e. of simpl. ab. groups.

2) fibrations:
Reedy fibrations $X \to Y$ s.t.

$$\pi_k(X_\cdot;P) \to \pi_k(Y_\cdot;P)$$

is a fibration of simpl. ab. groups. //

Write $s\mathcal{S}^E$ for this model cat., and

$$\pi_i^\natural(X_\cdot ; P) = \text{ho}\, s\mathcal{S}^E\left(\Delta[i]/\partial\Delta[i] \wedge P,\ X_\cdot\right)$$

$\llcorner$ simpl. set

Thm The groups $\pi_i^\natural(X_\cdot, P)$ are the D_2-grps for the spectral seq

$$[P, X_n] \Rightarrow [P, |X_\cdot|].$$

//

This gives the analog of the spiral exact sequence

$$\cdots \to \pi^\natural_{p-1}(X; \Sigma^{q+1}P) \to \pi^\natural_p(X, \Sigma^q P) \to \pi_p \pi_q(X, P) \to \pi^\natural_{p-2}(X, \Sigma^{q+1}P) \to \cdots$$

Weak eq. in $s\mathcal{S}^E$ are the $\pi^\natural_*(X; P)$-iso. s.

Check: $\pi^\natural_i(\Delta[n]/\partial\Delta[n] \wedge P, -) = 0$, $i < n$.

Using this we can build Postnikov sections in the simpl. direction, i.e.

given $X_\cdot$, we get

$$X_\cdot \to P_n X_\cdot \quad (= \tau_{\leq n} X_\cdot)$$

s.t. the induced map on $\pi_i^\natural$ is an iso- for $i \leq n$ and s.t. $\pi_i^\natural P_n X = 0$ for $i > n$. Moreover,

$$X \to \varprojlim_n P_n X.$$ //

Prop The forgetful functor

$$s\mathcal{S}^{A_\infty} \longrightarrow s\mathcal{S}$$

creates a simpl. model cat. str. on $s\mathcal{S}^{A_\infty}$, i.e. this is a model cat. with w.e. and fibr. being the maps which become w.e. and fibr. in $s\mathcal{S}$. //

Decomposition of the basic moduli problem:

$$X \in \mathcal{S} \subset s\mathcal{S}^E \qquad \text{const. simpl. spectrum.}$$

$$\rightsquigarrow P_n iX \in s\mathcal{S}^E$$

Let $\mathcal{M}_n(X)$ be the classifying space of the category with

ob: $R \in s\mathcal{S}^{A_\infty}$ $\quad R \xrightarrow{\sim} P_n X$ in $s\mathcal{S}^E$

mor:
$$\begin{array}{c} R \\ \downarrow \\ R' \end{array} \in s\mathcal{S}^{A_\infty} \qquad \begin{tikzcd} R \arrow[dr, "\sim"] \arrow[d] & \\ R' \arrow[r, "\sim"'] & P_n X \end{tikzcd}$$

Can form Postnikov sections in $s\mathcal{S}^{A_\infty, E}$ which are compatible with the Postnikov sections in $s\mathcal{S}^E$, so we get a map

$$\mathcal{M}_{n+1}(X) \to \mathcal{M}_n(X).$$

Prop $B(A_\infty / X) \xrightarrow{\sim} \underset{\leftarrow n}{\mathrm{holim}}\, \mathcal{M}_n(X)$. //

The obstruction th. is an analysis of the difference between $\mathcal{M}_{n+1}(X)$ and $\mathcal{M}_n(X)$.

Last time we introduced a model str., the resolution model str., on the category $s\mathcal{S}^{A_\infty}$ of simpl. A_∞-ring spectra. If E is a spectrum which satisfies the Adams condition that E is a filtr. colimit

$$E = \operatorname{colim}_\alpha DP_\alpha$$

with P_α in the class

$$\mathcal{P} = \left\{ \text{fin. } P \;\middle|\; \begin{array}{l} E_*P \text{ proj. } / E_* \\ [P,M] = \operatorname{Hom}_{E_*}(E_*P, M) \end{array} \right\}$$

Then a map $X. \to Y.$ in $s\mathcal{S}^{A_\infty}$ is a w.e. if the induced map

$$\pi_*\pi_*(X., P) \longrightarrow \pi_*\pi_*(Y., P) \qquad (E^2\text{-grps})$$

is an iso., for all $P \in \operatorname{ob}\mathcal{P}$, or equivalently, if

$$\pi_*^\natural(X., P) \longrightarrow \pi_*^\natural(Y., P) \qquad (D^2\text{-grps})$$

is an iso., for all $P \in \operatorname{ob}\mathcal{P}$. Moreover, If $X \to Y$ is a map of spectra, and if $\pi_*(X, P) \xrightarrow{\sim} \pi_*(Y, P)$, for all $P \in ob\,\mathcal{P}$, then $E_*X \xrightarrow{\sim} E_*Y$.

Def The E-resolution model str. on $s\mathcal{S}^{A_\infty}$ is the localization w.r.t. the class of maps $X_\cdot \to Y_\cdot$ s.t.

$$E_*X_\cdot \to E_*Y_\cdot$$

is a w.e. of simpl. ab. grps. //

Topological		Algebraic
X	$\longrightarrow$	E_*X
simpl. A_∞-ring spectrum		simpl. E_*-alg. simpl. algebra in E_*E-comodules

The algebraic side is a model cat. with w.e. = w.e. of simpl. ab. grps. On the top. side, $X_\cdot \to Y_\cdot$ is a w.e. if and only if $E_*X_\cdot \to E_*Y_\cdot$ is one. In algebra, have only π_*E_*, not $E_*^{\natural}$.

$\mathcal{C}$ model cat., $X \in \mathrm{ob}\,\mathcal{C}$

$$\mathcal{M}(X) = \text{nerve}\left\{\begin{array}{l}\text{ob: } Y \in \mathrm{ob}\,\mathcal{C} \text{ s.th. } Y \sim X \\ \text{mor: w.e.}\end{array}\right\}$$

$$= B\,\mathrm{haut}(X).$$

ex $\mathcal{C}$ = spaces, $X \to P_n X$ n'th Postnikov section. How does $\mathcal{M}(P_n X)$ relate to $\mathcal{M}(P_{n-1}X)$? For simple spaces

$$\begin{tikzcd} P_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & * \arrow[d] \\ P_{n-1} X \arrow[r] & K(\pi_n X, n+1) \end{tikzcd}$$

In general, there is a cartesian sq.

$$\begin{tikzcd} P_n X \arrow[r] \arrow[d] & P_1 X \arrow[d] \\ P_{n-1} X \arrow[r] & K(A, n+1) \end{tikzcd},$$

where $P_1 X$ is the nerve of the fund. groupoid of X, and where $A = \pi_n X$ as a module over the fund. groupoid.

The ~~space~~ $K(A, n+1)$ is obtained from the push-out by killing homotopy grps in degrees $\geq n+2$.

Given $Y \sim P_n X$, we form a cart. sq.

$$\begin{array}{ccc} Y & \to & P_1 Y \\ \downarrow & & \downarrow \\ P_{n-1} Y & \to & K \end{array}$$

from the push-out by killing htpy grps. in deg. $\geq n+2$. Then

$$\begin{array}{ccccc} P_{n-1} Y & \to & K & \leftarrow & P_1 Y \\ \wr & & \wr & & \wr \\ P_{n-1} X & \to & K(A, n+1) & \leftarrow & P_1 X \end{array}$$

so

$$\mathcal{M}(P_n X) \sim \mathcal{M}(P_{n-1} X \to K(A, n+1) \leftarrow P_1 X).$$

In the situation we will consider, the extra maps in the analog of the right-hand side can be expressed entirely by ~~algebraic~~ data....

"