4/22/2003

André

①

Seminar on the Hopkins Miller thm

On the construction of the EO_n's

$$\begin{pmatrix}\text{reduce}\\ \text{mod}\\ \mathfrak{m}\end{pmatrix} \cdot \pi_0 : \mathcal{A}_\infty^{LT} \longrightarrow \mathcal{FG}^{op}_{\text{sep. closed}}$$

is an equivalence of topological categories, i.e.

- essentially surjective
- fully faithfull in topological sense :

} have shown this in Vigleik's talk

$$\mathcal{A}_\infty^{LT}(E,F) \longrightarrow \mathcal{FG}((k_2,\Gamma_2),(k_1,\Gamma_1))$$

is a homotopy equivalence.

discrete topology

$$E_2 \in \mathcal{A}_\infty^{LT} \longmapsto (\mathbb{F}_4, \Gamma_2)$$

$$G = \tau^* \rtimes \mathbb{Z}/2$$

binary tetrahedral group

tetrahedral group (order 12) $\subset SO(3)$

look at preimage in univ. cover of $SO(3)$

i.e. in $SU(2)$,

this is a central extension (universal one)

there is

$$1 \to A \to \tilde{G} \to G \to 1$$

an action of $\mathbb{Z}/2$ on the tetrahedral group corresponding to the outer hom we get by exchanging vertices & faces. $\rightsquigarrow \rtimes$

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Also write G for the category $\cdot \mathcal{G}_G$.
We have $G \to \mathcal{F}\mathcal{G}^{op}$

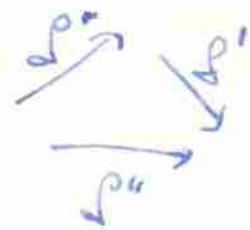
$$\begin{tikzcd} G \arrow[r] \arrow[dr, dashed] & \mathcal{F}\mathcal{G}^{op} \\ & \mathcal{A}_\infty^{LT} \arrow[u] \end{tikzcd}$$

question: can we lift this?

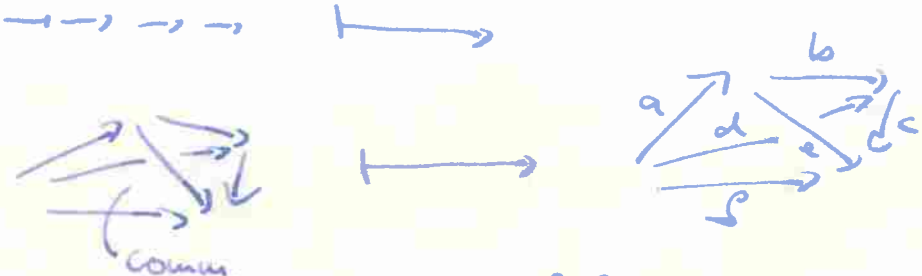
Once we lift it (for a good model of E_2 that will have strict action of G).
We can form $EO_2 = E_2^{hG} = F(EG_+, E_2)^G$.
$= \underset{G}{\operatorname{holim}}\, E_2$

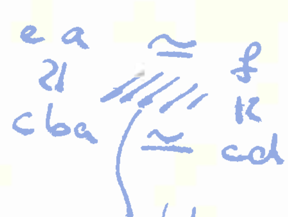
① Relax notion of functor

1) 



chosen homotopy $f'f \simeq f''$

2) 



htpys come from first step, square extra piece

3) ... cube ...

will allow us to def some notion of E^{hG}

② second idea:

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d, "\simeq"'] & \mathcal{A}_\infty^{LT} \arrow[d, "\simeq"] \\ G \arrow[r] & \mathcal{F}\mathcal{G}^{op} \end{tikzcd}$$

$\tilde{G} = \left\{ \begin{array}{l} 1 \text{ object} \\ \text{morphisms} = \mathcal{A}_\infty^{LT}(E_2, E_2) \end{array} \right\}$ that map to G

i.e. has 24 connected components ...

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$$EO_2 = \underleftarrow{\mathrm{holim}}_{\tilde G} E_2$$

(3) of course the best thing would be to really have an action on the nose

this is rather difficult but convenient for later.

<u>Recall:</u> $\underleftarrow{\mathrm{holim}}$ for spaces

$$J \longrightarrow \text{Spaces}$$

In indexing category

want $\underleftarrow{\mathrm{holim}}_{J} X_i$

a point in ______ is { pt x_i in each X_i; for every morphism $X_i \xrightarrow{f} X_j$ a path between x_j & $f(x_i)$;

$$\begin{tikzcd} X_i \arrow[r, "f"] \arrow[dr, "gf"'] & X_j \arrow[d, "g"] \\ & X_k \end{tikzcd}$$

a two simplex $gf(x_i)$, $g(x_j)$, x_k }

Now reformulate this for X_i = spectra & J ($\hat{=}\ \tilde G$) a topological category:

$$Y^0 \quad \prod_{i\in J} X_i$$

$$Y^1 \quad \prod_{i,j\in J} X_i^{\cancel{\Delta^1} \times \cancel{\mathrm{Map}(X_i,X_j)}\, J(i,j)}$$

$\vdots$

$$Y^{n-1} \quad \prod_{i_0,\dots,i_n} X_{i_n}^{\cancel{\Delta^n} \times J(i_0,i_1)\times J(i_1,i_2)\times\dots\times J(i_{n-1},i_n)}$$

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cosimplicial spectrum, $\mathrm{Tot}(-)$ gives back Δ's with all the nice compatibility conditions

$$\mathrm{Tot}(X^\bullet) =: \underset{\overleftarrow{J}}{\operatorname{holim}}\, X_i$$

At this point it becomes easy to fill in the dotted arrow above

$$\begin{tikzcd}
\tilde{J} \arrow[r, "\tilde{F}"] \arrow[d, "\wr"'] & \underline{\text{Spectra}} \\
J \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

homotopy commutes

F is defined by

$$F(i) = \underset{\substack{\overleftarrow{(i\to j)\text{ all arrows}} \\ \in (i\downarrow J)}}{\operatorname{holim}} \tilde{F}(j)$$

$\simeq \tilde{F}(i)$

i = initial object

Where are we and why?

We are ultimately interested in $\pi_* S^0$, or say $\pi_* S^0_{(p)}$. But the p-local stable homotopy category is already much more complicated than the rational one (latter $\simeq$ graded $\mathbb{Q}$ vs' $\to$ Mike Hill)

There is a theorem, the thick subcategory theorem that is making the sense in which this category is more complicated precise: it tells us that for each p there is a tower of non-trivial Bousfield localizations of $\mathcal{S}^{fin}_{(p)}$ (finite p-local spectra). Think of Bousfield localization at a homology theory $E_*(-)$ as

$$\text{"}\ \mathcal{S}^{fin}_{(p)\,E} = \mathcal{S}^{fin}_{(p)} \Big/ \underset{\text{spectra}}{E\text{-acyclic}} \quad \left(\begin{matrix} \text{i.e. } X \text{ s.t.} \\ E_*(X) = 0 \end{matrix}\right)\text{"}.$$

also denoted $\langle E \rangle$.

Fact: $L_E = L_F \iff \langle E \rangle = \langle F \rangle$, $\langle E \rangle \leq \langle F \rangle \Rightarrow L_E L_F = L_F$ / $L_E \to L_F$

Thick subcategory theorem: $\mathcal{S}^{fin}_{(p)}$ has a filtration (Hopkins, Smith)

$$(✿) \quad \mathcal{S}^{fin}_{(p)} = \mathcal{C}^0 \supsetneq \mathcal{C}^1 \supsetneq \mathcal{C}^2 \cdots \supsetneq \underbrace{\mathcal{C}^n}_{= K(n-1)\text{-acyclics}} \supsetneq \cdots$$

s.t. for any spectrum E, $\langle E \rangle = \mathcal{C}^n$ for some n.

Note: • $K(n) = n^{th}$ Morava K-theory, constructed by Baas & Sullivan, using bordism of mfs w/ singularities, coefficients are $\mathbb{F}_p[v_n^{\pm 1}]$ ($\sim$ Vigleik's first talk). $|v_n| = 2p^n - 2$ $\boxed{\text{carry Honda fgl.}}$

• The inclusions are a non-trivial statement, only true for finite spectra

• On finite spectra $\langle K(n) \rangle = \langle E(n) \rangle$, $E(n)$ = Johnson-Wilson spectra def. by LEFT &

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n^{\pm 1}].$$

"universal" height n fgl

(✿) This is called "the chromatic filtration of the stable homotopy category".

• For big enough p (fixed n), there are alg. models for $\mathcal{S}_{E(n)}$.

• $X \in \mathcal{C}^n - \mathcal{C}^{n+1}$ is called "of type n".

Since $\pi_* \mathbb{S}^0_{(p)}$ is too difficult for us to understand, we can try our luck in the localized categories and look at $\pi_*(L_E X)$. (L_E = (induced by the) (functorial) fibrant replacement in new model str (BL keeps cofib, extends w.eqs. & therefore has fewer fib) or : $\mathcal{S} \underset{R}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{S}_E$ has right adjoint, $L_E := R \circ \mathcal{L}$.) (Careful : it is not clear whether $\mathcal{S}^{fin}_{(p)}{}_E \subset \mathcal{S}^{fin}_{E(p)}$ is full $\rightsquigarrow$ telescope conj. (?))

The thick subcategory theorem tells us that $L_{K(0)}$ should be the easiest to understand (indeed $K(0) = H(-;\mathbb{Q})$), then $L_{K(1)}$ and so on. ($K(\infty) = H(-, \infty)$)

~~How far are people today~~

In other words : (thick subc. thm)

every finite spectrum admits a tower

$$\begin{tikzcd} L_{K(0)} X \arrow[d] & L_{K(1)} X \arrow[l] \arrow[d] & L_{K(2)} X \arrow[l] & \cdots \arrow[l] & X \simeq \operatorname{holim} L_{K(n)} X \arrow[l] \\ M_0 X \arrow[ur, "+1"', "*"] & M_1 X \arrow[ur, "+1"', "*"] & & & \end{tikzcd}$$

chromatic convergence thm (Hopkins Smith)

apply π_* to get an exact couple, the corresponding spectral sequence is called the geometric chromatic spectral sequence, it converges to the (geometric) chromatic filtration of the stable homotopy groups of X

$\ldots F^{n-1} \subseteq F^n = \ker(\pi_* X \to \pi_* L_{K(n)} X)$. $\in \pi_* X$

How far are we today?

$\pi_* L_{K(0)} \mathbb{S}^0$ ✓ well understood

$\pi_* L_{K(1)} \mathbb{S}^0$: ~~this~~ is ~~computed~~ : It is the image of J.

Therefore there is a geometric definition of these elements.

Recall from Mike Hopkins' talk that it is computed at

odd primes using some sort of spectral sequence that had to do with $K_p^\wedge$, and that at $p=2$, $KO_2^\wedge$ was a better starting point than $K_2^\wedge$ (compare also Adams original papers on the image of J IV).
As of today, people are still trying to get a good understanding of the second chromatic localization $L_{K(2)} S^0$. For this purpose,

$$E_2 := E_{\mathbb{F}_4,\, \Gamma = \text{Honda fgl}}$$

will play the rôle of $K_p^\wedge$-theory, and EO_2 will play the rôle of $KO_2^\wedge$ (at primes 2 and 3 ~~(2,3)~~). π_* 24 ☺

<u>More precisely</u>: To compute $L_{K(n)} S^0$, one uses the ~~E_n-ASS~~ $K(n)$-local E_n-Adams-spectral sequence (discussed in the Appendix of Dev. & Hopkin's paper on htpy fixed points spectra ~~of~~ for closed subgroups of the Morava stabilizer group), ~~Setup is like traditional~~ this is the E_n-ASS in the $K(n)$-local category, convergence follows from the fact that any $K(n)$-local spectrum X is E_n-nilpotent

def.: F_*-local E-nilp spectra smallest class $\mathcal{C}$ of F_*-local spectra s.t.

i) $L_F E \in \mathcal{C}$

ii) $L_F(N \wedge X) \in \mathcal{C}$ whenever $N \in \mathcal{C}$

iii) $\mathcal{C}$ is closed under retracts & cofibres

$E_n = E_{\mathbb{F}_{p^n}, \text{Honda fgl}}$. $\boxed{\text{Don't confuse with } E(n)!}$

Devinatz and Hopkins show that this spectral sequence is actually a homotopy fixed point spectral sequence (in a continuous sense) for $\pm$

$$L_{K(n)} \mathbb{S}^0 \simeq E_n^{hG_n}$$ full Morava stabilizer group ⋊ action of Galois group.

It's E_2-term is the same as that of the chromatic spectral sequence

$$\begin{array}{llcl} \text{htpy fixed pts s.s.} & H_c^*(G_n, E_{n*}) & \Longrightarrow & \pi_* L_{K(n)} \mathbb{S}^0 \\ & \quad\cong & \text{same s.s.} & \quad \| \\ K(n)\text{-local } E_n \text{ ASS} & \mathrm{Ext}_{E_{n*}E_n}(E_{n*}, \underbrace{E_{n*}L_{K(n)}\mathbb{S}^0}_{\overset{?}{=} E_{n*}\mathbb{S}^0}) & \Longrightarrow & \pi_* L_{K(n)} \mathbb{S}^0 \end{array}$$

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and I believe that that is not by coincidence, Ravenel, red book p. 80 (top) makes a sort of mysterious remark that sounds like this's E s.s. could indeed be the same as the ANSS for $L_{K(n)} \mathbb{S}^0$. Then we would have a picture of s.s. looking like this

$$\begin{array}{ccc} H_c^*(G_n, E_{n*}) & \Longrightarrow & \pi_* L_{K(n)} \mathbb{S}^0 \\ \Downarrow & & \\ \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) & \Longrightarrow & \pi_* \mathbb{S}^0 \end{array}$$

However, historically, the algebraic side was there first.

and possibly the ~~ss~~ left vertical spectral sequence is the same as the chromatic spectral sequence (?).

THIS MIGHT BE NOT TRUE, BUT THE TELESCOPE CONJECTURE?

~~That is what the EO~~

But this point of view makes clear, what the EO - theories are good for — the break up the homotopy fixed point spectral sequence into two steps, which are hopefully easier to compute.

Remark about elements that go wheeeh.

FOR THE REST OF THE SEMINAR $n=2$.

Mike Hill will tell us about one of the two steps, namely the computation of $(eo_{2*}, eo_{2*}eo_2)$; it is indeed possible to make some computations in $\pi_*(L_{K(2)} S^0)$ with this ($\to$ Goerss, Mahowald, Rezk ?).

But this is a different story from what I want to speak about today.

Lets stick with the picture that we want to understand chromatic level two phenomena, and that these should somehow be observed by height 2 cohom. theories.

Some Complex orientable cohom. theories with height ≤ 2 formal group laws

There is a geometric source of height ≤ 2 formal group laws:
Let $\mathcal{E}$ be an elliptic curve. Define the formal completion of $\mathcal{E}$ around 0 by

$$\boxed{X^{\wedge}_{Y} = (Y, \underleftarrow{\lim}\, \mathcal{O}_X / I(Y)^n)}$$
str. sheaf.

$$\mathcal{E}^{\wedge}_{0} = (\underset{\operatorname{spec} R}{\{0\}}, \underleftarrow{\lim}_n\, \mathcal{O}_X / I(0)^n)$$

Then $\mathcal{E}^{\wedge}_{0}$ is an (affine 1-dimensional) formal group: The ~~multipl~~ addition of $\mathcal{E}$ induces

$$\mathcal{E}^{\wedge}_{0} \times \mathcal{E}^{\wedge}_{0} \longrightarrow \mathcal{E}^{\wedge}_{0}$$

and $\mathcal{E}^{\wedge}_{0} \cong \hat{A}^1_0$; $\mathcal{O}_{\mathcal{E}^{\wedge}_0} \cong \mathcal{O}_{\hat{A}^1_0} = R[\![x]\!]$.

non canonically

ways to think about this formal completion:

- You are looking at all infinitesimal (nilpotent) neighbourhoods of 0 in $\mathcal{E}$ at once
- Picking a coordinate x around 0 will give an isomorphism of a neighbourhood of 0 with $A^1 = \operatorname{spec} R[x]$ ⇒ an isom $\mathcal{O}_{\mathcal{E}^{\wedge}_0} \cong R[\![x]\!]_{\text{x}}$. That is the reason for the terminology "x = coordinate of the formal group". ⇒ an fgl
- Think you are expanding the multiplication (around 0) as a power series using the chosen coordinate.

Height: $[p]_F$ is the Frobenius map!

<u>Theorem</u> (→Silverman): Formal groups of the form $\mathcal{E}^{\wedge}_{0}$ have height ≤ 2. (actually 1 or 2)

<u>Def:</u> If the height equals 2, we call the curve "super-singular".

At the prime 2 there is only one supersingular elliptic curve (up to autom) given by

$$C: \; y^2 + y = x^3$$

over $\mathbb{F}_4$.

At the prime 3:

Elliptic spectra:

Let E be an even periodic ring spectrum (i.e. $E^{odd} = 0$, $E^2 \ni$ unit). Then the AHSS for $E^*\mathbb{C}P^\infty$ collapses and it follows that E is complex orientable.

Definition (Ando, Hopkins, Strickland): An elliptic spectrum is a triple $(E, \mathcal{C}, t)$, where E is an even periodic ring spectrum, $\mathcal{C}$ is an elliptic curve over E^0 and t is an isomorphism between $\hat{\mathcal{C}}_0$ and the formal group over $E^0\mathbb{C}P^\infty$.

Note: We need to be careful and speak about formal groups rather than formal group laws, that is one reason why we look at even per. ring spectra and at E^0 rather than E^*. (No graded fgl's, no strict isoms.)

We do not want to choose a coordinate (= MPU-orientation ($\Rightarrow$ MU $\to$ MPU $\to E$)), it is not a natural thing to do if you come from elliptic curves and would destroy modularity properties.

A morphism of elliptic spectra is: guess what.

tmf

We would like to have a universal elliptic cohomology theory that maps into each of the other ones. Note that this can't exist, there is no such thing as a universal elliptic curve, and also, if we have a spectrum that maps into each elliptic spectrum naturally (wrt maps of ell spectra), it cannot be cx orientable ($\Rightarrow$ 2-periodic), because otherwise that would give us a coordinate on all ell. fgl's (already autom of $H(\ ;\hat{\mathbb{C}}A[u^{\pm 1}])$ fails).

Correction about last time:

geom. chrom. s.s.

$$\begin{tikzcd}
L_{E(0)} S^0 \arrow[dr, "+"'] & & L_{E(1)} S^0 \arrow[ll] \arrow[dr, "+"'] & & L_{E(2)} S^0 \arrow[ll] & \cdots \arrow[l] & S^0 \simeq \underset{\longleftarrow n}{\mathrm{holim}}\, L_{E(n)} S^0 \\
& M^0 \arrow[ur, "*"] & & M^1 \arrow[ur, "*"] & & &
\end{tikzcd}$$

$E(n)$ = Johnson Wilson spectra $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$
Landweber exact.

(don't confuse with E_n's)

<u>proof</u> that $\mathcal{E}^{\wedge}_0$ has height ≤ 2:

Topological modular forms

We would like to have a universal elliptic cohomology theory that maps into each of the other ones in a natural way. Note that for several reasons such a thing cannot exist:

1) There is no such thing as a universal elliptic curve $\rightsquigarrow$ Stacks ☺

2) Such a spectrum that maps into all the elliptic spectra naturally can't be even periodic: otherwise it would be complex orientable, and the universality would imply a coordinate on the formal group of every elliptic spectrum, natural under maps of elliptic spectra. But such a coordinate can't exist, ~~as~~ already the example $H^*(-;\mathbb{C}[u_2^{\pm 1}])$ has automorphisms that don't preserve any coordinate

3) Therefore we can certainly not hope for our spectrum to be Landweber exact.

~~But we will see that the Landweber exact functor theorem gets us pretty~~