The elliptic curves Hopf algebroid or the stack of elliptic curves

(Careful: this Hopf-algebroid will not be $(\mathrm{tmf}_*, \mathrm{tmf}_* \mathrm{tmf})$, but rather generate the E_2-term of a spectral sequence computing it.)

Let C be the plane projective curve

$$C: \; y^3 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

over
$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

A corepresents plane proj. curves in Weierstraß form. Such a curve is nonsingular if and only if the discriminant $\Delta \in A$ is invertible. I.e. $A[\Delta^{-1}]$ corepresents elliptic curves (and thus an elliptic curve with base-point at ∞: $e = [0:1:0]$) in Weierstraß form. An affine isomorphism of the plane that carries a curve in Weierstraß form into another curve in Weierstraß form (with possibly different a_i's) has over $k = \bar{k}$ the form

$$x \mapsto \lambda^2 (x + r) \qquad \lambda = \text{a unit.}$$
$$y \mapsto \lambda^3 (y + sx + t)$$

If $(x,y) \in C$, then $(\lambda^2(x+r), \lambda^3(y+sx+t))$ is a solution for

$$y^2 + \eta_R(a_1) xy + \eta_R(a_3) y^3 = x^3 + \eta_R(a_2) x^2 + \eta_R(a_4) x + \eta_R(a_6)$$

This defines a Hopf algebroid (A, Γ),

$$\Gamma = A[r,s,t][\lambda^{\pm 1}].$$

where the comultiplication encodes the composition and the conjugation the inverse isomorphism.

$\rightsquigarrow$ Stack of gen. ell. curves in W.F.: $\mathcal{M}_{Ell} := \mathcal{M}_{(A,\Gamma)}$ $\leftarrow$ this is the notation for the corresponding groupoid valued functor

The algebraic theory of modular forms

Write $\Lambda = A[r,s,t]$. (A,Λ) is also often referred to as the elliptic curves Hopf algebroid. Many notations for the same thing:

$$\Gamma(\mathcal{O}_{\mathcal{M}_{(A,\Lambda)}}) = H^0(\mathcal{M}_{(A,\Lambda)}) = \mathrm{Ext}^0_\Lambda(A,A) = \ker(\eta_L - \eta_R)$$

global sections on the stack of generalized elliptic curves in Weierstraß form and strict affine isoms $\lambda = 1$

$= H^0(H, A)$ Heisenberg gp $\left\{\begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right\}$

elements of A invariant under the action of the Heisenberg group; i.e. global sections of $\mathcal{O}_{\mathbb{A}^5_{\mathbb{Z}}}$ that are well defined on the "quotient space" $\mathbb{A}^5_{\mathbb{Z}}/H$" "moduli space..."

$$\cong \mathbb{Z}[c_4, c_6, \Delta]/(12^3\Delta = c_4^3 - c_6^2)$$

Deligne, Tate

$= mf_*$ weight: $a \in A$ has weight $i \Leftrightarrow a \mapsto a \otimes \lambda^{-i}$ $A \to A[\lambda^{\pm 1}]$

What about the higher cohomologies of (A,Λ)

$$A \rightrightarrows \Lambda \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \Lambda \otimes_A \Lambda \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \Lambda \otimes_A \Lambda \otimes_A A \quad \cdots \quad ?$$

"higher derived functors of modular forms"

E_2-term of MU-based ASS for tmf_* (?) $\to$ Mike Hill.

This Hopf algebroid is what we will use for computations. Here is another (equivalent) point of view on modular forms, that is closer to ~~the construction~~ how we like to think of them in the world of elliptic spectra:

$$\mathcal{M}f_k = \Gamma(\omega^k_{\mathcal{M}_{Ell}}), \quad \text{where } \omega_{\mathcal{M}_{Ell}} = p_* \Omega^1_{C/\mathcal{M}_{Ell}}$$

(line bdle on moduli space)

~~...~~ This is the same as a section of ω^k_A "invariant under the action of $H \rtimes \mathbb{G}_m$".

Let

$$u = \frac{dx}{2y + a_1 x + a_3} \quad \text{(invariant differential)},$$

this gives a trivialization for all the ω^k_A, and

$$\bigoplus_k \Gamma(\omega^k_A) = A[u^{\pm 1}]$$

and the "Γ-invariant sections" i.e. the modular forms are

$$H^0(A[u^{\pm 1}], \Gamma[u^{\pm 1}]) \quad \left(\begin{array}{l}\text{also higher cohoms}\\ \text{are the same}\end{array}\right)$$

recall $\Gamma = A[\lambda^{\pm 1}]$. Here A is considered as a non-graded ring, u has weight 1 or degree two if you speak to topologists,

Note that

$$\frac{d\,\lambda^2 x}{2\lambda^3 y + \lambda a_1' \lambda^2 x + a_3' \lambda^3} = \frac{u}{\lambda}$$

while the action of the Heisenberg group doesn't change the differential. (up to strict isom means this doesn't it?)

Remarks: 1) This should remind you of what happens if we are switching between MU and $MU[u^{\pm 1}]$, at the one side we have graded rings and strict isoms, classified by graded maps from (L, LB), at the other there are ordinary rings and isoms, classified by $(L, LB[\lambda^{\pm 1}])$, and the two Hopf algebroids (L, LB) and $(L[u^{\pm 1}], LB[\lambda^{\pm 1}][u^{\pm 1}])$ are equivalent.

$f(x) = \lambda x$
& strict isoms $\rightsquigarrow$ all isoms

2) The two points of view on modular forms are the same, because (A,Λ) corepresents gen. elliptic curves in W.F. together with a choice of differential.

3) I.e. Λ encodes exactly the isoms that induce a strict isom on the corresponding formal group, we have

$$(L, LB) \longrightarrow (A, \Lambda)$$

and

$$(L, LB[\lambda^{\pm 1}]) \longrightarrow (A, \Gamma) \quad \left(= (A, \Lambda[\lambda^{\pm 1}]) \right)$$

i.e.

$$\mathcal{M}_{Ell} \longrightarrow \mathcal{M}_{FG}$$
$$C \longmapsto \hat{C}_0$$

if we restrict to $\mathcal{M}_{(A,\Lambda)}$ we land in $\mathcal{M}_{FG,\text{ strict isoms}}$.

4) This is one reason to identify the higher E^*'s in an elliptic cohom. theory with sections in ω^k_{C/E^0}. Another way to say this is that we look at the associated graded of $I(0)^n \subseteq \ldots \subseteq \hat{C}_0$:

$$\begin{array}{lccc} & E^0\mathbb{CP}^\infty & \supset & I(0) \\ \text{thing:} & \| & & \| \\ & E^0[[x]] & & x\,E^0[[x]] \end{array}
\qquad
\begin{array}{ccccc} E^0\mathbb{CP}^\infty & \longrightarrow & E^0S^2 = E^0\mathbb{CP}^1 & = & I(0)/I(0)^2 \\ \| & & \| & & \\ E^0 MU(1) & & E^0(M(\mathbb{C})) & & \\ \| & & \| & & \\ x\,E^0[[x]] & & x\,E^0[[x]]/x^2 & & \end{array}$$

5) Mike claims this Hopf algebroid is split

This is how far we get using LEFT:

$$\mathcal{M}_{Ell} \longrightarrow \mathcal{M}_{FG} \qquad \left(\begin{array}{l}\text{representable :}\\ (L, LB[\Delta^{\pm 1}]) \to (A, \Gamma)\end{array}\right)$$

$$C \longmapsto \hat{C}_0$$

is not a flat map. However, $\mathcal{M}^\circ_{Ell} := c_4^{-1}\mathcal{M}_{Ell} \cup \Delta^{-1}\mathcal{M}_{Ell}$ is flat over $\mathcal{M}_{FG}$. (also representable)

$\mathcal{M}^\circ_{Ell}$ is $\mathcal{M}_{Ell}$ ~~minus~~ with the additive fibre removed, i.e. generalized elliptic curves w/ ht 1 or 2 fgl. Recall from Jacobs talk:

$$U = \mathrm{spec}(R) \longrightarrow \mathcal{M}_{FG} \quad \text{flat}$$

if and only if $R[u^{\pm 1}]$ satisfies the conditions of the LEFT. Kári's talk: no phantoms.

I.e. we get a ring spectrum (in a functorial way) for any

$$U = \mathrm{spec}(R) \xrightarrow{\text{flat}} \mathcal{M}^\circ_{Ell} \qquad E(U)$$

i.e. we have a presheaf (= contravariant functor) on the basis of affine opens in the flat topology on $\mathcal{M}^\circ_{Ell}$.

Restrict this to a presheaf on affine opens in the "quasi-étale" topology, i.e. maps $\mathrm{spec}(R) \to \mathcal{M}^\circ_{Ell}$ that are flat and whose cotangent complex is acyclic.

Then one can lift this presheaf to a presheaf with values in the category of A_∞-ring spectra.

(Hard work, done by Hopkins & Miller, using Dwyer-Kan obstruction theory; E_∞: Goerss-Hopkins).

~~sloppy notation: write $\mathcal{M}_{(A,\Gamma)}$ instead of $\mathcal{M}^{\circ}_{(A,\Gamma)}$ etc.~~

~~Look at affine covers~~

~~spec (A)~~

sloppy notation: write $\mathcal{M}_{(A,\Gamma)}$ for $\mathcal{M}^{\circ}_{(A,\Gamma)}$ ...

Look at the affine cover

$$\begin{tikzcd}
\operatorname{spec}(A^*)^{\circ} \arrow[r] & \mathcal{M}^{\circ}_{(A,\Gamma)} \\
\operatorname{spec}(\Gamma^*)^{\circ} \arrow[u, shift left] \arrow[u, shift right] \arrow[ur] & \\
\cdots \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2] &
\end{tikzcd}$$

it turns out that in the quasi-affine topology, the sheaf condition is already satisfied on the affine opens involved. Therefore, sheafification will simply be the equalizer

$$E^{sh}(\mathcal{U}) \longrightarrow E(\mathcal{U}|_{\operatorname{spec}(A)^{\circ}}) \rightrightarrows E(\mathcal{U}|_{\operatorname{spec}(\Gamma)^{\circ}})$$

we define tmf of the connected cover of

$$E^{sh}(\mathcal{M}^{\circ}_{Ell}) \qquad (= L_2\,\mathrm{tmf})$$

But this is nothing else than the ~~homotopy~~ "fixed points" of the action of $\Gamma[u^{\pm 1}]$ on $A[u^{\pm 1}]$

[if that's really split, otherwise the analogue of what André spoke about for non split groupoids.]

Remarks: 1) Of course we don't like strict fixed points. That's why we replace $E(-)$ by a fibrant A_∞-presheaf before we sheafify, i.e. hoequalizer, ho fixed points etc.

2) Sheafification in general

$$E^{sh}(U) = \lim_{\leftarrow}$$

this is the sense in which $E^{sh}(\mathcal{M}^{\circ}_{Ell})$ is the homotopy limit over "all" elliptic spectra.

3) A different affine cover

$$\begin{matrix} \{U_i\} \\ \uparrow\uparrow \\ \{U_i \cap U_j\} \\ \vdots \end{matrix} \longrightarrow \mathcal{M}_{(A,\Gamma)}$$

would be equivalent to

$$\begin{matrix} \operatorname{spec}(A)^{\circ} \\ \uparrow\uparrow \\ \operatorname{spec}(\Gamma)^{\circ} \\ \vdots \end{matrix}$$

as stacks

untangle what that says ⇝ our equalizer was enough.

4) We could (probably) interpret the spectral sequence

$$H^*(\mathcal{M}_{Ell}, \omega^{\otimes *}) \Longrightarrow tmf_*$$

as homotopy fixed point s.s.

But we will use another approach:

Let C be a generalized elliptic curve, i.e. a genus one curve with a marked smooth point e and at worst nodes as singularities (Δ^{-1} or c_4^{-1} (?))

A choice of Weierstraß parametrization on C ~~amounts~~ ~~to~~ is the same as a choice (x,y) of sections $x \in \Gamma(C, \mathcal{O}_C(2e))$ and $y \in \Gamma(C, \mathcal{O}_C(3e))$ s.t. $x^3 - y^2 \in \Gamma(C, \mathcal{O}_C(5e))$.

This again is the same as the choice of a coordinate on $\hat{C_0}$ ~~up to degree~~ modulo degree 5 ($F(x_1,x_2) \in R[[x_1,x_2]]/(x_1,x_2)^5$).

Pf: Claim $WF(C) \longrightarrow$ coord up to deg 4

is isom $(x,y) \longmapsto -\frac{x}{y} \quad (=: z)$

Claim : ~~eg. uniquely~~ : T' coord on $\hat{C_0}$, then $\exists\, x', y'$ s.t. $T' = \frac{x'}{y'}$, $T = T' \iff (x,y) = (x',y')$.

Rezk p. 35 :

Let $X(4)$ denote the Thom spectrum of

possibly need to make this 2-periodic

$$\Omega SU(4) \longrightarrow \Omega SU \simeq BU$$

Bott.

Then, for any even periodic ring spectrum E,

$E_0 X(4)$ corepresents coordinates up to degree 4 on G_E & $E_0(X(4)\wedge X(4))$ change of coord's by strict isom.

For E = an elliptic spectrum, (E, C, ϕ)

$(E\wedge X(4), C', \phi')$ ell. spectrum, $C' = C \otimes_{E_0} E_0 X(4)$,

$E \cong E\wedge S^0 \longrightarrow E\wedge X(4)$ map of elliptic spectra.

and ~~???~~ $\pi_0(E\wedge X(4))$ corepresents Weierstraß coords on C.

$\rightsquigarrow$ universal W. param on C'.

Corollary: (TMF = $\underleftarrow{\mathrm{holim}}$ some ell. spectra)

$$TMF \wedge X(4) = \underleftarrow{\mathrm{holim}}\left(\cdots \wedge X(4)\right)$$

$\rightsquigarrow$ corresponding Hopf algebroid is (A,Λ) or $(A[u^{\pm 1}], \Gamma[u^{\pm 1}])$.

Get spectral sequence for tmf, actually any elliptic spectrum.

TMF versus tmf

<u>Charles</u>: "$tmf \wedge \underset{\substack{\Uparrow \\ Y}}{X(4)}$" even periodic "

$$tmf \wedge D(A_1) = BP\langle 2\rangle .$$

Mike's email: This follows from

$$H^*(tmf; \mathbb{Z}/2) = \mathcal{A}/\!/\mathcal{A}_2$$

John's email:

Johan : ~~Serre~~ Tate :

$$\mathrm{Def}(C/_k) \cong \mathrm{Def}(\hat{C}/_k)$$

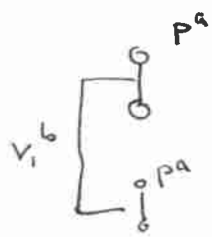
$\underline{L_{K(2)}\ tmf}$:

<u>Lemma</u>: $L_{K(2)}\, tmf \;=\; \underleftarrow{holim}\; v_2^{-1}\, tmf/p^a, v_1^b$.

<u>Proof</u>: $M(a,b) := (S^0 \cup_{p^a} e^1) \cup_{v_1^b} C\Sigma^{(2p-2)b}(S^0 \cup_{p^a} e^1)$

<u>Claim 1</u>: $\forall X$, $L_{K(2)}X = \underleftarrow{holim}_{a,b}\, L_{K(2)} X \wedge M(a,b)$

$\Uparrow$ smashing w/ fin spectra commutes w/ any localization



<u>Pf</u>:

$L_{K(2)}X \longrightarrow \underleftarrow{holim}\, L_{K(2)}X \wedge M(a,b)$

both sides are $K(2)$-local, need to show that $K(2)_*$-equiv.

Let F be a finite type 2 cx (ie. ...)

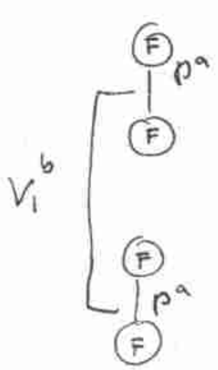
Künneth: it suffices to show

$X \wedge F \longrightarrow (\underleftarrow{holim}\ \ldots) \wedge F$ is $K(2)$ iso

$\| \Leftarrow \wedge$ finite commutes with $\underleftarrow{holim}$

$\underleftarrow{holim}(\ldots \wedge F)$.

$M(a,b) \wedge F$

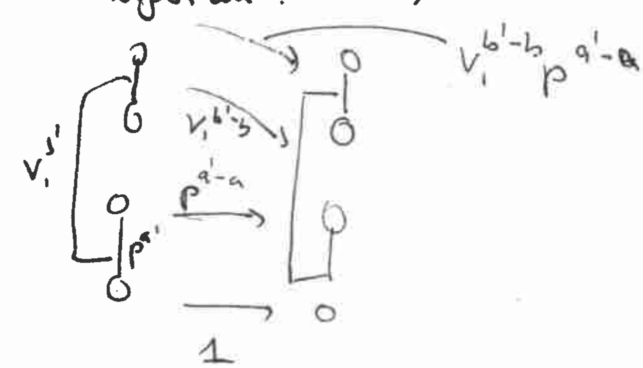


$a \gg 0$

$b \gg 0$

$\rightsquigarrow$

maps in direct system: $a, b \gg 0$



$\boxed{p, v_1 \text{ nilpotent on } F}$

$L_{K(2)}\, tmf = L_{K(2)} L_2\, tmf$?

$\Rightarrow$ contains cofinal system

$0 \xrightarrow{0} 0 \xrightarrow{0} 0$

$0 \xrightarrow{0} 0$ $\rightsquigarrow$ isom.

$0 \xrightarrow{1} 0$

<u>Claim 2:</u> If R is a BP-comodule, then $R \wedge v_2^{-1} M(a,b)$ is $K(2)$-local

reason: $\langle BP \wedge v_2^{-1} M(a,b) \rangle = \langle K(2) \rangle$

(a module over a ring spectrum E is local over it, b/c any acyclic maping into it factors over $\cancel{R}$ $E \wedge$ it ...

<u>Claim 3:</u> Assume there is a finite ~~[illegible]~~ Y of type 0 s.t. $R \wedge Y$ is BP-module. Then $R \wedge v_2^{-1} M(a,b)$ is $K(2)$-local.

<u>Pf:</u> $\{ Y'$ with this property $\}$ = thick $\ni Y$ = type 0

$\Rightarrow S^0 \in$; last claim $\square$

Together with ~~[illegible]~~ ; $R = tmf$ ~~[illegible]~~ $Y = D(A_1)$ $\square$