Mixed methods with weak symmetry for time dependent problems of elasticity and viscoelasticity

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Abstract

In this dissertation, we study numerical algorithms for time dependent problems in continuum mechanics using mixed finite element methods. We are particularly interested in linear elastodynamics and the Kelvin–Voigt, Maxwell, and generalized Zener models in linear viscoelasticity. We use mixed finite elements for elasticity with weak symmetry of stress, and show the a priori error analysis. A main contribution of our analysis is proving existence of a new elliptic projection map, called a weakly symmetric elliptic projection. In our analysis we prove that a priori error estimates for elastodynamics and viscoelasticity problems are as good as that of stationary elasticity problems. We present numerical results supporting our error analysis. We also present some basic numerical simulations which are more involved in physical situations.

Contents

Acknowledgements **i**

Abstract **ii**

List of tables **vi**

List of figures **vii**

1 Introduction **1**

1.1 Motivations 1

1.2 Mixed finite element methods for elasticity 3

1.3 Mixed methods for time dependent elasticity and viscoelasticity . 9

1.4 Overview of chapters 11

2 Preliminaries **14**

2.1 Notations and definitions 14

2.2 Continuum mechanics 17

2.2.1 Deformation, strain, momenta, and stress 17

2.2.2 Linear elasticity 20

2.2.3 Linear viscoelasticity 21

2.3 Mixed finite element methods 25

2.3.1 Saddle point problems 25

2.3.2 Mixed finite elements and Brezzi conditions 27

2.4 Mixed finite elements for linear elasticity with weak symmetry . 28

2.4.1 Mixed formulations of linear elasticity 29

2.4.2 Two families of stable mixed finite elements for elasticity 31

2.4.3 Error analysis for linear elasticity 34

2.4.4 A weakly symmetric elliptic projection 42

2.5 Miscellaneous preliminaries 43

2.5.1 Gronwall-type estimates 43
2.5.2 Well-posedness of differential algebraic equations 45
2.5.3 Regularity lemmas 46

3 Mixed methods for linear elastodynamics 49

3.1 Introduction 49
3.2 Weak formulations with weak symmetry 50
3.3 Semidiscrete problems 54
3.3.1 Existence and uniqueness of semidiscrete solutions 55
3.3.2 Decomposition of semidiscrete errors 56
3.3.3 Projection error estimates for the AFW elements 57
3.3.4 Approximation error estimates for the AFW elements . . 58
3.4 Full discretization 60
3.4.1 Well-definedness 61
3.4.2 Convergence 62
3.5 Error analysis for the GG elements 67
3.5.1 A priori error estimates 67
3.5.2 Postprocessing 70
3.6 Robustness for nearly incompressible materials 72
3.7 Numerical results 76

4 Mixed methods for the Kelvin–Voigt model of viscoelasticity 81

4.1 Introduction 81
4.2 Weak formulations with weak symmetry 82
4.3 Semidiscrete problems 86
4.3.1 Existence and uniqueness of semidiscrete solutions 86
4.3.2 Decomposition of semidiscrete errors 87
4.3.3 Projection error estimates for the AFW elements 89
4.3.4 Approximation error estimates for the AFW elements . . 90
4.4 Full discretization 93
4.4.1 Well-definedness 94
4.4.2 Convergence 95
4.5 Error analysis for the GG elements 101
4.6 Numerical results 104

5 Mixed methods for the Maxwell and generalized Zener models 107

5.1 Introduction 107

5.2 Weak formulations with weak symmetry 108
5.3 Semidiscrete problems 112
5.3.1 Existence and uniqueness of semidiscrete solutions 112
5.3.2 Decomposition of semidiscrete errors 113
5.3.3 Projection error estimates for the AFW elements 114
5.3.4 Approximation error estimates for the AFW elements . . 114
5.4 Full discretization 116
5.4.1 Well-definedness 117
5.4.2 Convergence 118
5.5 Error analysis for the GG elements 122
5.6 Numerical results 125

6 Numerical simulations 128
6.1 Elastodynamics 128
6.1.1 Wave propagation in homogeneous isotropic elastic media 128
6.1.2 Wave propagation in isotropic heterogeneous elastic media 129
6.1.3 Wave propagation in anisotropic media 130
6.2 Viscoelasticity of materials 134
6.2.1 Creep compliance 134
6.2.2 Attenuation of reflected waves 135

Bibliography 142

List of Tables

1.1 Mixed finite elements for elasticity with triangular meshes 7
1.2 Mixed finite elements for elasticity with rectangular meshes . . . 8
1.3 Previous studies of mixed methods for elastodynamics 10
1.4 Previous studies of mixed methods for viscoelasticity 11

3.1 Numerical result 1 of elastodynamics 77
3.2 Numerical result 2 of elastodynamics 78
3.3 Numerical result 3 of elastodynamics 78
3.4 The Butcher's table for general Runge–Kutta schemes 79
3.5 The Butcher's table for the 2-stage RadauIIA scheme 79
3.6 Numerical result 4 of elastodynamics 79

4.1 Numerical result 1 of the Kelvin–Voigt model 105
4.2 Numerical result 2 of the Kelvin–Voigt model 106
4.3 Numerical result 3 of the Kelvin–Voigt model 106

5.1 Numerical result 1 of the Zener model 125
5.2 Numerical result 2 of the Zener model 126
5.3 Numerical result 3 of the Zener model 126

List of Figures

1.1 The lowest order Arnold–Winther elements 5
1.2 The lowest order Arnold–Falk–Winther elements 5

2.1 Mechanical models of viscoelasticity 24

6.1 Elastic waves in homogeneous isotropic medium 129
6.2 Elastic wave in heterogeneous isotropic medium 1 131
6.3 Elastic wave in heterogeneous isotropic medium 2 132
6.4 Wave propagation in isotropic medium 133
6.5 Wave propagation in orthotropic medium 1 133
6.6 Wave propagation in orthotropic medium 2 133
6.7 Creep tests of viscoelastic materials 135
6.8 Regions of elastic and viscoelastic materials 135
6.9 Attenuation of reflected wave 136

Chapter 1

Introduction

1.1 Motivations

In this dissertation, we study numerical algorithms for linear elastodynamics and linear viscoelasticity using mixed finite elements for elasticity with weak symmetry.

Elastic and viscoelastic materials are of great interest in science and engineering because they are involved in many important problems in those areas with many applications. An *elastic* material is one of the most fundamental models of solids in engineering and physics. From the modeling point of view, an elastic material is regarded as a continuum consisting of infinitesimal springs. This is a straightforward way to extend a mechanical model to its continuum version, but it gives a good approximation for the behaviors of many solids when the deformation of the solids is within a certain small range. Therefore this elastic solid model is very useful for many important problems in solid mechanics and they have been used for a variety of practical applications. For example, to design a bridge, we need a good mathematical model for the bridge reflecting its kinematic features accurately. There are many other important areas that elastic materials are related, such as seismology in geophysics, so the study of elastic materials has been and is of great interest. A material is called *viscoelastic* when it shows kinematic features of both solids and fluids, often called elastic and viscous behaviors. Purely elastic or purely viscous behaviors of materials happen only for ideal solids or ideal fluids, respectively. In real life, all solids and fluids are not ideal, so they have viscoelastic features to some extent. Depending on the manner that the features of solids and fluids are combined, there are numerous different viscoelastic materials. In many important areas of engineering, the study of viscoelastic features of materials plays an important role. For example, most biological tissues show strong viscoelastic features mechanically, so if we want to develop an artificial organ to replace a tissue, we first have to understand the viscoelastic features of the tissue very well. Polymeric materials in rheology are also good examples of viscoelastic materials which have many important applications.

To study kinematic behaviors of a material theoretically, we express the behaviors of the material in mathematical forms. The kinematic behaviors of elastic and viscoelastic materials are formulated mathematically as partial differential equations (PDEs). However, a partial differential equation in material science is complicated in general even in simple physical models. Not surprisingly, solving the PDE analytically is very difficult, impossible in most cases. To find solutions of the PDE for practical purposes, we often need a numerical algorithm to find approximations of the solution with acceptable ranges of errors.

The numerical study of PDEs is one of the essential tools of modern science and for practical applications in engineering. For example, through a massive amount of numerical experiments people observe new phenomena which lead to improved models of weather prediction. Numerical analysis is also widely used in designing aircrafts, buildings, and electronic products. As scientists and engineers want to handle more complicated problems which require a large amount of computations, there is always a great need for faster and more accurate numerical algorithms.

During the last several decades, there have been many important advances in numerical analysis of PDEs. From the mathematical point of view, various new and improved numerical algorithms have been developed. The finite element method is among the most important approaches in the numerical study of solutions of PDEs. In this dissertation, we study numerical algorithms for time dependent problems of linear elastic and linear viscoelastic solids using *mixed finite element methods*. In our studies, we propose numerical algorithms and prove that the errors of our numerical solutions have the proposed error bounds.

1.2 Mixed finite element methods for elasticity

In this thesis we study mixed methods for time dependent problems of elasticity and viscoelasticity. Of course, these are based on existing mixed finite elements for stationary elasticity. Thus, we survey the development of mixed finite element methods for elasticity in this section. We will discuss them in more detail in chapter 2.

In the classical energy minimization form of linear elasticity problems, displacement is the only unknown of the equation and the numerical solution for stress is obtained using the numerical solution for displacement. In mixed methods for linear elasticity based on stress and displacement, there are two unknowns, stress and displacement. At first glance, this approach increases the number of unknowns and leads to a larger system of equations, but there are other benefits that make mixed methods attractive. A key advantage of mixed methods for linear elasticity is that they directly deliver the numerical solution for stress. Since stress is directly linked to destruction of materials, it is of great interest in engineering applications. Another advantage of mixed methods for elasticity is the robustness for nearly incompressible materials. In the displacement based approach, although the error for stress converges to zero as mesh size converges to zero, the error bound often contains a constant which is very large when a material is nearly incompressible, so we need a very small mesh size to get a sufficiently small error. However, the mixed methods we consider give uniform error bounds for nearly incompressible materials.

Since there are two unknowns, we need a pair of finite element spaces for mixed methods. One subtlety in mixed methods is to find a pair of finite elements which guarantee existence of numerical solutions with good approximation properties. A choice of mixed finite element spaces is called stable if it guarantees existence of numerical solutions. Necessary and sufficient conditions for stable mixed finite elements are known based on the foundational work of Babuška and Brezzi. However, finding stable mixed finite elements for elasticity has long been a difficult problem. A major obstacle to finding stable mixed finite elements for elasticity is the symmetry of stress. Since stress is symmetric, it is natural that the finite elements for stress are symmetric as well, but it is very difficult to find such stable mixed finite elements for elasticity.

The first mixed finite elements for elasticity were developed by Johnson and Mercier using composite triangles [36]. For a triangle, three subtriangles are obtained by connecting an interior point to the three vertices. In the Johnson–

Mercier elements, shape functions for stress are the piecewise linear polynomials adapted to the subtriangles satisfying normal continuity on the interior edges of subtriangles. Because of the construction using composite triangles, it is complicated to implement these elements. Moreover, generalizations of the Johnson–Mercier elements to higher order elements or to three dimensions are not obvious. In two dimensions, there is a family of higher order mixed finite elements for elasticity developed by Arnold, Douglas, and Gupta using composite triangles in [8]. They followed an analysis similar to that of Johnson and Mercier, but in a much more systematic manner using the exact sequence in linear elasticity involving the Airy operator and the divergence operator. Following the exact sequence in continuous level, they constructed finite element spaces which inherit the exact sequence structure from the continuous level, and used the exact sequence of finite elements for analysis. However, their implementations are still complicated due to the composite triangle construction.

Because the elements using composite triangles are very complicated, finding mixed finite elements for elasticity without using composite triangles was a question of great interest. This question remained unsolved for four decades, until the first example of such elements in two dimensions was developed by Arnold and Winther in 2002. In [11], Arnold and Winther used the exact sequence in linear elasticity which was used in [8]. For the construction of exact sequences of finite element spaces, they used the Argyris element and its generalizations for higher orders. They also showed that a piecewise polynomial finite element space for stress in this approach ought to have vertex degrees of freedom in a triangle. The vertex degrees of freedom give a main technical difficulty in analysis because the canonical interpolation operator is not well-defined for H^1 functions. They overcome this difficulty by constructing a new interpolation operator using the Clément interpolant in [23]. There are also three dimensional elements developed by Arnold, Awanou, and Winther following a similar approach [5]. Although these elements do not use composite triangles, they have a relatively large number of degrees of freedom, especially in three dimensions. For example, the lowest order Arnold–Awanou–Winther elements have 162 stress degrees of freedom for each tetrahedron. The lowest order Arnold–Winther elements have 24 stress degrees of freedom and 6 displacement degrees of freedom for each triangle (see Figure 1.1), which is a reasonable number of degrees of freedom, so they are indeed recommended for practical solid mechanics problems on the support of numerical experiments by Carstensen, Günther, Reininghaus, and Thiele [20]. However, there are some minor defects. One of them is that the full approximability of the Arnold–Winther elements, which is of order three for the lowest elements, is redundant when regularity of solutions is low. Another defect is that the hybridization in [6] is not available because of the vertex degrees of freedom.

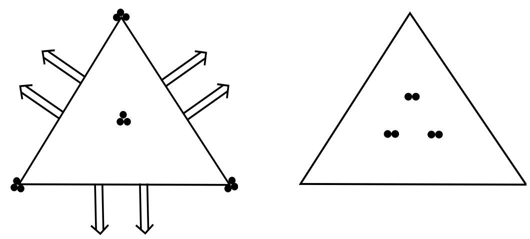

Figure 1.1: Element diagrams for the lowest order stress, displacement of the Arnold–Winther elements.

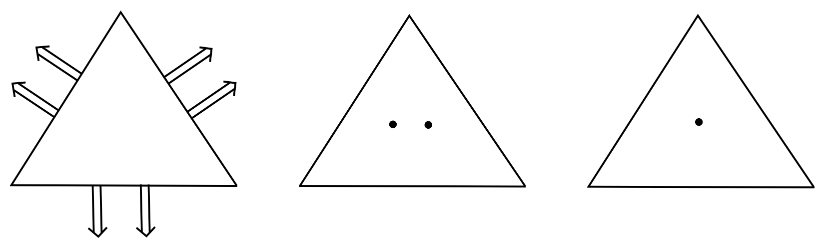

Figure 1.2: Element diagrams for the lowest order stress, displacement, and rotation elements of the Arnold–Falk–Winther elements.

An alternative approach to mixed methods for elasticity is to impose symmetry of stress weakly, by imposing orthogonality to spaces of skew-symmetric tensors. From another point of view, we introduce a skew-symmetric tensor, which is the Lagrange multiplier for the symmetry of stress, and rewrite the original elasticity problems with the Lagrange multiplier. The Lagrange multiplier is often called the rotation because it is the skew-symmetric part of the gradient of displacement. Therefore, in this approach, we have three unknowns, i.e., the stress tensor, the displacement vector, and the rotation. Historically, this weak symmetry idea was firstly suggested by Fraeijs de Veubeke in [26] and extended for higher orders by Amara and Thomas. The work of Amara and Thomas was not exactly written in a modern context of finite element methods[1], however, they observed and explained crucial concepts and ideas with a careful analysis.

[1]They did not use standard terminologies in finite element methods such as finite element spaces, stability, degrees of freedom, shape functions, and the inf-sup condition.

In [3], Amara and Thomas used a matrix-valued $H(\text{div})$ piecewise polynomial space for stress and a piecewise discontinuous polynomial space for rotation. Instead of a piecewise polynomial space for displacement, they used a piecewise polynomial space on edges which may correspond to the trace of displacement. For existence of numerical solutions, they used some additional terms for stress using bubble functions and proved error estimates using an interpolation map. The first finite elements of weak symmetry idea, described in mixed methods context, is the PEERS elements developed by Arnold, Brezzi, and Douglas in [7]. In the construction of PEERS elements, the vector-valued lowest order Raviart–Thomas elements augmented with additional terms using the bubble function, piecewise constants, and skew-symmetric piecewise linear functions are used for the shape functions for stress, displacement, and rotation, respectively. Following the weak symmetry idea and the approach of the PEERS elements, Stenberg constructed new finite elements in two and three dimensions and also for higher orders [47]. For displacement, he used the vector-valued discontinuous polynomials as in the PEERS elements. However, he used different spaces for stress and rotation. Instead of the Raviart–Thomas elements with additional terms using the bubble function and continuous skew-symmetric spaces, which were used in the PEERS elements, he used the Brezzi–Douglas–Marini–Nédélec elements with additional terms using bubble functions for the stress and discontinuous polynomials for the rotation such that both of them have one higher order approximation properties than the space for the displacement. He also observed that a postprocessing is eligible for the numerical solution for displacement, so a new numerical solution for displacement can be obtained, which has as same accuracy as the ones for stress and rotation. He also claimed that new finite elements using the Raviart–Thomas elements can be constructed with similar arguments straightforwardly. There are other extensions of the PEERS elements, done by Morley, to two dimensions for one higher order and to three dimensions. She used the Raviart–Thomas–Nédélec elements with additional terms using bubble functions as shape functions for stress, but she used nonconforming finite elements for rotation to avoid vertex degrees of freedom. She also observed the eligibility of postprocessing for the numerical displacement as Stenberg did. In [10], Arnold, Falk, and Winther introduced an exterior calculus framework for the study of mixed finite elements for elasticity. The framework is based on the elasticity complex which is constructed from the de Rham complex using the Bernstein–Gelfand–Gelfand resolution in representation theory by Eastwood [29]. As an application of the elasticity complex,

Table 1.1: Mixed finite elements for elasticity with triangular meshes. The σ, u, r denote the stress, the displacement, and the rotation, respectively. For all the finite elements that k is involved, we assume $k \geq 1$.

elements	symmetry	order of error			mesh & dimension
		σ	u	r	
JM [36]	strong	2	2	–	composite, 2D
ADG [8]	strong	$k+2$	$k+1$	–	composite, 2D
AW [11]	strong	$k+2$	$k+1$	–	2D
AAW [5]	strong	$k+2$	$k+1$	–	3D
AT [3]	weak	k	–	–	2D
PEERS [7]	weak	1	1	1	2D
Stenberg I [47]	weak	$k+1$	k	$k+1$	2D, 3D
Morley [40]	weak	2	2	2	2D, 3D
AFW [10]	weak	k	k	k	2D, 3D
CGG [24]	weak	$k+1$	k	$k+1$	2D, 3D
GG [32]	weak	$k+1$	k	$k+1$	2D, 3D

JM = Johnson–Mercier, ADG = Arnold–Douglas–Gupta, AW = Arnold–Winther,
AAW = Arnold–Awanou–Winther, AT = Amara–Thomas,
AFW = Arnold–Falk–Winther, CGG = Cockburn–Gopalakrishanan–Guzmán,
GG = Gopalakrishnan–Guzmán

they developed the Arnold–Falk–Winther elements. In the analysis, they used the elasticity complex to construct exact sequences of finite element spaces and constructed an interpolation operator with a commuting property. The Arnold–Falk–Winther elements are defined in two and three dimensions and for higher orders with simple descriptions (see Figure 1.2), and have small numbers of degrees of freedom. After this pioneering work, other elements were developed following the analysis of same exterior calculus framework. For example, Cockburn, Gopalakrishnan, and Guzmán constructed a family of elements such that the finite element spaces for stress are based on the Raviart–Thomas–Nédélec elements with additional terms using bubble functions [24]. These elements are similar to Stenberg's ones but have smaller degrees of freedom for same accuracy of errors. They also showed that the hybridization is available for their elements. More recently, another family of elements was developed by Gopalakrishnan and Guzmán [32], which have fewer degrees of freedom than their previous elements with same accuracy of errors. We refer Table 1.1 for some features of these elements.

There are also rectangular and quadrilateral mixed finite elements for elasticity with both strong and weak symmetry. For strong symmetry elements,

Table 1.2: Mixed finite elements for elasticity with rectangular or quadrilateral meshes. The σ, u, r denote the stress, the displacement, and the rotation, respectively. For all the finite elements that k is involved, we assume $k \geq 1$.

elements	symmetry	order of error			mesh & dimension
		σ	u	r	
JM [36]	strong	2	2	–	composite, quad., 2D
ADG [8]	strong	$k+2$	$k+1$	–	composite, quad., 2D
PS [41]	strong	3/2	3/2	–	rect., 2D
		2	2	–	composite, rect. 2D
Stenberg II [46]	strong	2	3	–	rect., 2D
		3	4	–	
BJT [15]	strong	k	k	–	rect., 2D, 3D
AA [4]	weak	k	k	k	rect., 2D
Morley [40]	weak	2	2	2	rect., 2D
Awanou [13]	weak	k	k	k	rect., 2D, 3D

JM = Johnson–Mercier, ADG = Arnold–Douglas–Gupta, PS = Pitkäranta–Stenberg
AA = Arnold–Awanou, quad. = quadrilateral, rect. = rectangular

Johnson and Mercier constructed quadrilateral finite elements with linear polynomials using composite quadrilaterals [36]. In [8], Arnold, Douglas, and Gupta also constructed quadrilateral elements for higher orders using composite quadrilaterals. Pitkäranta and Stenberg showed the error analysis of two mixed finite elements in two dimensions [41]. Stenberg constructed some low order rectangular mixed finite elements in two dimensions and showed error analysis in [46]. There is a family of rectangular elements in two and three dimensions and also for higher orders developed by Bécache, Joly, and Tsogka in [15]. For shape functions for the stress and the displacement, they use the symmetric tensors that each entry belongs to Q_{k+1}, and the vectors that each entry belongs to Q_k, respectively, where Q_k is the standard tensor product space of the polynomials of degree less than or equal to k. To make the divergence operator is well-defined on the finite element space for stress, they used a nonstandard choice of degrees of freedom that each entry of the stress tensor is continuous along specific one or two directions. Since the definition of degrees of freedom strongly relies on the rectangular structure of meshes, it seems to be difficult to extend their approach to triangular meshes. More recently, in [4], Arnold and Awanou developed rectangular finite elements with strong symmetry in two dimensions based on the idea of [11]. For weak symmetry elements, Morley constructed rectangular elements in her generalization of the PEERS elements in [40]. In [13], Awanou developed a family of rectangular elements with weak symmetry in two and three dimensions and for higher orders. His elements have fewer degrees of freedom than Morley's ones. Some features of these elements are summarized in Table 1.2. Rectangular elements are very useful for problems with domains of special geometry, however, it is difficult to use them to the problems which have general shape domains.

To summarize, after intensive studies of four decades, there are many mixed finite elements for elasticity. Among them, the weak symmetry elements are advantageous because they are defined in two and three dimensions and for higher orders. Moreover, they have relatively simple descriptions with small number of degrees of freedom.

1.3 Mixed methods for time dependent elasticity and viscoelasticity

In continuum mechanics, there are many problems for which stress is of primary interest. For example, to design and construct an earthquake resistant building, the stress exerted on the building is one of most important quantities to consider. Based on this philosophy, we use mixed finite element methods to study time dependent problems of elasticity and viscoelasticity.

As we have seen in the previous section, mixed finite elements for elasticity with weak symmetry have relatively few degrees of freedom and are relatively easy to implement in both two and three dimensions. Thus we shall use the weak symmetry elements for our studies of continuum mechanics problems. In this section we briefly survey previous studies of elastodynamics and viscoelasticity problems using mixed methods.

Mixed methods for linear elastodynamics have been studied by various researchers. In [27], Douglas and Gupta used a displacement-stress formulation of elastodynamics equations and the mixed finite elements using composite triangles developed in [8]. For the error analysis of semidiscrete solutions, they use an asymptotic expansion of solutions using the quasi-projection. As a consequence of the error analysis, they showed that the errors for stress and displacement are of same orders as for stationary elasticity problems. The superconvergence result in their work is based on the superconvergence in the error analysis of stationary problems but the error analysis for fully discrete solutions was not shown. In [39], Makridakis proposed two approaches for linear elastodynamics, the displacement-stress formulation used in [27] and a velocity-stress formulation. The velocity-stress formulation is based on the work of Geveci for scalar wave equations. In [31], Geveci suggested a velocity-flux formulation for scalar wave equations and showed a unified error analysis for the Raviart–Thomas and the Brezzi–Douglas–Marini elements. He also pointed out that a similar analysis can be adapted to the corresponding velocity-stress formulation of elastodynamics. In the work of Makridakis, he only assumed that the finite elements are stable, strongly symmetric, have a good approximability, and have interpolation maps satisfying a certain commutativity property, so his analysis is valid for many finite elements including the composite elements in [8, 36] and the rectangular elements developed in [41, 46]. For the error analysis, Makridakis used the elliptic projection approach, which was introduced in [49] for heat equations. Using the elliptic projection, and an energy estimate, he simplified the error analysis significantly than the one of Douglas and Gupta. He also considered fully discrete solutions with general time discretization based on the Padé approximation. In [15], Bécache, Joly, and Tsogka constructed new rectangular finite elements, which can be extended to three dimensions and for higher orders, and applied them for linear elastodynamics. They used the velocity-stress formulation and the elliptic projection for error analysis as in the work of Makridakis.

Table 1.3: Comparison of the previous studies and the work in this thesis for elastodynamics. (disp.-stress = displacement-stress, vel.-stress = velocity-stress) Finite elements are denoted by using the abbreviations in Table 1.1 and Table 1.2.

	Douglas Gupta	Makridakis	Bécache Joly Tsogka	this thesis (chapter 3)
formulation	disp.-stress	disp.-stress vel.-stress	vel.-stress	vel.-stress
finite elements	ADG	JM, ADG, PS Stenberg II	BJT	AFW, GG
time scheme	–	Pade	–	Crank–Nicolson

In contrast to elastodynamics, there are not many previous works on mixed methods for viscoelasticity. In [16], Bécache, Ezziani, and Joly used their rectangular elements developed in [15] for the generalized Zener model of linear viscoelasticity. To have a mixed form of equations, they took three unknowns, the displacement, the total stress, and the difference of the total stress and the elastic stress. Rewriting equations, a system of equations consisting of an al-

Table 1.4: Comparison of the previous studies and the works in the thesis for linear viscoelasticity. (disp.-stress = displacement-stress, vel.-stress = velocity-stress) Finite elements are denoted by using the abbreviations in Table 1.1 and Table 1.2.

	Bécache Ezziani Joly	Rognes Winther	this thesis (chapters 4, 5)
formulation	disp.-stress	vel.-stress	vel.-stress
finite elements	BJT	AFW	AFW, GG
viscoelastic model	gZ	qM& qKV	M, KV, gZ
time scheme	leap-frog type	BDF2	Crank–Nicolson
symmetry of finite elements	strong	weak	weak

M=Maxwell, KV=Kelvin–Voigt, gZ=generalized Zener, q=quasistatic

gebraic equation, and equations with one and two time derivatives. For time discretization, they chose a leap-frog type scheme and proved that the scheme is stable when a certain CFL condition holds. Some discussions on mass lumping, PML adaptation, and upper bounds of CFL condition were presented. In [44], Rognes and Winther studied mixed methods for the Kelvin–Voigt and the Maxwell models of linear viscoelasticity but for quasistatic problems, i.e., the problems that mass densities are vanishing. They suggested a unified framework for general viscoelasticity models and applied it to the specific two problems. A key idea of the unified framework is using two stresses, the viscous and the elastic ones, and generalize the velocity-stress formulation for elastodynamics in the context of viscoelasticity equations. For mixed finite elements, they used the Arnold–Falk–Winther elements and a variant of them for the lowest order by Falk. Due to the weak symmetry of finite elements, the equations of the Maxwell and the Kelvin–Voigt models had to be rewritten in weak symmetry form. They used the skew-symmetric part of the gradient of velocity as the Lagrange multiplier for symmetry of stress, and obtained differential algebraic equations for the semidiscrete solutions. For full discretization, they used the second backward differentiation formula and applied a known result in general theory of the numerical analysis of differential algebraic equations for convergence. They also presented numerical results for the Zener model.

1.4 Overview of chapters

The organization of this thesis is as follows.

In chapter 2, we develop background materials which will be needed in the rest of this thesis. We present notations, definitions, a brief survey of mixed methods, and expository descriptions of the Arnold–Falk–Winther (AFW) and the Gopalakrishnan–Guzmán (GG) elements, which we will use in our studies. We revisit improved error estimates and postprocessing results, proposed in [47, 35, 10] for these two families of elements with complete proofs. We also introduce some results on evolutionary equations and regularity of functions, that we need in later chapters, with their complete proofs.

In chapter 3, we study mixed methods for linear elastodynamics, which represents wave propagation in elastic media, using a velocity-stress mixed formulation. This is the first work of mixed methods for elastodynamics using weak symmetry elements. We use the Crank–Nicolson scheme for time discretization and propose error bounds by a priori error analysis. A key idea for the error analysis is our weakly symmetric elliptic projection which will be explained in chapter 2. By a careful analysis, we can obtain error bounds in elastodynamics, similar to the ones in stationary elasticity problems, using the AFW and GG elements. We also prove that robustness for nearly incompressible materials still holds in elastodynamics. Some numerical results which support our analysis are presented at the end of this chapter.

In chapter 4, we consider mixed methods for the Kelvin–Voigt model of linear viscoelasticity, which is a fundamental unit to construct models of viscoelastic solids. We study the full dynamic Kelvin–Voigt model with a nonvanishing mass density using a velocity-stress mixed formulation, the AFW and GG elements, and the Crank–Nicolson scheme for time discretization. The semidiscrete problem of the Kelvin–Voigt model leads to a system of differential algebraic equations, so initial data for numerical computation should be carefully chosen to achieve stability of time discretization. We show an error analysis for fully discrete solutions and propose error bounds. There are also numerical results which support our error analysis.

In chapter 5, we consider mixed methods for the Maxwell and the generalized Zener models of linear viscoelasticity. Since the Maxwell and the Zener models can be written in a unified form and the Maxwell model is a special case of the Zener model, we show careful error analysis only for the Zener model. Extending the analysis to the generalized Zener model is straightforward. As in elastodynamics and the Kelvin–Voigt model problems, we use a velocity-stress formulation, the AFW and GG elements, and the Crank–Nicolson scheme for time discretization. We show that the error bounds of our numerical scheme are stable as the parameters, which determine viscoelastic features of the material, decays to zero. As a consequence, our numerical algorithm can be used for a material which is a composition of elastic and viscoelastic solids. Another benefit of our method is that no time integration is needed in the computation of each time step. Based on the displacement, the solution of Zener model includes a convolution term with a kernel depending on material parameters and time (see e.g., [37]), so a numerical solution also needs a numerical time integration for all the past time intervals at each time step. In mixed methods using our velocity-stress formulation, the equations of the Zener model is written as differential equations (see e.g., [16, 33]), henceforth no numerical time integration is required and an implementation of the algorithm is easier. The payment for these advantages is a larger system of equations. However, the number of degrees of freedom increases almost linearly to the number of mesh components, i.e, the triangles, the edges, and the vertices. Since the computational cost increases linearly on the number of degrees of freedom in advanced linear algebraic solvers, the increment of computational cost is not a big obstacle. In contrast to this, the number of time intervals for numerical integration is not limited and therefore the computation cost for numerical integration can be very large unless there is a good argument to justify that a truncation of the time interval for numerical integration is reasonable. As in previous chapters, we present numerical results supporting our error analysis.

Finally, in chapter 6, we show numerical results which are more interesting from the physical point of views. In elastodynamics, there are examples showing that the different material parameters influence differently on the propagation of P and S waves in homogeneous and heterogeneous isotropic media. We also present numerical results which show wave propagation in anisotropic media. In viscoelasticity, we use our numerical schemes for creep compliances of viscoelastic materials. We present a simple schematic model which compares reflected waves in a purely elastic medium and a medium including viscoelastic regions.

Chapter 2

Preliminaries

In this chapter, we will survey preliminary backgrounds for our discussions in the rest of this dissertation. The contents of this chapter are organized as follows.

In section 2.1, we introduce notations and definitions. In section 2.2, we survey continuum mechanics backgrounds which are necessary to derive our governing equations later. In section 2.3, we introduce a general theory of mixed finite element methods. In section 2.4, we survey mixed methods for linear elasticity and introduce two families of mixed finite elements for elasticity. For those families, we present some technical details including a priori error estimates, robustness for nearly incompressible materials, postprocessing, and the existence of an elliptic projection which preserves weak symmetry. Finally, in section 2.5, we prove miscellaneous lemmas which are needed for error estimates and regularity of weak solutions.

2.1 Notations and definitions

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n = 2$ or 3. If a range is not specified, then indices i, j span $1, \cdots, n$. We use ∂_i to denote the partial derivative for the i-th variable in $\mathbb{R}^n$.

We use $\mathbb{V}$ and $\mathbb{M}$ to denote $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$. We also use $\mathbb{S}$ and $\mathbb{K}$ to denote the spaces of symmetric and skew-symmetric $n \times n$ matrices, respectively. For $\sigma : \Omega \to \mathbb{M}$ and $u : \Omega \to \mathbb{V}$, their components are denoted by σ_{ij} and u_i, respectively. Let us define

$$(\sigma, \tau) = \int_\Omega \sigma : \tau \, dx := \int_\Omega \sum_{i,j} \sigma_{ij} \tau_{ij} \, dx,$$
$$(v, w) = \int_\Omega v \cdot w \, dx := \int_\Omega \sum_i v_i w_i \, dx,$$

for $\sigma, \tau : \Omega \to \mathbb{M}$ and $v, w : \Omega \to \mathbb{V}$. It is easy to check that these are inner products. We can define norms by $\|\sigma\|^2 = (\sigma, \sigma)$, $\|u\|^2 = (u, u)$ and define two Hilbert spaces

$$L^2(\Omega; \mathbb{M}) = \{\sigma : \Omega \to \mathbb{M} \mid \|\sigma\| < \infty\}, \quad L^2(\Omega; \mathbb{V}) = \{u : \Omega \to \mathbb{V} \mid \|u\| < \infty\}.$$

For $\sigma : \Omega \to \mathbb{M}$ and $u : \Omega \to \mathbb{V}$, $\operatorname{div} \sigma$ and $\operatorname{grad} u$ are defined by the row-wise divergence and the row-wise gradient

$$\operatorname{div} \sigma = \sum_j \partial_j \sigma_{ij}, \qquad (\operatorname{grad} u)_{ij} = \partial_j u_i,$$

respectively, where $\partial_j \sigma_{ij}, \partial_j u_i$ are understood in the sense of distributions [45]. For $\sigma : \Omega \to \mathbb{M}$, the symmetric and skew-symmetric parts of σ are

$$\operatorname{sym} \sigma = \frac{\sigma + \sigma^T}{2}, \qquad \operatorname{skw} \sigma = \frac{\sigma - \sigma^T}{2},$$

where σ^T is the transpose of σ. If $\sigma : \Omega \to \mathbb{M}$ and $\operatorname{div} \sigma \in L^2(\Omega; \mathbb{V})$, we define $\|\sigma\|_{\operatorname{div}}^2 = \|\sigma\|^2 + \|\operatorname{div} \sigma\|^2$ and for a subspace $\mathbb{X}$ of $\mathbb{M}$,

$$H(\operatorname{div}, \Omega; \mathbb{X}) = \{\sigma \in L^2(\Omega; \mathbb{X}) \mid \|\sigma\|_{\operatorname{div}} < \infty\}.$$

We also define function spaces M, S, V, and K by

$$\begin{aligned} M &= H(\operatorname{div}, \Omega; \mathbb{M}), \quad S = H(\operatorname{div}, \Omega; \mathbb{S}), \\ V &= L^2(\Omega; \mathbb{V}), \quad K = L^2(\Omega; \mathbb{K}). \end{aligned} \tag{2.1}$$

For a nonnegative integer $0 \le m < \infty$, we use $C^m(\overline{\Omega})$ to denote the set of functions defined on Ω such that the functions and all their partial derivatives of order less than or equal to m are continuous and can be continuously extended to $\overline{\Omega}$. We use $C^\infty(\overline{\Omega})$ to denote the intersection of all $C^m(\overline{\Omega})$ for $m \ge 0$. We also use $C_0^m(\Omega)$ to denote the functions in $C^m(\overline{\Omega})$ whose supports are compact sets in Ω.

A multi-index α is a sequence of nonnegative integers $(\alpha_1, \cdots, \alpha_n)$ and the degree of α, denoted by $|\alpha|$, is defined by $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For $u \in C^m(\overline{\Omega})$, we define

$$\|u\|_m^2 = \sum_{|\alpha| \le m} \|\partial^\alpha u\|^2, \qquad \text{where} \qquad \partial^\alpha u := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u.$$

The Sobolev space $H^m(\Omega)$ is the Banach space which is the completion of $C^m(\overline{\Omega})$ with the norm $\|\cdot\|_m$. We define $\mathring{H}^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ and it becomes a subspace of $H^1(\Omega)$. For $\mathbb{X} = \mathbb{V}, \mathbb{M}, \mathbb{K}$, or $\mathbb{S}$, $H^m(\Omega; \mathbb{X})$ is the space of $\mathbb{X}$-valued functions such that each component of the function is in $H^m(\Omega)$. If $\mathbb{X}$ is clear in context, $H^m(\Omega)$ is used as an abbreviation of $H^m(\Omega; \mathbb{X})$.

For a Banach space $\mathcal{X}$ and $0 < T_0 < \infty$, $C^0([0, T_0]; \mathcal{X})$ denotes the set of functions $f : [0, T_0] \to \mathcal{X}$ which are continuous in $t \in [0, T_0]$. For an integer $m \ge 1$ we define

$$C^m([0, T_0]; \mathcal{X}) = \{f \,|\, \partial^l f/\partial t^l \in C^0([0, T_0]; \mathcal{X}),\, 0 \le l \le m\},$$

where $\partial^l f/\partial t^l$ is the l-th time derivative in the sense of the Fréchet derivative in $\mathcal{X}$ (see e.g., [50]). For a function $f : [a, b] \to \mathcal{X}$, we define the space-time norm

$$\|f\|_{L^p([a,b];\mathcal{X})} = \begin{cases} \left(\int_a^b \|f\|_{\mathcal{X}}^p \, dt\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t \in [a,b]} \|f\|_{\mathcal{X}}, & p = \infty. \end{cases}$$

If the time interval is fixed as $[0, T_0]$, then we use $L^p\mathcal{X}$ to denote $L^p([0, T_0]; \mathcal{X})$ for simplicity. We define the space-time Sobolev spaces $W^{m,p}([0, T_0]; \mathcal{X})$ for nonnegative integer m and $1 \le p \le \infty$ as the closure of $C^m([0, T_0]; \mathcal{X})$ with the norm $\|u\|_{W^{m,p}\mathcal{X}} = \sum_{l=0}^m \|\partial^l u/\partial t^l\|_{L^p\mathcal{X}}$. We adopt a convention $\|f, g\|_{\mathcal{X}}$ to denote $\|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}}$ for the norm of a Banach space $\mathcal{X}$. For simplicity of notations, $\dot{f}$ is used to denote the time derivative of f and similarly, $\ddot{f}$, $\dddot{f}$ are used to denote $\partial^2 f/\partial t^2$, $\partial^3 f/\partial t^3$, respectively.

For a triangle or a tetrahedron T, a vector space $\mathbb{X}$, and a nonnegative integer k, $\mathcal{P}_k(T; \mathbb{X})$ is the space of $\mathbb{X}$-valued polynomials defined on T of degree less than or equal to k. If T is a triangle, the space $N_k(T)$, $k \ge 1$ is

$$N_k(T) = \mathcal{P}_{k-1}(T; \mathbb{R}^2) + \{(-w x_2, w x_1) \,|\, w \in \mathcal{P}_{k-1}(T)\}, \tag{2.2}$$

which is the usual space of shape functions for the (rotated) Raviart–Thomas elements [9, 42]. We define $\tilde{N}_k(T)$ as the space consisting of all τ in $\mathcal{P}_k(T;\mathbb{R}^{2\times 2})$ such that each row of τ is in $N_k(T)$. We will use this space when we define the degrees of freedom for our mixed finite elements for elasticity in section 2.4.2.

For the domain Ω, $\mathcal{T}_h$ denotes a shape-regular quasi-uniform triangulation of Ω for which the maximum diameter of triangles (or tetrahedra) is h. For an integer $k \geq 0$ and a vector space $\mathbb{X}$, $\mathcal{P}_k(\mathcal{T}_h;\mathbb{X})$ is the space of piecewise $\mathbb{X}$-valued polynomials adapted to $\mathcal{T}_h$ of degree less than or equal to k. If $\mathbb{X}$ is a subspace of $\mathbb{M}$, then $\mathcal{P}_k(\mathcal{T}_h, \mathrm{div};\mathbb{X}) = \mathcal{P}_k(\mathcal{T}_h;\mathbb{X}) \cap H(\mathrm{div},\Omega;\mathbb{X})$.

Let $\Delta t > 0$ such that $T_0 = N\Delta t$ for an integer N, and $t_j = j\Delta t$ for $j = 0, 1, \cdots, N$. For a continuous function f defined on $[0, T_0]$, we define $f^j = f(t_j)$ and $f^{j+1/2} = f(t_j + \Delta t/2)$. For example, σ^j, $\sigma_h^{P,j}$, $e_\sigma^{P,j}$ denote $\sigma(t_j)$, $\sigma_h^P(t_j)$, $e_\sigma^P(t_j)$ for the functions σ, σ_h^P, e_σ^P defined on $[0, T_0]$, respectively. For a sequence $\{f^j\}_{j\geq 0}$, we define

$$\begin{aligned} \bar{\partial}_t f^{j+\frac{1}{2}} &= \frac{f^{j+1} - f^j}{\Delta t}, \qquad \hat{f}^{j+\frac{1}{2}} = \frac{f^j + f^{j+1}}{2}, \\ \bar{\partial}_t^2 f^j &= \frac{f^{j+1} - 2f^j + f^{j-1}}{\Delta t^2}. \end{aligned} \tag{2.3}$$

Note that for f defined on $[0, T_0]$, $\hat{f}^{j+1/2} \neq f^{j+1/2}$ in general.

2.2 Continuum mechanics

We survey basic continuum mechanics which is necessary to derive the governing equations of our problems.

Continuum mechanics is a way to formulate kinematic behavior of materials mathematically. In continuum mechanics, a material body is regarded as a continuum and the microscopic structures of the material are neglected. In many macroscopic scale problems, it is a good approximation of real physical phenomena.

2.2.1 Deformation, strain, momenta, and stress

If a continuum body occupies a bounded domain in $\mathbb{R}^n$ where $n = 2, 3$, then the occupied domain is called a *configuration*. For simplicity, we assume all configurations have sufficiently smooth boundaries. Let Ω be the domain that a continuum body occupies at initial state, which is called the *reference config-*

uration. The *deformation map* $\Phi : \Omega \times [0, T_0] \to \Omega' \subset \mathbb{R}^n$ is a map which is continuously differentiable, homeomorphic, and orientation preserving. The image of Ω under $\Phi(\cdot, t)$ for $t \in [0, T_0]$ is called the *deformed configuration* at time t and denoted by Ω_t. The gradient of deformation map is called the *deformation gradient* and denoted by F.

The *rigid deformations* are the deformation maps of the form $x \mapsto A(t)x+b(t)$ for $x \in \mathbb{R}^n$ where $t \mapsto A(t)$, $t \mapsto b(t)$ are continuous maps to the space of orthogonal matrices of positive determinant and the space $\mathbb{R}^n$, respectively. In continuum mechanics, rigid deformations are not interesting because when a deformation map is a rigid deformation, all kinematic quantities of deformed configuration are obtained by composing the inverse of the rigid deformation and the corresponding kinematic quantities of reference configuration. A C^1 deformation map Φ is a rigid deformation if and only if $F^TF = I$ (see [22], p. 44), so we call $(F^TF - I)/2$ the (Green–St.Venant) *strain* or *strain tensor* where I is the identity matrix in $\mathbb{R}^{n\times n}$.

In many problems, it is convenient to work with the difference of the deformed and reference configurations rather than the deformed configuration itself. The *displacement* $u : \Omega \to \mathbb{R}^n$ is defined by $u(x, t) = \Phi(x, t) - x$ for $x \in \Omega$, $t \in [0, T_0]$. Then the gradient of displacement is $\operatorname{grad} u = F - I$ and the strain tensor can be written

$$\begin{aligned}\frac{1}{2}(F^TF - I) &= \frac{1}{2}((\operatorname{grad} u + I)^T(\operatorname{grad} u + I) - I)\\ &= \frac{1}{2}((\operatorname{grad} u)^T(\operatorname{grad} u) + (\operatorname{grad} u)^T + \operatorname{grad} u).\end{aligned} \tag{2.4}$$

We use v to denote $\partial u/\partial t$, the velocity field and $\rho(x)$ to denote the mass density at $x \in \Omega$. Then the *linear momentum* and *angular momentum* (about the origin) on a subregion ω are defined by

$$\int_\omega \rho v \, dx, \qquad \int_\omega \rho \vec{x} \times v \, dx,$$

where $\vec{x}$ is the position vector defined by the coordinate x. If $n = 2$, we can still define the angular momentum by extending all two dimensional vectors to three dimensional ones which have zero third coordinate.

We now consider an internal surface force on a surface in a continuum body. For a surface in a continuum body, there is a force acting between two continuum subbodies along the surface. In a continuum sense, this force is proportional to surface area, and at a point on the surface it is defined as the limit of force on shrinking surface regions divided by the surface area of the regions. We call this internal surface force as the *stress vector* or *traction*.

Let ω_0, ω_1 be two subregions in a continuum body Ω with contacting surface $\mathcal{S}$. If we let ν be the unit normal vector of $\mathcal{S}$ at point x which is outward from ω_0, then the surface force that ω_0 exerts on ω_1 at x is denoted by $T(x, \nu) \in \mathbb{R}^n$. Thus, the surface force that ω_1 exerts on ω_0 at x is $T(x, -\nu)$, and $T(x, -\nu) = -T(x, \nu)$ by Newton's third law of motion.

We assume that the balance laws of linear and angular momenta, which are

$$\frac{d}{dt}\int_\omega \rho v \, dx = \int_{\partial\omega} T(x, \nu)\, dS + \int_\omega f \, dx,$$
$$\frac{d}{dt}\int_\omega \rho \vec{x} \times v \, dx = \int_{\partial\omega} \vec{x} \times T(x, \nu)\, dS + \int_\omega \vec{x} \times f \, dx,$$

hold for any subregion $\omega \subset \Omega_t$, where ν is the outward unit normal vector field on $\partial\omega$ and f is a body force. Here we state an important result on stress vectors which was proved by Cauchy. For its proof, see [34], chapter 5.

Theorem 2.1 (Cauchy's theorem)**.** *If the balance laws of linear and angular momenta hold, then there exists a matrix valued function σ from Ω_t to $\mathbb{S}$ such that $T(x, \nu) = \sigma(x)\nu$ for all $x \in \Omega_t$ where the right-hand side is the matrix-vector multiplication.*

The σ in the Cauchy's theorem is called the (Cauchy) *stress tensor* or simply *stress*. In the proof of the above theorem, the symmetry of the stress tensor is due to the balance law of angular momentum.

Let ω be a subregion of Ω and f be an external body force acting on ω. By the divergence theorem,

$$\int_{\partial\omega} \sigma\nu \, dS = \int_\omega \operatorname{div} \sigma \, dx.$$

Thus the integration of surface traction exerted to ω on $\partial\omega$ is same as the force obtained by integrating $-\operatorname{div}\sigma$ on ω. By using the balance law of linear momentum, conservation of mass, the fact that ω is arbitrary, we have

$$\frac{d}{dt}(\rho v) - \operatorname{div}\sigma = f \quad \text{in } \Omega.$$

We refer to [34] for derivation of the above equation.

2.2.2 Linear elasticity

A material is called *elastic* if its stress tensor at a certain time is solely determined by the deformed configuration at that time. From a physical point of view, a key feature of elastic materials is that the shape of material deformed by a stress vector returns to the original shape when the stress vector which caused deformation is removed. In an elastic material, the stress and strain tensors satisfy a relation determined by the kinematic properties of the material. This relation governing kinematic behavior of a material is called a *constitutive law*.

We confine our discussion to elastic materials for which the constitutive laws are linear equations relating the stress and strain tensors, and we also use the linearized strain tensor, which is the linear approximation of strain tensor, instead of the original one. These linearization assumptions are acceptable in many applications when deformations of material are relatively small compared to the scale of whole kinematic system.

From the definition of strain tensor in (2.4), the linearized strain tensor $\epsilon = \epsilon(u) : \Omega \to \mathbb{S}$ is defined by

$$\epsilon(u) = \frac{1}{2}(\operatorname{grad} u + (\operatorname{grad} u)^T), \quad \textit{i.e.}, \quad \epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad 1 \le i, j \le n,$$

for given displacement $u : \Omega \to \mathbb{V}$.

From our assumption that the constitutive equations are linear, the stress tensor σ and the linearized strain tensor $\epsilon(u)$ are related by

$$\sigma(x) = C(x)(\epsilon(u)(x)), \tag{2.5}$$

where $C(x) : \mathbb{S} \to \mathbb{S}$ is symmetric positive definite and uniformly bounded above and below. The *stiffness tensor* or *elasticity tensor* C is a rank 4 tensor with components $C_{ijkl} : \Omega \to \mathbb{R}$, $1 \le i, j, k, l \le n$ such that

$$C_{ijkl} = C_{jikl} = C_{klij}, \tag{2.6}$$

which may be determined by measuring the kinematic properties of elastic medium with experiments. For simplicity, the stress-strain relation (2.5) will be denoted by $\sigma = C\epsilon(u)$. From the uniform boundedness of $C(x)$, the map $C : L^2(\Omega; \mathbb{S}) \to L^2(\Omega; \mathbb{S})$ is a symmetric positive definite bounded linear operator.

The *compliance tensor* $A(x)$ is defined by $A(x) = C(x)^{-1}$. Thus $A(x) : \mathbb{S} \to \mathbb{S}$ is symmetric positive definite and uniformly bounded above and below. An elastic medium is called *isotropic* if kinematic properties of the material at each point is same in any direction. If an elastic medium is isotropic, then $C\tau$ and $A\tau$ have the forms

$$C\tau = 2\mu\tau + \lambda \operatorname{tr}(\tau)I, \qquad A\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda} \operatorname{tr}(\tau)I\right), \tag{2.7}$$

where μ, λ are positive scalar functions defined on Ω, called the Lamé coefficients, and $\operatorname{tr}(\tau)$ is the trace of τ.

2.2.3 Linear viscoelasticity

Viscoelastic materials

In constitutive laws of elastic materials, the time dependence of strain is not involved. However, in many fluids the stress tensor is related to the *strain rate tensor*, which is the time derivative of strain tensor. Such a dependence is called *viscosity* of materials.

A material is called *viscoelastic* if the material has both elastic and viscous kinematic features. Many polymers, biomedical tissues, and geophysical materials are viscoelastic, so it is important to understand the behavior of viscoelastic materials in science and engineering.

In order to model viscoelastic materials, we need a constitutive law which describes the relation of stress, strain, and strain rate tensors. If we confine our attention to linear viscoelastic materials, then there is a unified framework to describe all possible constitutive laws by using convolution integrals in time with some kernels. This integral form to describe constitutive laws, called the *hereditary approach*, is useful for analysis from the viewpoint of PDE but it creates difficulties for numerical computation because the numerical time integration of convolution is not easy to implement in an efficient way. Therefore we shall use differential forms of constitutive laws, say *differential constitutive laws*, for our study of numerical methods for viscoelasticity problems. The differential constitutive laws are not available for all linear viscoelastic materials. Some materials need constitutive laws with fractional time derivatives, which are not local operators and cannot be written as differential operators [14]. However, differential constitutive laws are obtained for the *mechanical models* of viscoelastic materials, which will be introduced later, and mechanical models include many important models of viscoelastic materials. The equivalence of integral and differential forms of constitutive laws under some assumptions is discussed in [33].

Hereditary approach

We briefly introduce the hereditary approach because it is related to the two fundamental characteristics of viscoelastic materials, the *creep compliance* and the *relaxation modulus*.

Before we define the creep compliance and relaxation modulus mathematically, let us describe those properties in a physical sense. If a material is purely elastic, then the dependence of strain on stress is instantaneous and strain is not changed as long as stress is constant in time. However, in viscoelastic materials, the dependence is not instantaneous and strain changes in time even if stress is held constant. We can see it clearly when we push a foam pillow with a force which is constant in time. This kinematic behavior is called *creep*. Conversely, suppose we push an elastic material, deform it up to a certain distance, and then keep the state. The stress response remains constant. In viscoelastic materials, when we do the same action, the stress is the strongest at the beginning moment and decays in time. This is explained by the fact that the molecules of viscoelastic material are rearranged by stress and the rearrangement of molecules requires some time. This kinematic behavior is called *relaxation*.

Now, in a one dimensional model, we introduce rigorous definitions of the creep compliance and relaxation modulus and describe the hireditary approach of linear viscoelasticity. Let $\sigma(t)$ be the stress and $\epsilon(t)$ be the linear strain, which are scalars in the one dimensional case. For constitutive laws, we assume *invariance of time translation* and *causality* of material properties. Invariance of time translation means if the input at certain time t_0 induces output at time $t_0+\delta$, $\delta>0$, then the same input at time t_0+d induces the output at time $t_0+d+\delta$ which is same as the output at time $t=t_0+\delta$. Causality is the property that the output at time t_1 is completely determined by the inputs in the time range $t \leq t_1$.

Let $\Theta(t)$ be the Heaviside function, i.e., the function defined on $\mathbb{R}$ which is 1 for $t>0$ and 0 for $t<0$. The creep test is to set $\sigma(t)=\Theta(t)$ and observe the corresponding $\epsilon(t)$ which is called the *creep compliance* and is denoted by $J(t)$. The relaxation test is to set $\epsilon(t)=\Theta(t)$ and observe the corresponding $\sigma(t)$ which is called the *relaxation modulus* and is denoted by $G(t)$. These two functions are called *materials functions*. From causality, $J(t) = G(t) = 0$ for $t < 0$. In experiments, $G, J \geq 0$ (or symmetric positive definite in higher than one dimension) and on $0 < t < +\infty$, J is non-decreasing and G is non-increasing. Suppose $J(t)$ is differentiable and increasing in time. Then for $t > 0$, $\dot{J} > 0$ and $0 \leq J(0+) < J(t) < J(+\infty) \leq +\infty$. Similarly, under the assumption $\dot{G} < 0$, $+\infty \geq G(0+) > G(t) > G(+\infty) \geq 0$.

By using the material functions, the stress and strain are described by the Riemann–Stiltjes integrals

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau)d\sigma(\tau), \qquad \sigma(t) = \int_{-\infty}^{t} G(t-\tau)d\epsilon(\tau).$$

They are called creep and relaxation representations, respectively. The above formulas are justified by the Boltzmann superposition principle which will be explained below.

Suppose a constant amount of stress σ_1 is exerted from time τ_1, i.e., $\sigma(t) = \sigma_1\Theta(t-\tau_1)$. Then the corresponding strain $\epsilon(t)$ is $\sigma_1 J(t-\tau_1)$. Suppose the stresses $\Delta\sigma_i = \sigma_{i+1} - \sigma_i$ are added at time τ_i for $i = 2, \cdots, n$. $\tau_1 < \tau_2 < \cdots < \tau_n$. Then the strain is $\epsilon(t) = \sum_{i=1}^{n} \Delta\sigma_i J(t-\tau_i)$. In this manner, for continuous $\sigma(t)$, the corresponding strain is obtained as the limit of the summation by increasing n and letting the maximum of time intervals converge to zero. In a similar way, the formula of $\sigma(t)$ is obtained.

Mechanical models

In mechanical models, a viscoelastic material is understood as a continuum of infinitesimal elements consisting of a combination of infinitesimal springs and dashpots. For example, the special case of a linear elastic material is modeled by a continuum of elements consisting of infinitesimal springs. In a mechanical model of viscoelastic materials, for each spring and dashpot unit, the elastic stress σ_e and viscous stress σ_v are related to the strain tensor and strain rate tensor by

$$\sigma_e = C\epsilon(u), \qquad \sigma_v = C'\epsilon(\dot{u}), \tag{2.8}$$

where C and C' are rank 4 tensors satisfying (2.6) and are uniformly bounded from above and below. By combining spring and dashpot units in series or parallel, we can make infinitely many mechanical models of viscoelastic materials.

In Figure 2.1, we illustrate the spring-dashpot combination of some elementary models. The Kelvin–Voigt and Maxwell models are obtained by combining one spring and one dashpot in parallel and in series, respectively. The Zener model is the parallel combination of Maxwell component and one spring, and the generalized Zener model is a generalization of Zener model with multiple Zener components.

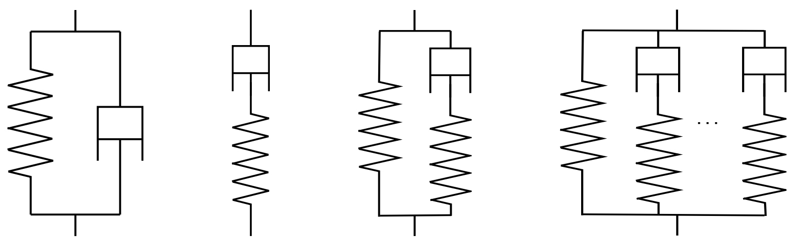

Figure 2.1: Examples of mechanical models of viscoelastic materials. The Kelvin–Voigt, Maxwell, Zener (or standard linear solid), and generalized Maxwell (or Weichert) models.

In the Kelvin–Voigt model, the elastic and viscous stresses σ_e, σ_v, are related to $\epsilon(u)$ and $\epsilon(\dot{u})$ by the spring and dashpot units as $\sigma_e = C\epsilon(u)$, $\sigma_v = C'\epsilon(\dot{u})$. The total stress is the sum of elastic and viscous stresses, so a constitutive equation is

$$\sigma = C\epsilon(u) + C'\epsilon(\dot{u}).$$

In the Maxwell model, we consider the decomposition of displacement $u = u_e + u_v$ where u_e and u_v are the parts of displacement involved with the spring and dashpot units. By (2.8), the stresses related to the spring and dashpot components are $C\epsilon(u_e)$ and $C'\epsilon(\dot{u}_v)$. However, by Newton's third law, $C\epsilon(u_e) = C'\epsilon(\dot{u}_v)$, which is the total stress tensor σ. If we let $A = C^{-1}$, $A' = C'^{-1}$, then $A\sigma = \epsilon(u_e)$, $A'\sigma = \epsilon(\dot{u}_v)$. Thus a constitutive equation for the Maxwell model is

$$A\dot{\sigma} + A'\sigma = \epsilon(\dot{u}_e) + \epsilon(\dot{u}_v) = \epsilon(\dot{u}).$$

The constitutive equations of the Zener and the generalized Zener models are obtained with similar arguments. The derivation of equations of the Zener model will be discussed in detail in Chapter 5. A similar approach can be applied to the generalized Maxwell and generalized Zener models.

Before moving to the next section, we remark that the viscoelastic features of one material can be described by more than one mechanical models, i.e., two different mechanical models may show kinematics of exactly same creep compliance and relaxation modulus. For instance, there is another description of the Zener model (see [48]), which is the serial combination of one Kelvin–Voigt component and a spring. In the hereditary approach, we have a unique constitutive law for given creep compliance and the relaxation modulus. However, as we have seen in the examples of the Maxwell and Kelvin–Voigt models, differential constitutive laws include quantities which are strongly motivated by the structure of mechanical models which may not be intrinsic in the sense of physics. The differential constitutive laws from different mechanical models may have very different forms of equations nonetheless they describe same kinematic features. In our study of the Zener model, we use the generalized Maxwell form of mechanical model because it is easier to analyze than the model of generalized Kelvin–Voigt form even if they have same kinematic features.

2.3 Mixed finite element methods

In this section, we introduce basics of saddle point problems and mixed finite element methods. For more information about mixed finite element methods, see [19].

2.3.1 Saddle point problems

Let Σ, V be Hilbert spaces and suppose that $a : \Sigma \times \Sigma \to \mathbb{R}$, $b : \Sigma \times V \to \mathbb{R}$ are bounded bilinear forms. We denote the dual spaces of Σ, V by Σ^*, V^*. We now consider a variational problem with constraints.

Constrained minimization problem. For $F \in \Sigma^*$, $G \in V^*$, find $\sigma \in \Sigma$ which minimizes

$$J(\sigma) = \frac{1}{2}a(\sigma, \sigma) - F(\sigma),$$

subject to the constraint $b(\sigma, v) = G(v)$ for all $v \in V$.

Instead of this minimization problem, we find a critical point $(\sigma, u) \in \Sigma \times V$ of

$$\mathcal{L}(\tau, v) = \frac{1}{2}a(\tau, \tau) + b(\tau, v) - F(\tau) - G(v), \qquad (\tau, v) \in \Sigma \times V, \tag{2.9}$$

by using the Lagrange multiplier u. By the Fréchet derivative computation, we see that a critical point (σ, u) of (2.9) should satisfy

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= F(\tau), \quad \tau \in \Sigma, \\ b(\sigma,v) &= G(v), \quad v \in V. \end{aligned} \tag{2.10}$$

Note that for any $(\tau, v) \in \Sigma \times V$, the inequalities

$$\mathcal{L}(\sigma, v) \le \mathcal{L}(\sigma, u) \le \mathcal{L}(\tau, u),$$

hold, so the variational problem of (2.9) is called a *saddle point problem.*

We will discuss necessary and sufficient conditions for the well-posedness of problem (2.10). Define $A : \Sigma \to \Sigma^*$, $B : \Sigma \to V^*$ by

$$(A\sigma)(\tau) = a(\sigma,\tau), \quad (B\sigma)(v) = b(\sigma, v), \qquad \tau \in \Sigma,\ v \in V.$$

Then we can rewrite (2.10) as

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{2.11}$$

In order to show well-posedness of (2.10), let

$$Z = \{\tau \in \Sigma \mid b(\tau, v) = (B\tau)(v) = 0, \quad \forall v \in V\}, \tag{2.12}$$

which is the null space of B. Since Z is a closed subspace of Σ, we have the orthogonal decomposition $\Sigma = Z + Z^\perp$, and $\Sigma^* = Z^* + (Z^\perp)^*$. Let π_{Z^*}, $\pi_{Z^{\perp *}}$ be the projections from Σ^* onto Z^*, $(Z^\perp)^*$, and define $A_{ZZ} : Z \to Z^*$, $A_{Z\perp} : Z \to (Z^\perp)^*$, $A_{\perp Z} : Z^\perp \to Z^*$, $A_{\perp\perp} : Z^\perp \to (Z^\perp)^*$ by

$$\begin{aligned} A_{ZZ} &:= \pi_{Z^*} \circ A|_Z, & A_{\perp Z} &:= (\pi_{Z^*} A)|_{Z^\perp}, \\ A_{Z\perp} &:= (\pi_{Z^{\perp *}} A)|_{Z^\perp}, & A_{\perp\perp} &:= (\pi_{Z^{\perp *}} A)|_{Z^\perp}. \end{aligned} \tag{2.13}$$

The following theorem for the well-posedness of (2.10) was proved by Brezzi. See [19] for its proof.

Theorem 2.2. *Suppose that A_{ZZ} in (2.13) is an isomorphism and B is onto. Then (2.10) has a unique solution $(\sigma, u) \in \Sigma \times V$ and there exists $c > 0$ such* *that*

$$\|\sigma\|_\Sigma + \|u\|_V \le c(\|F\|_{\Sigma^*} + \|G\|_{V^*}).$$

Remark 2.3. In the above theorem, the two conditions that A_{ZZ} is isomorphic and B is onto, are called the first and second Brezzi conditions. Since b is a bounded bilinear form, the second Brezzi condition is equivalent to the inf-sup condition

$$\inf_{0\neq v\in V} \sup_{0\neq \tau\in\Sigma} \frac{b(\tau,v)}{\|\tau\|_\Sigma \|v\|_V} \ge \gamma > 0, \tag{2.14}$$

by the closed range theorem in functional analysis [50].

If the bilinear form a is symmetric, then the first Brezzi condition is obtained from another inf-sup condition

$$\inf_{0\neq \sigma\in Z} \sup_{0\neq \tau\in Z} \frac{a(\sigma,\tau)}{\|\sigma\|_\Sigma \|\tau\|_\Sigma} \ge \gamma' > 0. \tag{2.15}$$

Its proof is obtained by identifying Σ and Σ^* by the Riesz representation theorem and using the closed range theorem.

2.3.2 Mixed finite elements and Brezzi conditions

We will discuss the numerical solution of saddle point problems. Since the bilinear form a is symmetric in most important problems, we assume that a is symmetric for simplicity.

In order to solve the saddle point problems numerically with finite elements, we use finite element spaces $\Sigma_h \subset \Sigma$, $V_h \subset V$ and consider the following discrete form of problem (2.10): Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$\begin{aligned} a(\sigma_h,\tau) + b(\tau,u_h) &= F(\tau), \quad \tau\in\Sigma_h, \\ b(\sigma_h,v) &= G(v), \quad v\in V_h. \end{aligned} \tag{2.16}$$

The pair of finite element spaces (Σ_h, V_h) is called *mixed finite elements* and the numerical methods of solving the discrete saddle point problem (2.16) with mixed finite elements are called *mixed finite element methods* or *mixed methods.*

We consider the conditions that the problem (2.16) is well-posed. If we apply Theorem 2.2 to (2.16), then we only need to check the first and second Brezzi conditions for (Σ_h, V_h). Let

$$Z_h = \{\tau \in \Sigma_h \mid b(\tau, v) = 0, \quad v \in V_h\}. \tag{2.17}$$

By the symmetry assumption of bilinear form a, as we pointed out in Remark 2.3,the Brezzi conditions are obtained from the two inf-sup conditions

$$\inf_{0 \neq \sigma \in Z_h} \sup_{0 \neq \tau \in Z_h} \frac{a(\sigma, \tau)}{\|\sigma\|_{\Sigma_h} \|\tau\|_{\Sigma_h}} \geq \alpha_h > 0, \tag{2.18}$$

$$\inf_{0 \neq v \in V_h} \sup_{0 \neq \tau \in \Sigma_h} \frac{b(\tau, v)}{\|\tau\|_{\Sigma_h} \|v\|_{V_h}} \geq \beta_h > 0. \tag{2.19}$$

Let us consider a family of mixed finite elements $\{(\Sigma_h, V_h)\}_{h>0}$ with parameter h and suppose that there are α_h, β_h for each h. Then $\{(\Sigma_h, V_h)\}_{h>0}$ is called *stable* if α_h, β_h are bounded below by some positive constants independent of h. For simplicity, we usually use (Σ_h, V_h) to denote the family of finite elements $\{(\Sigma_h, V_h)\}_{h>0}$. If (Σ_h, V_h) is stable, then the following quasi-optimal error estimate is straightforwardly obtained from Theorem 2.2.

$$\|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_V \leq c \left(\inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_\Sigma + \inf_{v \in V_h} \|u - v\|_V \right). \tag{2.20}$$

2.4 Mixed finite elements for linear elasticity with weak symmetry

From the balance law of angular momentum, the stress tensor is symmetric and the symmetry of stress must be enforced when we find a numerical approximation of the stress. The symmetry of stress can be imposed on the numerical solution strongly by using symmetric finite elements for stress. However, it is not easy to find stable families of such mixed finite elements and they have a fairly large number of degrees of freedom in general. This is a nontrivial drawback in practical computations. An alternative is to enforce symmetry of stress weakly by requiring that σ_h, the numerical approximation of stress, to be orthogonal to a skew-symmetric finite element space. To our knowledge, this weak symmetry idea firstly appeared in [26] and extended to higher orders in [3]. The first finite elements in the context of mixed finite elements were the PEERS elements developed in [7]. Thereafter, numerous mixed finite elements for linear elasticity were developed with the weak symmetry idea [10, 17, 24, 32, 46, 47].

In general, this weak symmetry elements are easier to implement and have have fewer degrees of freedom for low order elements than strong symmetry elements [5, 8, 11, 36].

2.4.1 Mixed formulations of linear elasticity

In our discussions, we only consider problems with the homogeneous displacement boundary conditions $u = 0$ on $\partial\Omega$ for simplicity but all discussions can be extended easily to problems with inhomogeneous displacement boundary conditions.

From the balance of linear momentum, $-\operatorname{div} C\epsilon(u) = f$ at equilibrium for an external body force $f \in V$. In order for mixed formulations, we introduce a new variable $\sigma : \Omega \to S$ for the stress $C\epsilon(u)$ and have $A\sigma = \epsilon(u)$. From the homogeneous displacement boundary conditions $u = 0$ on $\partial\Omega$, by integration by parts, we obtain the following Hellinger–Reissner formulation of linear elasticity which seeks (σ, u) in $S \times V$ so that

$$(A\sigma, \tau) + (\operatorname{div} \tau, u) = 0, \qquad \tau \in S, \tag{2.21}$$

$$-(\operatorname{div} \sigma, w) = (f, w), \qquad w \in V. \tag{2.22}$$

We can also use a modified Hellinger–Reissner formulation with weak symmetry of stress. We first extend the A operator, originally defined only on symmetric tensors, to be the identity map on skew-symmetric tensors. If we set $r = \operatorname{skw} \operatorname{grad} u$, then $A\sigma = \epsilon(u) = \operatorname{grad} u - r$. By integration by parts, using the homogeneous displacement boundary conditions, we have

$$(A\sigma, \tau) = (\operatorname{grad} u - r, \tau) = -(u, \operatorname{div} \tau) - (r, \tau), \quad \tau \in M.$$

Now we seek (σ, u, r) in $M \times V \times K$ satisfying

$$(A\sigma, \tau) + (\operatorname{div} \tau, u) + (r, \tau) = 0, \qquad \tau \in M, \tag{2.23}$$

$$-(\operatorname{div} \sigma, w) = (f, w), \qquad w \in V, \tag{2.24}$$

$$(\sigma, q) = 0, \qquad q \in K. \tag{2.25}$$

One can check that the two formulations, (2.21–2.22) and (2.23–2.25), are equivalent but there are significant differences between the discrete problems deduced from them.

To discretize (2.21–2.22) with mixed finite elements, we select finite element spaces $S_h \subset S$, $V_h \subset V$. The discrete problem is then to seek (σ_h, u_h) in $S_h \times V_h$ so that

$$
\begin{aligned}
(A\sigma_h, \tau) + (\operatorname{div} \tau, u_h) &= 0, && \tau \in S_h, \\
-(\operatorname{div} \sigma_h, w) &= (f, w), && w \in V_h.
\end{aligned}
$$

By Theorem 2.2, this problem is well-posed when the Brezzi conditions of mixed finite element methods are satisfied. However, it is not easy to find simple mixed finite elements satisfying the symmetry condition $S_h \subset S$ and Brezzi conditions. There are some known finite elements using composite triangles [8, 36]. Finding finite elements without using composite triangles had been an open question for four decades until the first family of elements were discovered by Arnold and Winther [11]. Although there are now some known finite elements with symmetric stresses [5, 11], they have a large number of degrees of freedom and their application to practical problems is limited, especially in three dimensions.

As an alternative approach, we consider the discrete problem of (2.23–2.25). Let $M_h \subset M$, $V_h \subset V$, $K_h \subset K$ be finite element spaces and we find (σ_h, u_h, r_h) in $M_h \times V_h \times K_h$ so that

$$
\begin{aligned}
(A\sigma_h, \tau) + (\operatorname{div} \tau, u_h) + (r_h, \tau) &= 0, && \tau \in M_h, && (2.26) \\
-(\operatorname{div} \sigma_h, w) &= (f, w), && w \in V_h, && (2.27) \\
(\sigma_h, q) &= 0, && q \in K_h. && (2.28)
\end{aligned}
$$

This is a saddle point problem in the sense of (2.10) if we take M for Σ, and $V \times K$ for V. The bilinear forms a and b in (2.10) are

$$
a(\sigma, \tau) = (A\sigma, \tau), \qquad b(\sigma, (u, r)) = (\operatorname{div} \sigma, u) + (\sigma, r). \tag{2.29}
$$

Now we need the Brezzi conditions (2.18) and (2.19) for the mixed finite elements M_h, V_h, and K_h, where the spaces Σ_h, V_h in (2.18), (2.19) are M_h and $V_h \times K_h$ in our elasticity formulation. We will introduce two families of stable mixed finite elements in the next section.

2.4.2 Two families of stable mixed finite elements for elasticity

In this section, among the finite elements for elasticity with weak symmetry of stress, we introduce two families developed by Arnold, Falk, and Winther [10], and by Gopalakrishnan and Guzmán [32]. The Arnold–Falk–Winther (AFW) elements have simple shape functions and so are easy to implement, but the accuracy of stress is suboptimal. The Gopalakrishnan–Guzmán (GG) elements, which are more recently developed, give optimal order accuracy of errors but shape functions are more complicated. The GG elements allow for increased accuracy of the displacement approximation by postprocessing.

For simplicity, we will describe these two families of elements in detail only in the two dimensional case. For the descriptions of three dimensional elements, see [10] and [32].

The Arnold–Falk–Winther elements

Let T be a triangle in the triangulation $\mathcal{T}_h$. In the Arnold–Falk–Winther elements of degree $k \geq 1$, the shape functions of M_h, V_h, and K_h are

$$\mathcal{P}_k(T;\mathbb{M}), \quad \mathcal{P}_{k-1}(T;\mathbb{V}), \quad \mathcal{P}_{k-1}(T;\mathbb{K}), \tag{AFW}$$

respectively. The local degrees of freedom (DOFs) of M_h are

$$\sigma \mapsto \int_e w \cdot (\sigma\nu)\, dS, \quad w \in \mathcal{P}_k(e;\mathbb{V}), \qquad \sigma \mapsto \int_T \tau : \sigma\, dx, \quad \tau \in \tilde{N}_{k-1}(T),$$

where e is an edge of T and $\tilde{N}_k(T)$ is the set consisting of pairs of the shape functions of the rotated Raviart–Thomas elements, defined right after (2.2) in section 2.1. The local DOFs of V_h and K_h are

$$v \mapsto \int_T v \cdot w\, dx, \quad w \in \mathcal{P}_{k-1}(T;\mathbb{V}), \qquad r \mapsto \int_T r : q\, dx, \quad q \in \mathcal{P}_{k-1}(T;\mathbb{K}).$$

It is easy to check unisolvency of shape functions of V_h and K_h for these local DOFs. Unisolvency of shape functions of M_h for the above local DOFs is a consequence of unisolvency of the Brezzi–Douglas–Marini (BDM) elements because M_h is the space which is a pair of BDM elements. For more details on BDM elements, see [9, 19] for instance.

For the triangulation $\mathcal{T}_h$, we define M_h, V_h, K_h by the assembled finite element spaces from these local DOFs, respectively. Let $\Pi_{h,T}$, $P_{h,T}$, $P'_{h,T}$ be the interpolation maps determined by the local DOFs of each space M_h, V_h, and K_h, respectively. The interpolation maps Π_h, P_h, and P'_h are defined by

$$\Pi_h|_T := \Pi_{h,T}, \quad P_h|_T := P_{h,T}, \quad P'_h|_T := P'_{h,T}.$$

Since V_h and K_h do not have any DOFs on edges or vertices, they are piecewise polynomial spaces adapted to $\mathcal{T}_h$ without any interelement continuity and one can check that P_h and P'_h are the orthogonal L^2 projections onto V_h and K_h, respectively. Therefore,

$$\|v - P_h v\| \le ch^m \|v\|_m, \quad 0 \le m \le k, \qquad \|r - P'_h r\| \le ch^m \|r\|_m, \quad 0 \le m \le k,$$

for $v \in H^m(\Omega; \mathbb{V})$ and $r \in H^m(\Omega; \mathbb{K})$. In contrast to V_h and K_h, the DOFs for M_h impose some interelement continuity. For any $\tau \in M_h$, the normal component of each row of τ on each edge is continuous from the DOFs assigned on edges. Therefore $M_h \subset H(\operatorname{div}; \mathbb{M})$, and $\operatorname{div} M_h$ is well-defined.

Now we state properties of Π_h and the spaces $M_h \times V_h \times K_h$ proved in [10].
(A1) We have $\operatorname{div} M_h = V_h$, and for any $\sigma \in H^1(\Omega; \mathbb{M})$, $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma$,

$$\|\Pi_h \sigma\| \le c\|\sigma\|_1, \quad \|\sigma - \Pi_h \sigma\| \le ch^m \|\sigma\|_m, \quad 1 \le m \le k+1.$$

(A2) (inf-sup condition) There exists $c > 0$ which is uniform in the maximum diameter of mesh triangulation h, so that for any $(u, r) \in V_h \times K_h$, there is $\tau \in M_h$ satisfying

$$\operatorname{div} \tau = u, \quad (\tau, q) = (r, q), \quad \forall q \in K_h, \qquad \|\tau\|_{\operatorname{div}} \le c(\|u\| + \|r\|).$$

We claim that **(A1)**, **(A2)** imply (2.18), (2.19), and so the Arnold–Falk–Winther elements are stable mixed finite elements. Recall that the bilinear forms a and b for our elasticity problem are as in (2.29).

In order to show (2.18), we first note that the space Z_h defined by (2.17) is

$$Z_h = \{\sigma \in M_h \,|\, (\operatorname{div} \sigma, w) + (\sigma, q) = 0, \quad \forall (w, q) \in V_h \times K_h\}.$$

If $\sigma \in Z_h$, then $\operatorname{div} \sigma \in V_h$ because $\operatorname{div} M_h = V_h$ from **(A1)** and $Z_h \subset M_h$ from the definition of Z_h. Furthermore, $\sigma \in Z_h$ implies that $\operatorname{div} \sigma = 0$ because $(\operatorname{div} \sigma, w) = 0$ for all $w \in V_h$ from the definition of Z_h. Therefore for any $\sigma \in Z_h$, using the coercivity of A and the fact that $\operatorname{div}\sigma = 0$, we obtain

$$a(\sigma,\sigma) = (A\sigma,\sigma) \geq c_0\|\sigma\|^2 = c_0\|\sigma\|^2_{\operatorname{div}}, \qquad c_0 > 0.$$

This implies (2.18) with $\alpha_h = c_0$. Thus (2.18) holds with $\alpha_h \geq c_0$.

To prove (2.19) from **(A2)**, we rewrite (2.19) with $M_h \times V_h \times K_h$, which is

$$\inf_{0\neq(u,r)\in V_h\times K_h} \sup_{0\neq\tau\in M_h} \frac{(\operatorname{div}\tau, u) + (\tau, r)}{\|\tau\|_{\operatorname{div}}(\|u\| + \|r\|)} \geq \beta > 0.$$

For given $(u, r) \in V_h \times K_h$, if we take a τ which satisfies the conditions in **(A2)**, then

$$\frac{(\operatorname{div}\tau, u) + (\tau, r)}{\|\tau\|_{\operatorname{div}}(\|u\| + \|r\|)} \geq \frac{\|u\|^2 + \|r\|^2}{c(\|u\| + \|r\|)^2} \geq \frac{1}{2c},$$

where the last inequality is due to the arithmetic-geometric mean inequality, so the second Brezzi condition (2.19) holds.

For future reference, here we state a simple corollary.

Corollary 2.4. *For any $r \in K_h$, there exists a $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(\tau, r) = \|r\|^2$, and $\|\tau\| \leq c\|r\|$.*

Its proof is obvious if we take $u = 0$ and $q = r$ in the conditions of **(A2)**.

The Gopalakrishnan–Guzmán elements

For a triangle T, let v_i, e_i, $i = 0, 1, 2$ be the vertices and edges of T, numbered so that v_i is not an endpoint of e_i. For each i, there is a unique linear polynomial λ_i which has value 1 at v_i and vanishes on e_i, and we call such λ_i, $i = 0, 1, 2$, the barycentric coordinates on T. The bubble function b_T on T is defined by $b_T = \lambda_0\lambda_1\lambda_2$, the product of all barycentric coordinates on T. We also define rot and curl as

$$\operatorname{rot}\begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix} := \begin{pmatrix}\partial_1\tau_{12} - \partial_2\tau_{11}\\ \partial_1\tau_{22} - \partial_2\tau_{21}\end{pmatrix}, \qquad \operatorname{curl}\begin{pmatrix}u_1\\ u_2\end{pmatrix} := \begin{pmatrix}\partial_2 u_1 & -\partial_1 u_1\\ \partial_2 u_2 & -\partial_1 u_2\end{pmatrix}.$$

Let $k \geq 1$ be fixed. In the Gopalakrishnan–Guzmán (GG) elements of degree k, the shape functions of M_h are

$$\mathcal{P}_k(T;\mathbb{M}) + B(\mathcal{P}_k(T;\mathbb{K})), \qquad B(\eta) := \operatorname{curl}((\operatorname{rot}\eta)b_T), \quad \eta \in \mathcal{P}_k(T;\mathbb{M}),$$

and the shape functions of V_h and K_h are

$$\mathcal{P}_{k-1}(T;\mathbb{V}), \qquad \mathcal{P}_k(T;\mathbb{K}).$$

Let $\tilde{\mathcal{P}}_k(T;\mathbb{K}) = \{r \in \mathcal{P}_k(T;\mathbb{K}) \,|\, (r,q) = 0, \forall q \in \mathcal{P}_{k-1}(T;\mathbb{K})\}$. The local DOFs of M_h are

$$\begin{aligned}
\sigma &\mapsto \int_e w \cdot (\sigma\nu)\, dS, \quad w \in \mathcal{P}_k(e;\mathbb{V}),\\
\sigma &\mapsto \int_T \tau : \sigma \, dx, \quad \tau \in \tilde{N}_{k-1}(T) + \tilde{\mathcal{P}}_k(T;\mathbb{K}).
\end{aligned}$$

The local DOFs of V_h and K_h are

$$v \mapsto \int_T v \cdot w \, dx, \quad w \in \mathcal{P}_{k-1}(T;\mathbb{V}), \qquad r \mapsto \int_T r : q \, dx, \quad q \in \mathcal{P}_k(T;\mathbb{K}).$$

In [32], Gopalakrishnan and Guzmán proved unisolvency of these DOFs. If we use Π_h, P_h, P_h' again to denote the interpolation maps determined by the DOFs of M_h, V_h, K_h, then the maps P_h, P_h' are the orthogonal L^2 projections onto V_h, K_h, respectively, and the estimates

$$\begin{aligned}
\|v - P_h v\| &\le c h^m \|v\|_m, \quad 0 \le m \le k,\\
\|r - P_h' r\| &\le c h^m \|r\|_m, \quad 0 \le m \le k+1,
\end{aligned} \tag{2.30}$$

hold for $v \in H^m(\Omega;\mathbb{V})$ and $r \in H^m(\Omega;\mathbb{K})$. Note that the best L^2 approximation in K_h is one degree higher than the one in V_h because of the one degree higher shape functions. It is also proved in [32] that the conditions **(A1)**, **(A2)** in the previous section hold, and the GG elements are stable.

2.4.3 Error analysis for linear elasticity

As a consequence of Theorem 2.2, stability of mixed finite elements implies the quasi-optimal error estimate (2.20). However, when we use the AFW or GG elements, the error analysis can be improved by utilizing the features of finite elements in **(A1)** and **(A2)**. In this section, we will prove improved error analysis following [30], [35], and show a postprocessing technique following [32, 47].

Throughout this section, (M_h, V_h, K_h) will denote either the AFW or GG elements and (Π_h, P_h, P_h') will denote the interpolation maps given by the degrees of freedom without distinguishing the two families.

An improved error analysis

We claim the following.

Theorem 2.5. *Suppose that*

$$c_0\|\tau\|^2 \le \|\tau\|_A^2 \le c_1\|\tau\|^2, \quad \tau \in L^2(\Omega;\mathbb{M}), \tag{2.31}$$

with positive constants c_0*,* c_1 *and let* (σ, u, r) *be the exact solution of* (2.23–2.25)*. Then for the solution* (σ_h, u_h, r_h) *of* (2.26–2.28)*, the following improved error estimates hold:*

$$\|\operatorname{div}\sigma - \operatorname{div}\sigma_h\| = \|\operatorname{div}\sigma - P_h\operatorname{div}\sigma\|, \tag{2.32}$$

$$\|\sigma-\sigma_h\| + \|u_h - P_h u\| + \|r - r_h\| \le c(\|\sigma - \Pi_h\sigma\| + \|r - P_h' r\|), \tag{2.33}$$

with c *depending on* c_0*,* c_1*.*

Proof. The error equations are

$$(A(\sigma-\sigma_h),\tau) + (\operatorname{div}\tau, u-u_h) + (r-r_h,\tau) = 0, \qquad \tau \in M_h, \tag{2.34}$$

$$(\operatorname{div}(\sigma-\sigma_h), v) = 0, \qquad v \in V_h, \tag{2.35}$$

$$(\sigma-\sigma_h, q) = 0, \qquad q \in K_h. \tag{2.36}$$

Since $\operatorname{div}\sigma_h \in V_h$, (2.35) implies that $\operatorname{div}\sigma_h = P_h\operatorname{div}\sigma$, so (2.32) holds.

For the proof of (2.33), we first prove that

$$\|r_h - P_h' r\| \le c(\|\sigma - \Pi_h\sigma\| + \|\sigma_h - \Pi_h\sigma\| + \|r - P_h' r\|). \tag{2.37}$$

By Corollary 2.4, there exists a $\tau \in M_h$ such that $\operatorname{div}\tau = 0$ and

$$(r_h - P_h' r, \tau) = \|r_h - P_h' r\|^2, \qquad \|\tau\| \le c\|r_h - P_h' r\|.$$

If we choose such τ in (2.34), we have $(A(\sigma-\sigma_h),\tau) + (r - r_h, \tau) = 0$. By splitting $r - r_h = r - P_h' r + P_h' r - r_h$ and invoking (2.31), we get

$$\begin{aligned} \|r_h - P_h' r\|^2 &= (A(\sigma-\sigma_h),\tau) + (r - P_h' r, \tau) \\ &\le c(\|\sigma-\sigma_h\|_A + \|r - P_h' r\|)\|r_h - P_h' r\|. \end{aligned} \tag{2.38}$$

By the triangle inequality with $\|\cdot\|_A$ norm, we have

$$\|r_h - P_h' r\| \le c(\|\sigma - \Pi_h \sigma\|_A + \|\Pi_h \sigma - \sigma_h\|_A + \|r - P_h' r\|), \tag{2.39}$$

and (2.37) follows from (2.31).

Next, choose $\tau = \sigma_h - \Pi_h \sigma$ in (2.34), $q = r_h - P_h' r$ in (2.36). Since we have shown that $\operatorname{div} \sigma_h = P_h \operatorname{div} \sigma$ and by the fact $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma$ in **(A1)**, we have $\operatorname{div}(\sigma_h - \Pi_h \sigma) = 0$, and

$$\begin{aligned} (A(\sigma - \sigma_h), \sigma_h - \Pi_h \sigma) + (r - r_h, \sigma_h - \Pi_h \sigma) &= 0, \\ (\sigma - \sigma_h, r_h - P_h' r) &= 0. \end{aligned}$$

By writing $\sigma - \sigma_h$ as $(\sigma - \Pi_h \sigma) + (\Pi_h \sigma - \sigma_h)$ in the second equation, we have an equality $(\sigma - \Pi_h \sigma, r_h - P_h' r) = (\sigma_h - \Pi_h \sigma, r_h - P_h' r)$. If we substitute $r - r_h$ by $(r - P_h' r) + (P_h' r - r_h)$ in the first equation, and use the previously obtained equality to replace $(P_h' r - r_h, \sigma_h - \Pi_h \sigma)$ by $(P_h' r - r_h, \sigma - \Pi_h \sigma)$, then we obtain

$$(A(\sigma - \sigma_h), \sigma_h - \Pi_h \sigma) + (r - P_h' r, \sigma_h - \Pi_h \sigma) + (P_h' r - r_h, \sigma - \Pi_h \sigma) = 0,$$

or, equivalently,

$$\begin{aligned} (A(\sigma_h - \Pi_h \sigma), \sigma_h - \Pi_h \sigma) &= (A(\sigma - \Pi_h \sigma), \sigma_h - \Pi_h \sigma) \\ &\quad + (r - P_h' r, \sigma_h - \Pi_h \sigma) + (P_h' r - r_h, \sigma - \Pi_h \sigma). \end{aligned} \tag{2.40}$$

From (2.31), the Cauchy–Schwarz inequality, and Young's inequality, we get, for any $\epsilon > 0$,

$$\begin{aligned} & c_0 \|\sigma_h - \Pi_h \sigma\|^2 \\ & \le (A(\sigma - \Pi_h \sigma), \sigma_h - \Pi_h \sigma) + (r - P_h' r, \sigma_h - \Pi_h \sigma) + (P_h' r - r_h, \sigma - \Pi_h \sigma) \\ & \le C_\epsilon (\|\sigma - \Pi_h \sigma\|^2 + \|r - P_h' r\|^2) + \epsilon (\|\sigma_h - \Pi_h \sigma\|^2 + \|r_h - P_h' r\|^2). \end{aligned}$$

By moving $\epsilon \|\sigma_h - \Pi_h \sigma\|^2$ term to the left-hand side, one gets

$$(c_0 - \epsilon) \|\sigma_h - \Pi_h \sigma\|^2 \le C_\epsilon (\|\sigma - \Pi_h \sigma\|^2 + \|r - P_h' r\|^2) + \epsilon \|r_h - P_h' r\|^2.$$

If we use (2.37) to bound the last term on the right-hand side of this inequality, and take ϵ sufficiently small, then we obtain

$$\|\sigma_h - \Pi_h \sigma\|^2 \le c(\|\sigma - \Pi_h \sigma\|^2 + \|r - P_h' r\|^2), \tag{2.41}$$

for some $c > 0$. By applying this bound to (2.37), we then get

$$\|r_h - P_h' r\| \le c(\|\sigma - \Pi_h \sigma\| + \|r - P_h' r\|). \tag{2.42}$$

By the triangle inequality and inequalities (2.41) and (2.42), a partial result of (2.33) is obtained by

$$\begin{aligned}
&\|\sigma - \sigma_h\| + \|r - r_h\| \\
&\le \|\sigma - \Pi_h \sigma\| + \|\Pi_h \sigma - \sigma_h\| + \|r - P_h' r\| + \|P_h' r - r_h\| \\
&\le C(\|\sigma - \Pi_h \sigma\| + \|r - P_h' r\|).
\end{aligned}$$

For $\|u_h - P_h u\|$ in (2.33), choose τ in (2.34) such that $\operatorname{div} \tau = u_h - P_h u$, $(\tau, q) = 0$ for all $q \in K_h$, and $\|\tau\| \le c\|u_h - P_h u\|$ using **(A2)**. Note that

$$(\operatorname{div} \tau, u - u_h) = (\operatorname{div} \tau, P_h u - u_h) = -\|P_h u - u_h\|^2,$$

and (2.34) gives

$$\begin{aligned}
\|P_h u - u_h\|^2 &= (A(\sigma - \sigma_h), \tau) + (r - r_h, \tau) \\
&\le c(\|\sigma - \sigma_h\| + \|r - r_h\|)\|P_h u - u_h\|,
\end{aligned}$$

which completes proof of (2.33) by the previous estimates of $\|\sigma - \sigma_h\|$ and $\|r - r_h\|$. □

Robustness for nearly incompressible materials

Now we consider the error analysis for homogeneous isotropic nearly incompressible materials. If a material is homogeneous isotropic, then the compliance tensor A is of the form (2.7) with constants μ and λ. From a physical point of view, λ is very large when the material is nearly incompressible, which is an important class of materials. In nearly incompressible materials, A is still coercive on $L^2(\Omega; \mathbb{M})$ but the coercivity constant c_0 in (2.31) decays to zero as $\lambda \to +\infty$, so the constant c in (2.33), obtained from the argument we have used, may grow to infinity as $\lambda \to +\infty$.

We claim that even though c_0 decays to zero as $\lambda \to +\infty$, the constant c in (2.33) is uniformly bounded. This will be demonstrated by a more careful analysis using the following lemmas proved in [8]. We use τ^D to denote the deviatoric part of τ in $L^2(\Omega; \mathbb{M})$ which is defined by $\tau^D := \tau - (1/n)\,\mathrm{tr}(\tau) I$.

Lemma 2.6. *Let $\tau \in H(\mathrm{div}, \Omega; \mathbb{M})$ satisfy $\int_\Omega \mathrm{tr}(\tau)\, dx = 0$. Then the estimate*

$$\|\tau\| \le c(\|\tau^D\| + \|\operatorname{div}\tau\|_{-1}), \tag{2.43}$$

holds with $c > 0$ independent of τ.

Lemma 2.7. *For $\tau \in L^2(\Omega; \mathbb{M})$ and A of the form in (2.7), the inequality*

$$\|\tau^D\|^2 \le c(A\tau, \tau) =: c\|\tau\|_A^2, \tag{2.44}$$

holds with c depending only on μ and n.

Theorem 2.8. *Suppose that A is of the form (2.7) with positive constants λ, μ and the triples (σ, u, r), (σ_h, u_h, r_h) are solutions of (2.23–2.25) and (2.26–2.28), respectively. Then the constant c in (2.33) is uniformly bounded as $\lambda \to +\infty$.*

Proof. We begin with the same error equations (2.34–2.36) and prove (2.33) without using the coercivity of A. Since we will see many formulas in the proof of Theorem 2.5 again, we will not repeat them but simply refer to them in the proof.

We first prove (2.33) for $\|\sigma - \sigma_h\|$ without using the coercivity of A. Note that (2.39) still holds with c independent of the coercivity of A. For the estimate of $\|\sigma - \sigma_h\|$, we first claim

$$\|\sigma - \sigma_h\| \le c(\|\sigma - \Pi\sigma_h\| + \|\sigma_h - \Pi_h\sigma\|_A). \tag{2.45}$$

To show this inequality, choose $\tau = I$ in (2.34). Since $\operatorname{div}\tau = 0$ and both r and r_h are skew symmetric, we get

$$0 = (A(\sigma - \sigma_h), I) = \frac{1}{2\mu + n\lambda}\int_\Omega \mathrm{tr}(\sigma - \sigma_h)\, dx,$$

where the last equality comes from the definition of A in (2.7). By applying Lemma 2.6 to $\sigma - \sigma_h$,

$$\|\sigma - \sigma_h\| \le c(\|(\sigma - \sigma_h)^D\| + \|\operatorname{div}(\sigma - \sigma_h)\|_{-1}).$$

Since $\operatorname{div}\sigma_h = \operatorname{div}\Pi_h\sigma$ in V_h, one can see that

$$\|\operatorname{div}(\sigma - \sigma_h)\|_{-1} = \|\operatorname{div}(\sigma - \Pi_h\sigma)\|_{-1} \le \|\sigma - \Pi_h\sigma\|, \tag{2.46}$$

and by the triangle inequality and Lemma 2.7,

$$\begin{aligned} \|(\sigma - \sigma_h)^D\| &\le \|(\sigma - \Pi_h\sigma)^D\| + \|(\Pi_h\sigma - \sigma_h)^D\| \\ &\le c(\|\sigma - \Pi_h\sigma\| + \|\Pi_h\sigma - \sigma_h\|_A), \end{aligned} \tag{2.47}$$

so (2.45) is obtained.

Now we prove an estimate for $\|\sigma_h - \Pi_h\sigma\|_A$. Since $P_h'r$ and r are skew-symmetric, we have an equality

$$(r - P_h'r, \sigma_h - \Pi_h\sigma) = (r - P_h'r, (\sigma_h - \Pi_h\sigma)^D).$$

If we use this to replace $(r - P_h'r, \sigma_h - \Pi_h\sigma)$ in (2.40), then by the Cauchy–Schwarz inequality, Young's inequality, Lemma 2.7, and the fact that c_1 in (2.31) is uniformly bounded when $\lambda \to +\infty$, we get

$$\begin{aligned} \|\sigma_h - \Pi_h\sigma\|_A^2 &= (A(\sigma - \Pi_h\sigma), \sigma_h - \Pi_h\sigma) + (r - P_h'r, (\sigma_h - \Pi_h\sigma)^D) \\ &\quad + (P_h'r - r_h, \sigma - \Pi_h\sigma) \\ &\le \|\sigma - \Pi_h\sigma\|_A\|\sigma_h - \Pi_h\sigma\|_A + \|r - P_h'r\|\|(\sigma_h - \Pi_h\sigma)^D\| \\ &\quad + \|P_h'r - r_h\|\|\sigma - \Pi_h\sigma\| \\ &\le C_\epsilon(\|\sigma - \Pi_h\sigma\|^2 + \|r - P_h'r\|^2) \\ &\quad + \epsilon(\|P_h'r - r_h\|^2 + \|\sigma_h - \Pi_h\sigma\|_A^2). \end{aligned}$$

By using (2.39) and absorbing the terms with the coefficient ϵ into the left-hand side, we have

$$\|\sigma_h - \Pi_h\sigma\|_A^2 \le C(\|\sigma - \Pi_h\sigma\|^2 + \|r - P_h'r\|^2).$$

Thus, combining the above with (2.45), the estimate (2.33) for $\|\sigma - \sigma_h\|$ is obtained.

The estimate (2.33) for $\|r - r_h\|$ is obtained by applying Lemma 2.7 to (2.39), using the above result on $\|\sigma_h - \Pi_h\sigma\|_A$, and the triangle inequality.

The estimate (2.33) for $\|u_h - P_hu\|$ is easily obtained from (2.33) for $\|\sigma - \sigma_h\|$ and $\|r - r_h\|$, by the same argument in the proof of Theorem 2.5. □

Postprocessing

By postprocessing we mean a technique with relatively small computational costs to find an improved numerical solution from the previously obtained numerical solution. In this section, we show a postprocessing technique to improve the error estimate of $\|u - u_h\|$ for the exact solution u and the numerical solution u_h obtained in (2.26–2.28). More precisely, we will find u_h^*, a piecewise polynomial function of higher degree than u_h with a relatively simple procedure and show that the order of accuracy of $\|u - u_h^*\|$ is higher than that of $\|u - u_h\|$. For completeness, we present a proof of postprocessing developed in [47] but in a slightly modified form which is suggested in [32]. This postprocessing is available only for the GG elements. The reason for this restriction will be clear when we describe the assumptions of postprocessing later.

Recall that $V_h = \mathcal{P}_{k-1}(\mathcal{T}_h; \mathbb{V})$ for the GG elements. We define

$$V_h^* = \mathcal{P}_k(\mathcal{T}_h; \mathbb{V}), \qquad \tilde{V}_h = \{w \in V_h^* \mid w \perp V_h\},$$

and denote the orthogonal L^2 projections of $V = L^2(\Omega; \mathbb{V})$ onto V_h^* and $\tilde{V}_h$ by P_h^* and $\tilde{P}_h$, respectively. It is obvious that $P_h^* = \tilde{P}_h + P_h$. Let (σ, u, r) and (σ_h, u_h, r_h) be solutions of (2.23–2.25) and (2.26–2.28), respectively, and define $u_h^* \in V_h^*$ by

$$\begin{aligned} (\operatorname{grad}_h u_h^*, \operatorname{grad}_h w) &= (A\sigma_h + r_h, \operatorname{grad}_h w), \quad && w \in \tilde{V}_h, \\ (u_h^*, v) &= (u_h, v), && v \in V_h, \end{aligned} \tag{2.48}$$

where grad_h is the piecewise gradient operator adapted to $\mathcal{T}_h$.

We show that u_h^* in (2.48) is well-defined. From $V_h^* = V_h \oplus \tilde{V}_h$ and counting number of equations and unknowns, it is easy to see that (2.48) is a system of linear equations with same number of equations and unknowns. In order to show that u_h^* is well-defined by (2.48), it is enough to show that $u_h^* = 0$ if the right-hand sides vanish. From the second equation in (2.48), with vanishing right-hand side, u_h^* should be in $\tilde{V}_h$ and by taking $w = u_h^*$ in the first equation in (2.48), one can conclude $u_h^* = 0$ because $\tilde{V}_h$ does not include constants and therefore grad_h is an injective operator on $\tilde{V}_h$.

Theorem 2.9. *Let* (σ, u, r), (σ_h, u_h, r_h) *be the solutions of* (2.23–2.25) *and* (2.26–2.28)*, respectively. Let* (M_h, V_h, K_h) *be the* GG *elements of degree* k *and* *assume that* $\sigma \in H^{k+1}(\Omega; \mathbb{M})$*,* $r \in H^{k+1}(\Omega; \mathbb{K})$*. Then*

$$\|\sigma - \sigma_h\| + \|r - r_h\| + \|P_h u - u_h\| \le ch^{k+1}(\|\sigma\|_{k+1} + \|r\|_{k+1}), \tag{2.49}$$

and, for u_h^* *defined by* (2.48)*,* $\|u - u_h^*\| \le ch^{k+1}\|\sigma, u, r\|_{k+1}$*.*

Remark 2.10. Since $\sigma = C\epsilon(u)$, $\sigma \in H^{k+1}$ implies $u \in H^{k+2}$.

Proof. The estimate (2.49) follows directly from (2.33) for the GG elements of degree k. Since $\|u - P_h^* u\| \le ch^{k+1}\|u\|_{k+1}$, for the error estimate $\|u - u_h^*\| \le ch^{k+1}$, we need only estimate $\|P_h^* u - u_h^*\|$. Let $\tilde{u}_h = \tilde{P}_h u_h^*$. Since $\|P_h^* u - u_h^*\|^2 = \|P_h u - u_h\|^2 + \|\tilde{P}_h u - \tilde{u}_h\|^2$, by orthogonality of V_h and $\tilde{V}_h$, and $\|P_h u - u_h\|$ is already bounded by ch^{k+1}, it suffices to estimate $\|\tilde{P}_h u - \tilde{u}_h\|$. Recall that $\operatorname{grad} u = A\sigma + r$, so

$$(\operatorname{grad} u, \operatorname{grad}_h w) = (A\sigma + r, \operatorname{grad}_h w), \quad w \in \tilde{V}_h.$$

Considering the difference of this equation and the first equation in (2.48), we get

$$(\operatorname{grad}_h(u - u_h^*), \operatorname{grad}_h w) = (A(\sigma - \sigma_h) + r - r_h, \operatorname{grad}_h w), \quad w \in \tilde{V}_h. \tag{2.50}$$

Since $P_h^* u = P_h u + \tilde{P}_h u$ and $u_h^* = u_h + \tilde{u}_h$, one sees that

$$u - u_h^* = (u - P_h^* u) + (P_h^* u - u_h^*) = (u - P_h^* u) + (P_h u - u_h) + (\tilde{P}_h u - \tilde{u}_h),$$

and if we use this to rewrite (2.50), then we have

$$\begin{aligned}
&(\operatorname{grad}_h(\tilde{P}_h u - \tilde{u}_h), \operatorname{grad}_h w) \\
&\qquad = -(\operatorname{grad}_h(u - P_h^* u), \operatorname{grad}_h w) - (\operatorname{grad}_h(P_h u - u_h), \operatorname{grad}_h w) \\
&\qquad\qquad + (A(\sigma - \sigma_h) + r - r_h, \operatorname{grad}_h w).
\end{aligned}$$

By taking $w = \tilde{P}_h u - \tilde{u}_h$, we have

$$\begin{aligned}
&\|\operatorname{grad}_h(\tilde{P}_h u - \tilde{u}_h)\| \\
&\le c(\|\operatorname{grad}_h(u - P_h^* u)\| + \|\operatorname{grad}_h(P_h u - u_h)\| + \|\sigma - \sigma_h\| + \|r - r_h\|).
\end{aligned} \tag{2.51}$$

Now we note that there is a uniform constant c such that

$$h\|\operatorname{grad}_h w\| \le c\|w\|, \qquad w \in V_h^*, \tag{2.52}$$

$$\|w\| \le ch\|\operatorname{grad}_h w\|, \qquad w \in \tilde{V}_h, \tag{2.53}$$

where (2.52) is an inverse estimate (see [18], p. 110) and (2.53) is a result of the fact that $\tilde{V}_h$ is orthogonal to piecewise constants and a discrete version of Poincaré inequality with scaling. By using these inequalities and (2.51), we get

$$\begin{aligned}\|\tilde{P}_h u - \tilde{u}_h\| &\le ch\|\operatorname{grad}_h(\tilde{P}_h u - \tilde{u}_h)\| \\ &\le ch(\|\operatorname{grad}_h(P_h u - u_h)\| + \|\operatorname{grad}_h(P_h^* u - u)\| \\ &\qquad + \|\sigma - \sigma_h\| + \|r - r_h\|) \\ &\le c(\|P_h u - u_h\| + h^{k+1}\|u\|_{k+1} + \|\sigma - \sigma_h\| + \|r - r_h\|),\end{aligned}$$

where the last inequality is due to (2.52) and the Bramble–Hilbert lemma. By (2.49), we can conclude $\|\tilde{P}_h u - \tilde{u}_h\| \le ch^{k+1}$ and the proof is completed. □

Remark 2.11. The assumption (2.49) does not hold for the AFW elements of degree k because K_h is the space of piecewise polynomials of degree less than or equal to $k-1$.

2.4.4 A weakly symmetric elliptic projection

For the error analysis of time dependent problems which will be discussed later, we define a bounded projection $\tilde{\Pi}_h : H^1(\Omega;\mathbb{M}) \to M_h$ and describe its properties.

Lemma 2.12. *Let M_h be the finite element stress space in the* AFW *or* GG *elements of degree $k \ge 1$. For $\sigma \in H^1(\Omega;\mathbb{M})$, we consider a solution $(\sigma_h, u_h, r_h) \in M_h \times V_h \times K_h$ of the system*

$$(\sigma_h, \tau) + (\operatorname{div}\tau, u_h) + (\tau, r_h) = (\sigma, \tau), \qquad \tau \in M_h, \tag{2.54}$$

$$(\operatorname{div}\sigma_h, w) = (\operatorname{div}\sigma, w), \qquad w \in V_h, \tag{2.55}$$

$$(\sigma_h, q) = (\sigma, q), \qquad q \in K_h, \tag{2.56}$$

and define $\tilde{\Pi}_h\sigma$ as σ_h. Then $\tilde{\Pi}_h : H^1(\Omega;\mathbb{M}) \to M_h$ is a projection such that $\|\tilde{\Pi}_h\sigma\| \le c\|\sigma\|_1$ with $c > 0$ which is uniformly bounded in h. Furthermore, for

$\sigma \in H^1(\Omega; \mathbb{M})$,

$$\|\sigma - \tilde{\Pi}_h \sigma\| \le c \|\sigma - \Pi_h \sigma\|, \tag{2.57}$$

$$\operatorname{div} \tilde{\Pi}_h \sigma = P_h \operatorname{div} \sigma, \qquad (\tilde{\Pi}_h \sigma, q) = (\sigma, q), \quad q \in K_h, \tag{2.58}$$

with $c > 0$ which is independent of h.

Proof. From the stability of the mixed finite elements, the linear system (2.54 – 2.56) has a unique solution, so $\tilde{\Pi}_h \sigma$ is well-defined and $\tilde{\Pi}_h$ is a linear operator.

Note that $(\sigma, 0, 0)$ is the exact solution of the problem, in which M_h, V_h, K_h are replaced by M, V, K. From (2.33) with the fact $r = 0$, we have

$$\|\sigma - \tilde{\Pi}_h \sigma\| \le c(\|\sigma - \Pi_h \sigma\| + \|r - P_h' r\|) = c\|\sigma - \Pi_h \sigma\|,$$

with $c > 0$ independent of h, so (2.57) is proved. Since $\|\Pi_h \sigma\| \le c' \|\sigma\|_1$ by **(A1)**, we have $\|\tilde{\Pi}_h \sigma\| \le c'' \|\sigma\|_1$ for some $c'' > 0$ which is independent of h.

The properties in (2.58) follow from (2.55) with **(A1)**, and (2.56) . □

Remark 2.13. The first equality in (2.58) is a commutativity property of $\tilde{\Pi}_h$. The second equation in (2.58) implies that $\tilde{\Pi}_h \sigma$ is weakly symmetric if σ is symmetric. One can see that $\|\tilde{\Pi}_h \sigma\| \le c\|\sigma\|_1$ by the triangle inequality, (2.57), and **(A1)**.

2.5 Miscellaneous preliminaries

In this section we prove lemmas that will be used for evolutionary equations and regularity of solutions.

2.5.1 Gronwall-type estimates

For error estimates of time dependent problems, we need a Gronwall-type inequality.

Lemma 2.14. *Let $F, G, Q : [0, T_0] \to \mathbb{R}$ be continuous, nonnegative functions. Suppose that $Q(t)$ is continuously differentiable and satisfies*

$$\frac{1}{2} \frac{d}{dt} Q^2 \le FQ + G, \tag{2.59}$$

for all $t \in [0, T_0]$*. Then for* $t \in [0, T_0]$*,*

$$Q(t) \le Q(0) + \max\left\{4\int_0^t F\,ds,\ 2\left(\int_0^t G\,ds\right)^{\frac{1}{2}}\right\}. \tag{2.60}$$

Proof. Since T_0 is arbitrary, it suffices to show the inequality for $t = T_0$. Suppose that $Q(t)$ attains its maximum at $t_M \in [0, T_0]$. If $t_M = 0$, then there is nothing to prove, so we assume $t_M > 0$. If the inequality (2.60) holds for t_M, then

$$\begin{aligned} Q(T_0) \le Q(t_M) &\le Q(0) + \max\left\{4\int_0^{t_M} F\,ds,\ 2\left(\int_0^{t_M} G\,ds\right)^{\frac{1}{2}}\right\}, \\ &\le Q(0) + \max\left\{4\int_0^{T_0} F\,ds,\ 2\left(\int_0^{T_0} G\,ds\right)^{\frac{1}{2}}\right\}. \end{aligned}$$

Hence, without loss of generality, we may assume $t_M = T_0$. If we integrate (2.59) from 0 to T_0, then we get

$$Q(T_0)^2 - Q(0)^2 \le 2\int_0^{T_0} (FQ + G)\,ds. \tag{2.61}$$

It is obvious that one of the following inequalities holds:

$$\int_0^{T_0} G\,ds \le \int_0^{T_0} FQ\,ds, \qquad \int_0^{T_0} FQ\,ds < \int_0^{T_0} G\,ds.$$

Suppose that $\int_0^{T_0} G\,ds \le \int_0^{T_0} FQ\,ds$ is the case. Then (2.61) gives

$$Q(T_0)^2 - Q(0)^2 \le 4\int_0^{T_0} FQ\,ds.$$

If we divide both sides by $Q(T_0)$, the maximum of $Q(t)$ on $[0, T_0]$, then we get

$$Q(T_0) \le Q(0) + 4\int_0^{T_0} F\,ds,$$

which implies (2.60). To complete the proof, suppose that $\int_0^{T_0} FQ\,ds < \int_0^{T_0} G\,ds$. Then (2.61) gives

$$Q(T_0)^2 \le Q(0)^2 + 4\int_0^{T_0} G\,ds,$$

which implies that

$$Q(T_0) \leq Q(0) + 2\left(\int_0^{T_0} G\,ds\right)^{\frac{1}{2}}.$$

This completes the proof. ☐

If we assume $G \equiv 0$ in Lemma 2.14, then the same argument gives a sharper result.

Corollary 2.15. *Let $F, Q : [0, T_0] \to \mathbb{R}$ be continuous, nonnegative functions and suppose that Q is continuously differentiable. Suppose also that*

$$\frac{1}{2}\frac{d}{dt}Q^2 \leq FQ,$$

holds for all $t \in [0, T_0]$. Then for $t \in [0, T_0]$,

$$Q(t) \leq Q(0) + 2\int_0^t F(s)\,ds.$$

A discrete version of Lemma 2.14 also holds.

Lemma 2.16. *Let $\{Q_i\}$, $\{F_i\}$, $\{G_i\}$ be sequences of nonnegative numbers with $0 \leq i \leq N$ and let $\Delta t > 0$ be given. If*

$$Q_{i+1}^2 - Q_i^2 \leq \Delta t((Q_i + Q_{i+1})F_i + 2G_i),$$

for all $0 \leq i \leq N$, then

$$Q_i \leq Q_0 + \max\left\{4\Delta t\sum_{j=0}^{i-1} F_j, 2\left(\Delta t\sum_{j=0}^{i-1} G_j\right)^{\frac{1}{2}}\right\}.$$

The proof is similar to that of the continuous case, so details are omitted.

2.5.2 Well-posedness of differential algebraic equations

Now we prove a lemma for well-posedness of linear differential algebraic equations. This will be used in chapter 4.

Lemma 2.17. *Let E, F be matrices in $\mathbb{R}^{m\times m}$ of the form*

$$E = \begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

with $E_0, F_{11} \in \mathbb{R}^{l\times l}$ for some $1 \le l < m$, with E_0, F_{22} nonsingular. Let $f : [0, T_0] \to \mathbb{R}^m$ be continuous. Let $X \in C^1([0, T_0]; \mathbb{R}^m)$ and consider the differential algebraic equation

$$E\dot{X} = FX + f. \tag{2.62}$$

If we let X_1 be the first l tuple of X and X_2 be the complementary $m - l$ tuple of X, then for any given initial data $X_1(0) \in \mathbb{R}^l$, (2.62) has a unique solution which satisfies the initial condition.

Proof. Let f_1, f_2 be the first l tuple and final $m - l$ tuple of f. If we rewrite (2.62) using the block matrix forms of E and F, we have

$$E_0\dot{X}_1 = F_{11}X_1 + F_{12}X_2 + f_1, \qquad F_{21}X_1 + F_{22}X_2 + f_2 = 0. \tag{2.63}$$

Thus $X_2 = -F_{22}^{-1}F_{21}X_1 - F_{22}^{-1}f_2$ from the second equation in (2.63). Substituting X_2 in the first equation of (2.63) yields

$$E_0\dot{X}_1 = (F_{11} - F_{12}F_{22}^{-1}F_{21})X_1 - F_{12}F_{22}^{-1}f_2 + f_1 =: BX_1 + \tilde{f}_1. \tag{2.64}$$

The equation (2.64) is a system of linear ordinary differential equations, so has a unique solution for given $X_1(0) \in \mathbb{R}^l$ (see [25]). When X_1 is obtained, X_2 is uniquely determined from X_1 and f_2 by the second equation in (2.63). This proves existence and uniqueness of solutions. □

2.5.3 Regularity lemmas

We prove lemmas which are needed later when we discuss regularity of weak solutions. Let ν denote the outward unit normal vector field on $\partial\Omega$.

Lemma 2.18. *The set $\{\tau\nu \,|\, \tau \in C^1(\overline{\Omega}; \mathbb{S})\}$ is dense in $L^2(\partial\Omega; \mathbb{V})$.*

Proof. Suppose that the lemma is not true. Then there exists $0 \neq v \in L^2(\partial\Omega; \mathbb{V})$

such that

$$\int_{\partial\Omega} v \cdot \tau\nu \, dS = 0, \tag{2.65}$$

for all $\tau \in C^1(\overline{\Omega}; \mathbb{S})$. If we rewrite (2.65) using the components of v, τ, and ν, then we have

$$\int_{\partial\Omega} \sum_{1 \le i,j \le n} v_i \tau_{ij} \nu_j \, dS = 0,$$

for all $\tau \in C^1(\overline{\Omega}; \mathbb{S})$. Let us suppose that, for $1 \le i,j \le n$, only the (i,j) and (j,i) entries of τ are possibly nonvanishing. Since the set of traces of all $C^1(\overline{\Omega})$ functions is dense in $L^2(\partial\Omega)$, we obtain that $v_i\nu_j + v_j\nu_i = 0$ almost everywhere (a.e.) on $\partial\Omega$ for all $1 \le i,j \le n$. In particular, $v_i\nu_i = 0$ a.e. when $i = j$. If we multiply v_i by the equality $v_i\nu_j + v_j\nu_i = 0$, then $v_i^2\nu_j + v_iv_j\nu_i = 0$ almost everywhere. Since $v_i\nu_i = 0$ a.e., we can see that $v_i^2\nu_j = 0$ a.e. and therefore $v_i\nu_j = 0$ a.e. for any $1 \le i,j \le n$. From this equality, we can see that $v \equiv 0$ a.e. because ν is a unit vector field and therefore $\nu \neq 0$ almost everywhere. The proof is completed. □

Using the above lemma, we now obtain a regularity result for weak solutions.

Lemma 2.19. *Let $\sigma \in L^2(\Omega; \mathbb{S})$, $v \in V$ and suppose that*

$$(\sigma, \tau) + (\operatorname{div} \tau, v) = 0, \qquad \tau \in S, \tag{2.66}$$

holds. Then $v \in \mathring{H}^1(\Omega; \mathbb{V})$ and $\epsilon(v) = \sigma$ in $L^2(\Omega; \mathbb{M})$. Conversely, if $\sigma = \epsilon(v)$ for $v \in \mathring{H}^1(\Omega; \mathbb{V})$, then (2.66) holds.

Proof. Suppose that (2.66) holds with the given assumptions on σ, v. By integration by parts,

$$(\sigma, \tau) = (\operatorname{grad} v, \tau) = (\epsilon(v), \tau), \quad \tau \in C_0^\infty(\Omega; \mathbb{S}),$$

so $\sigma = \epsilon(v)$ in the sense of distributions. By Korn's inequality (see [18], chapter 11), $v \in H^1(\Omega; \mathbb{V})$ and thus $\sigma = \epsilon(v)$ almost everywhere. Then for any $\tau \in C^1(\overline{\Omega}; \mathbb{S})$, we also have

$$(\sigma, \tau) = -(\operatorname{div} \tau, u) = \int_{\partial\Omega} v \cdot \tau\nu \, dS + (\operatorname{grad} v, \tau) = \int_{\partial\Omega} v \cdot \tau\nu \, dS + (\epsilon(v), \tau),$$

and we have $\int_{\partial\Omega} v \cdot \tau\nu \, dS = 0$ for any $\tau \in C^1(\overline{\Omega}; \mathbb{S})$. Since $\{\tau\nu \,|\, \tau \in C^1(\overline{\Omega}; \mathbb{S})\}$ is dense in $L^2(\partial\Omega; \mathbb{V})$, we obtain $v|_{\partial\Omega} = 0$ almost everywhere and therefore $v \in \mathring{H}^1(\Omega; \mathbb{V})$. For the other direction, suppose $v \in \mathring{H}^1(\Omega; \mathbb{V})$. Then (2.66) is obvious from integration by parts. $\square$

Corollary 2.20. *Let $\sigma \in L^2(\Omega; \mathbb{S})$, $r \in K$, $v \in V$ and suppose that*

$$(\sigma, \tau) + (\operatorname{div} \tau, v) + (r, \tau) = 0, \qquad \tau \in M, \tag{2.67}$$

holds. Then $v \in \mathring{H}^1(\Omega; \mathbb{V})$ and $\epsilon(v) = \sigma$, $\operatorname{skw} \operatorname{grad} v = r$ in $L^2(\Omega; \mathbb{M})$. Conversely, if $\sigma = \epsilon(v)$ and $r = \operatorname{skw} \operatorname{grad} v$ for $v \in \mathring{H}^1(\Omega; \mathbb{V})$, then (2.67) holds.

Proof. Suppose that (2.67) holds with the given assumptions of σ, v, r. Since $S \subset M$, and $(r, \tau) = 0$ for $\tau \in S$, we have $\sigma = \epsilon(v)$, $v \in \mathring{H}^1(\Omega; \mathbb{V})$ by Lemma 2.19. Furthermore, by integration by parts, one can see

$$(\sigma + r, \tau) = (\operatorname{grad} v, \tau), \quad \tau \in C^1(\Omega; \mathbb{M}),$$

from (2.67), so $r = \operatorname{grad} v - \sigma = \operatorname{skw} \operatorname{grad} v$.

For the other direction, suppose $v \in \mathring{H}^1(\Omega; \mathbb{V})$ and $\sigma = \epsilon(v)$, $r = \operatorname{skw} \operatorname{grad} v$. Then (2.67) is obvious from integration by parts. $\square$

Chapter 3

Mixed methods for linear elastodynamics

3.1 Introduction

We consider the numerical solution of linear elastodynamics with mixed finite element methods. The linear elastodynamics equation is an evolutionary partial differential equation describing wave propagation in an elastic medium. It has the form

$$\rho \ddot{u} - \operatorname{div} C\epsilon(u) = f \quad \text{in } \Omega, \tag{3.1}$$

where $u : \Omega \to \mathbb{R}^n$ is the displacement vector field, C is the stiffness tensor of the elastic medium, $\epsilon(u)$ is the linearized strain tensor of displacement, ρ is the mass density, and f is an external body force. In the equation we omit the time variable t for simplicity but both u and f depend on time and the equation is interpreted as holding for all $t \in [0, T_0]$. It is known that the equation (3.1), with appropriate boundary conditions and initial data $(u(0), \dot{u}(0))$, has one and only one weak solution (see e.g., [28], Theorem 4.1).

Numerical solutions of linear elastodynamics with mixed finite element methods have been studied by various researchers [15, 27, 39]. In [27], Douglas and Gupta studied linear elastodynamics in a planar domain using the mixed finite elements for stationary linear elasticity developed in [8] and a displacement-stress weak formulation. In [39], Makridakis studied linear elastodynamics in two and three dimensional domains using displacement-stress and velocity-stress weak formulations and the finite elements developed in [8, 36, 47]. Makridakis also studied higher order time discretization in his work and proved a priori error estimates. In [15], Bécache, Joly, and Tsogka developed a new family of rectangular mixed finite elements and carried out the a priori error analysis for the velocity-stress formulation of linear elastodynamics.

In general, mixed methods for linear elastodynamics are based on the development of mixed finite elements for stationary linear elasticity. To our knowledge, mixed finite elements for linear elasticity with weak symmetry of stress, for instance the AFW and GG elements that we introduced in section 2.4.2, have not been used for linear elastodynamics. Since these elements are advantageous in computational costs and implementations, it is worth to study how to use them for linear elastodynamics.

The rest of this chapter is organized as follows. In section 3.2, we derive a velocity-stress weak formulation of linear elastodynamics with weak symmetry of stress. A priori error estimates of the semidiscrete and fully discrete solutions for the AFW elements are discussed in sections 3.3 and 3.4, respectively. In section 3.5, we prove that error bounds, better than the ones in section 3.4, can be obtained for the GG elements with a more careful error analysis. In section 3.6, we consider numerical solutions in nearly incompressible homogeneous isotropic materials and prove that our numerical solutions are locking-free, i.e., the constants of error bounds do not grow to infinity as the Lamé coefficient λ goes to infinity. Finally, numerical results verifying our analysis are presented in section 3.7.

3.2 Weak formulations with weak symmetry

The goal of this section is to derive a velocity-stress formulation of linear elastodynamics with weakly imposed symmetry of stress. For simplicity of error analysis, we only consider the homogeneous displacement boundary conditions $u \equiv 0$ on $\partial\Omega$ for all time. We assume that the mass density ρ satisfies $0 < \rho_0 \leq \rho \leq \rho_1 < \infty$ for constants ρ_0, ρ_1.

In order to have a mixed form with velocity and stress, we set $v = \dot{u}$, $\sigma = C\epsilon(u)$ in (3.1), and get a system of equations

$$\rho \dot{v} - \operatorname{div} \sigma = f, \qquad A\dot{\sigma} = \epsilon(v), \tag{3.2}$$

where $A = C^{-1}$. Note that the boundary conditions $u \equiv 0$ on $\partial\Omega$ and the initial data $(u(0), \dot{u}(0))$ in (3.1) give boundary conditions $v \equiv 0$ on $\partial\Omega$ and the initial data $\sigma(0) = C\epsilon(u(0))$, $v(0) = \dot{u}(0)$.

Let us consider well-posedness of (3.2). We first rewrite (3.2) as

$$\begin{pmatrix} \dot{\sigma} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & C\epsilon \\ \rho^{-1} \operatorname{div} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \rho^{-1} f \end{pmatrix}.$$

For well-posedness of this system, let us recall the Hille–Yosida theorem. For a Hilbert space $\mathcal{X}$ and a closed, densely defined operator $\mathcal{L}$ on $\mathcal{X}$ with domain $D(\mathcal{L})$, we consider an evolution equation $\dot{U} = \mathcal{L}U + F$ with initial data $U(0)$. The operator $\mathcal{L}$ is called an m-dissipative operator if, for some $\lambda > 0$, $\|(I - \lambda\mathcal{L})u\|_{\mathcal{X}} \geq \|u\|_{\mathcal{X}}$ for $u \in D(\mathcal{L})$ and $I - \lambda\mathcal{L} : D(\mathcal{L}) \to \mathcal{X}$ is surjective. If $\mathcal{L}$ is an m-dissipative operator, and $F \in W^{1,1}([0, T_0]; \mathcal{X})$, $U(0) \in D(\mathcal{L})$, then the evolution equation has a unique solution $U \in C^0([0, T_0]; D(\mathcal{L}) \cap C^1([0, T_0]; \mathcal{X})$ (see [21], Proposition 4.1.6).

In our elastodynamics problem, let $\mathcal{X} = L^2(\Omega; \mathbb{S}) \times V$ be the Hilbert space with the inner product

$$((\sigma, v), (\tau, w))_{\mathcal{X}} := (\sigma, \tau)_A + (v, w)_\rho = (A\sigma, \tau) + (\rho v, w).$$

We define a linear operator $\mathcal{L}$ as $\mathcal{L}(\sigma, v) = (C\epsilon(v), \rho^{-1} \operatorname{div} \sigma)$. Note that $\mathcal{L}$ is an unbounded operator on $\mathcal{X}$ and its domain $D(\mathcal{L}) = S \times \mathring{H}^1(\Omega; \mathbb{V})$ is dense in $\mathcal{X}$.

To apply the Hille–Yosida theorem, we need to check that $\mathcal{L}$ is an m-dissipative operator, which means that for some $\lambda > 0$,

$$\|(I - \lambda\mathcal{L})(\tau, w)\|_{\mathcal{X}} \geq \|(\tau, w)\|_{\mathcal{X}}, \qquad (\tau, w) \in D(\mathcal{L}), \tag{3.3}$$

and also $I - \lambda\mathcal{L} : D(\mathcal{L}) \to \mathcal{X}$ is surjective ([21], Definition 2.2.2). To check these conditions, let $\lambda > 0$, $(\tau, w) \in D(\mathcal{L})$. Then one sees that

$$\begin{aligned} \|(I - \lambda\mathcal{L})(\tau, w)\|_{\mathcal{X}} \|(\tau, w)\|_{\mathcal{X}} &\geq ((I - \lambda\mathcal{L})(\tau, w), (\tau, w))_{\mathcal{X}} \\ &= \|(\tau, w)\|_{\mathcal{X}}^2 - \lambda((C\epsilon(w), \rho^{-1} \operatorname{div} \tau), (\tau, w))_{\mathcal{X}} \\ &= \|(\tau, w)\|_{\mathcal{X}}^2, \end{aligned}$$

where the last equality is due to $(\epsilon(w), \tau) + (\operatorname{div} \tau, w) = 0$ by integration by parts. Hence, by dividing both sides by $\|(\tau, w)\|_{\mathcal{X}}$, (3.3) holds. To check the surjectivity of $I - \lambda\mathcal{L}$, for any given $(\eta, p) \in \mathcal{X}$, we prove that a solution of the equation,

$$((I - \lambda\mathcal{L})(\sigma, v), (\tau, w))_{\mathcal{X}} = ((\eta, p), (\tau, w))_{\mathcal{X}}, \qquad (\tau, w) \in D(\mathcal{L}),$$

exists. Since $D(\mathcal{L})$ is dense in $\mathcal{X}$, a solution (σ, v) of this equation satisfies $(I - \lambda\mathcal{L})(\sigma, v) = (\eta, p)$, so $I - \lambda\mathcal{L}$ is surjective. To show existence of solutions of the above equation, we rewrite it as

$$(\sigma - \lambda C\epsilon(v), \tau)_A = (\eta, \tau)_A, \qquad \tau \in S, \tag{3.4}$$

$$(v - \lambda\rho^{-1} \operatorname{div} \sigma, w)_\rho = (p, w)_\rho, \qquad w \in \mathring{H}^1(\Omega; \mathbb{V}). \tag{3.5}$$

Rewriting (3.5) in a weak form,

$$(\rho v, w) + \lambda(\sigma, \epsilon(w)) = (\rho p, w), \quad w \in \mathring{H}^1(\Omega; \mathbb{V}). \tag{3.6}$$

The equation (3.4) gives a constraint $\sigma - \lambda C\epsilon(v) = \eta$, and substituting σ in (3.6) by $\lambda C\epsilon(v) + \eta$, we obtain

$$(\rho v, w) + \lambda^2(C\epsilon(v), \epsilon(w)) = (\rho p, w) - (\eta, \epsilon(w)), \quad w \in \mathring{H}^1(\Omega; \mathbb{V}).$$

By Korn's inequality and the Lax–Milgram lemma, this equation has a unique solution $v \in \mathring{H}^1(\Omega; \mathbb{V})$. One can easily see that $\sigma = \lambda C\epsilon(v) + \eta$ is in $L^2(\Omega; \mathbb{S})$, and also in S because the equation (3.6) implies that $\operatorname{div} \sigma$ is well-defined in the sense of distributions. Consequently, we have checked that $\mathcal{L}$ is m-dissipative, and we can use the Hille–Yosida theorem.

As a consequence of the Hille–Yosida theorem, if $\sigma(0) \in S$, $v(0) \in \mathring{H}^1(\Omega; \mathbb{V})$, and $f \in W^{1,1}([0, T_0]; \mathbb{V})$, then there exist

$$\sigma \in C^0([0, T_0]; S) \cap C^1([0, T_0]; L^2(\Omega; \mathbb{S})),$$
$$v \in C^0([0, T_0]; \mathring{H}^1(\Omega; \mathbb{V})) \cap C^1([0, T_0]; V),$$

satisfying (3.2) with the given initial data.

Now we describe a weak formulation of (3.2) with weak symmetry of stress. We assume that $\sigma(0) = C\epsilon(u(0))$ for some $u \in \mathring{H}^1(\Omega; \mathbb{V})$. If we define $u(t) = u(0) + \int_0^t v(s)\, ds$, then, using $A\dot{\sigma} = \epsilon(v)$ and the fundamental theorem of calculus, we get $A\sigma = \epsilon(u)$. If we set $r = \operatorname{skw} \operatorname{grad} u$, then $\dot{r} = \operatorname{skw} \operatorname{grad} v$. Integrating the second equation of (3.2) by parts with the boundary conditions $v \equiv 0$ on $\partial\Omega$, we get $(A\dot{\sigma}, \tau) = (\epsilon(v), \tau) = (\operatorname{grad} v - \dot{r}, \tau) = -(v, \operatorname{div} \tau) - (\dot{r}, \tau)$ for all

$\tau \in M$, which is equivalent to

$$(A\dot{\sigma}, \tau) + (\operatorname{div} \tau, v) + (\dot{r}, \tau) = 0, \qquad \tau \in M. \tag{3.7}$$

From the first equation of (3.2), we get $(\rho \dot{v}, w) - (\operatorname{div} \sigma, w) = (f, w)$ for $w \in V$. Finally, the symmetry of σ gives $(\dot{\sigma}, q) = 0$ for $q \in K$. Consequently, a weak formulation with weak symmetry of stress is to seek

$$\begin{aligned} &\sigma \in C^0([0, T_0]; M) \cap C^1([0, T_0]; L^2(\Omega; \mathbb{M})), \\ &v \in C^1([0, T_0]; V)), \quad r \in C^1([0, T_0]; K), \end{aligned} \tag{3.8}$$

such that

$$(A\dot{\sigma}, \tau) + (\operatorname{div} \tau, v) + (\dot{r}, \tau) = 0, \qquad \tau \in M, \tag{3.9}$$

$$(\rho \dot{v}, w) - (\operatorname{div} \sigma, w) = (f, w), \qquad w \in V, \tag{3.10}$$

$$(\dot{\sigma}, q) = 0, \qquad q \in K, \tag{3.11}$$

with given initial data $(\sigma(0), v(0), r(0)) = (C\epsilon(u(0)), \dot{u}(0), \operatorname{skw} \operatorname{grad} u(0))$.

If (σ, v) is a solution of (3.2) obtained by the Hille–Yosida theorem, then we can show that there exist $u \in C^0([0, T_0]; \mathring{H}^1(\Omega))$, $r \in C^1([0, T_0]; K)$ such that (σ, v, r) is a solution of (3.9–3.11), $r(t) = \operatorname{skw} \operatorname{grad} u(t)$, and $\dot{r} = \operatorname{skw} \operatorname{grad} v$. The proof is similar to the derivation of (3.9–3.11), so we omit details.

Theorem 3.1. *For given initial data* $(\sigma(0), v(0), r(0)) \in S \times \mathring{H}^1(\Omega; \mathbb{V}) \times K$ *such that* $\sigma(0) = C\epsilon(u(0))$, $r(0) = \operatorname{skw} \operatorname{grad} u(0)$ *for some* $u(0) \in \mathring{H}^1(\Omega; \mathbb{V})$, *the system* (3.9–3.11) *has a unique solution* (σ, v, r) *satisfying* (3.8).

Proof. For existence, let $(\sigma(0), v(0), r(0))$ be a given initial data satisfying the assumptions in the theorem. By the Hille–Yosida theorem, the equation (3.2) has a unique solution (σ, v) with the initial data $(\sigma(0), v(0))$. We have seen that there exists r such that the triple (σ, v, r) satisfies (3.9–3.11), and $r(t) = \operatorname{skw} \operatorname{grad} u(t)$ for some $u \in C^0([0, T_0]; \mathring{H}^1(\Omega; \mathbb{V}))$.

For uniqueness, suppose that there are two solutions of (3.9–3.11) with same initial data, and denote their difference by (σ^d, v^d, r^d). Then this triple satisfies

$$\begin{aligned} (A\dot{\sigma}^d, \tau) + (\operatorname{div} \tau, v^d) + (\dot{r}^d, \tau) &= 0, \qquad && \tau \in M, \\ (\rho \dot{v}^d, w) - (\operatorname{div} \sigma^d, w) &= 0, && w \in V, \\ (\dot{\sigma}^d, q) &= 0, && q \in K, \end{aligned} \tag{3.12}$$

with $\sigma^d(0) = v^d(0) = r^d(0) = 0$. Now we set $\tau = \sigma^d$, $w = v^d$ in the first two equations and add them. Since $\sigma^d \perp K$ and $\dot{r} \in K$, we have $(\dot{r}, \sigma) = 0$, so the sum of two equations gives

$$\frac{1}{2}\frac{d}{dt}\|\sigma^d\|_A^2 + \frac{1}{2}\frac{d}{dt}\|v^d\|_\rho^2 = 0.$$

By the fundamental theorem of calculus, we get

$$\|\sigma^d(t)\|_A^2 + \|v^d(t)\|_\rho^2 = \|\sigma^d(0)\|_A^2 + \|v^d(0)\|_\rho^2 = 0,$$

so $\sigma^d = v^d \equiv 0$. From these facts and (3.12), one sees that $\dot{r}^d \equiv 0$. Since $r^d(0) = 0$, $r^d \equiv 0$ by the fundamental theorem of calculus, so uniqueness is proved. □

We can generalize our velocity-stress formulation for mixed boundary conditions in a straightforward way. Let $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \neq \emptyset$, and $\Gamma_D \cap \Gamma_N = \emptyset$. Suppose the boundary conditions of (3.2) are given as $v = g$ on Γ_D, $\sigma\nu = G$ on Γ_N for all time $t \in [0, T_0]$, and we call them mixed boundary conditions. We define $M_{\Gamma_N} = \{\tau \in M \,|\, \tau\nu = 0 \text{ on } \Gamma_N\}$. Then a velocity-stress formulation with weak symmetry is to seek (σ, v, r) satisfying (3.8) such that $\sigma\nu = G$ on Γ_N and

$$\begin{aligned}
(A\dot{\sigma}, \tau) + (\operatorname{div}\tau, v) + (\dot{r}, \tau) &= \int_{\Gamma_D} g \cdot \tau\nu \, ds, && \tau \in M_{\Gamma_N}, \\
(\rho\dot{v}, w) - (\operatorname{div}\sigma, w) &= (f, w), && w \in V, \\
(\dot{\sigma}, q) &= 0, && q \in K,
\end{aligned} \tag{3.13}$$

with initial data $(\sigma(0), v(0), r(0)) \in M \times H^1(\Omega; \mathbb{V}) \times K$ satisfying $\sigma(0)\nu = G(0)$ on Γ_N and $v(0) = g(0)$ on Γ_D.

3.3 Semidiscrete problems

In this section we consider spatial discretization of problem (3.8–3.11) with given initial data. We show existence and uniqueness of semidiscrete solutions and discuss the semidiscrete error analysis.

For the error analysis, we follow a standard approach: representatives of (σ, v, r) are used to split the semidiscrete errors into the projection errors and the approximation errors, and their bounds are achieved by the a priori error analysis.

3.3.1 Existence and uniqueness of semidiscrete solutions

Let $M_h \times V_h \times K_h$ be the AFW elements of degree $k \geq 1$. For $\tau \in M_h$, we use $\tau \perp K_h$ to denote $(\tau, q) = 0$ for any $q \in K_h$.

Definition 3.2. For initial data $(\sigma_h(0), v_h(0), r_h(0))$ in $M_h \times V_h \times K_h$, a semidiscrete solution of (3.9–3.11) is

$$\sigma_h \in C^1([0, T_0]; M_h), \quad v_h \in C^1([0, T_0]; V_h), \quad r_h \in C^1([0, T_0]; K_h),$$

satisfying the system of equations

$$(A\dot{\sigma}_h, \tau) + (\operatorname{div} \tau, v_h) + (\dot{r}_h, \tau) = 0, \qquad \tau \in M_h, \tag{3.14}$$

$$(\rho \dot{v}_h, w) - (\operatorname{div} \sigma_h, w) = (f, w), \qquad w \in V_h, \tag{3.15}$$

$$(\dot{\sigma}_h, q) = 0, \qquad q \in K_h, \tag{3.16}$$

for all time $t \in [0, T_0]$ with the given initial data.

Theorem 3.3. *For given $\sigma_h(0) \in M_h$, $v_h(0) \in V_h$, and $r_h(0) \in K_h$, the system* (3.14–3.16) *has a unique solution.*

Proof. Let $\{\phi_i\}$, $\{\psi_i\}$, $\{\chi_i\}$ be bases of M_h, V_h, and K_h, respectively. We use $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$, $\mathscr{D}$ to denote the matrices whose (i, j)-entries are

$$(A\phi_j, \phi_i), \quad (\operatorname{div} \phi_j, \psi_i), \quad (\phi_j, \chi_i), \quad (\rho\psi_j, \psi_i), \quad (\psi_j, \psi_i),$$

respectively. We write $\sigma_h = \sum_i \alpha_i \phi_i$, $v_h = \sum_i \beta_i \psi_i$, $r_h = \sum_i \gamma_i \chi_i$, $P_h f = \zeta_i \psi_i$, and use α, β, γ, ζ to denote the coefficient vectors. Then we may rewrite (3.14–3.16) in a matrix equation form,

$$\begin{pmatrix} \mathscr{A} & 0 & \mathscr{C}^T \\ 0 & \mathscr{M} & 0 \\ \mathscr{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} 0 & -\mathscr{B}^T & 0 \\ \mathscr{B} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ \mathscr{D}\zeta \\ 0 \end{pmatrix}.$$

The above matrix equation is a linear system of ordinary differential equations. Note that the coefficient matrix on the left-hand side is invertible because $\mathscr{A}$ and $\mathscr{M}$ are positive definite and $\mathscr{C}^T$ is injective from the inf-sup condition **(A2)**. By the standard ODE theory (see [25], p.75), the matrix equation is well-posed as an initial value problem, so the existence and uniqueness of solutions of (3.14–3.16) follow. □

3.3.2 Decomposition of semidiscrete errors

In this section, we state the main result of semidiscrete error analysis and explain a decomposition of the semidiscrete errors into the projection and the approximation errors. The proof of the main result will be given in the remainder of this section.

Theorem 3.4. *Let* (M_h, V_h, K_h) *be the* AFW *elements of degree* $k \geq 1$ *and let* m *be an integer such that* $1 \leq m \leq k$. *Suppose* $\sigma, v, r \in W^{1,1}([0, T_0]; H^m)$ *and let* (σ_h, v_h, r_h) *be the semidiscrete solution of* (3.14–3.16) *with initial data* $(\sigma_h(0), v_h(0), r_h(0))$ *such that*

$$\operatorname{div} \sigma_h(0) = P_h \operatorname{div} \sigma(0), \qquad v_h(0) = P_h v(0), \tag{3.17}$$

$$(A\sigma_h(0), \tau) + (r_h(0), \tau) = 0, \quad \text{for } \tau \in M_h,\ \operatorname{div} \tau = 0, \qquad \sigma_h(0) \perp K_h, \tag{3.18}$$

$$\|\sigma(0) - \sigma_h(0), r(0) - r_h(0)\| \leq ch^m \|\sigma(0), r(0)\|_m, \tag{3.19}$$

for $1 \leq m \leq k$. *Then we have*

$$\|\sigma - \sigma_h, v - v_h, r - r_h\|_{L^\infty L^2} \leq ch^m \|\sigma, v, r\|_{W^{1,1} H^m},$$

where c *depends on the compliance tensor* A, *and the lower and upper bounds of the mass density* ρ_0, ρ_1, *but is independent of* h.

We are able to find initial data satisfying (3.17–3.19) when $\sigma(0)$, $v(0)$ are given. Since there is $u(0) \in \mathring{H}^1(\Omega; \mathbb{V})$ such that $\sigma(0) = C\epsilon(u(0))$ and $r(0) = \operatorname{skw} \operatorname{grad} u(0)$, we have $(A\sigma(0), \tau) + (\operatorname{div} \tau, u(0)) + (r(0), \tau) = 0$ for $\tau \in M$. Now we consider a solution $(\sigma_h(0), u_h(0), r_h(0))$ of the system,

$$\begin{aligned} (A\sigma_h(0), \tau) + (\operatorname{div} \tau, u_h(0)) + (r_h(0), \tau) &= 0, && \tau \in M_h, \\ (\operatorname{div} \sigma_h(0), w) &= (\operatorname{div} \sigma(0), w), && w \in V_h, \\ (\sigma_h(0), q) &= 0, && q \in K_h. \end{aligned}$$

Then (3.18) is obvious from the first and third equations, and (3.19) follows from the estimate (2.33). The first condition in (3.17) is a consequence of (2.32), so all conditions in (3.17–3.19) are satisfied.

For our error analysis, we denote the semidiscrete errors, i.e., the difference of the exact solution (σ, v, r) and the semidiscrete solution (σ_h, v_h, r_h), by

$$e_\sigma = \sigma - \sigma_h, \quad e_v = v - v_h, \quad e_r = r - r_h.$$

Then, by taking differences of equations (3.9–3.11) and (3.14–3.16), we get

$$(A\dot{e}_\sigma, \tau) + (\operatorname{div} \tau, e_v) + (\dot{e}_r, \tau) = 0, \qquad \tau \in M_h, \tag{3.20}$$

$$(\rho \dot{e}_v, w) - (\operatorname{div} e_\sigma, w) = 0, \qquad w \in V_h, \tag{3.21}$$

$$(\dot{e}_\sigma, q) = 0, \qquad q \in K_h. \tag{3.22}$$

Recall that $\tilde{\Pi}_h$ is the weakly symmetric elliptic projection in Lemma 2.12 and P_h, P_h' are the orthogonal L^2 projections onto V_h and K_h, respectively. We decompose the semidiscrete errors (e_σ, e_v, e_r) into

$$\begin{aligned} e_\sigma &= e_\sigma^P + e_\sigma^h := (\sigma - \sigma_h^P) + (\sigma_h^P - \sigma_h), \\ e_v &= e_v^P + e_v^h := (v - v_h^P) + (v_h^P - v_h), \\ e_r &= e_r^P + e_r^h := (r - r_h^P) + (r_h^P - r_h), \end{aligned} \tag{3.23}$$

where $\sigma_h^P := \tilde{\Pi}_h \sigma$, $v_h^P := P_h v$, and $r_h^P := P_h' r$. We call the e^P terms the projection errors and the e^h terms the approximation errors, respectively. We remark that, by **(A1)** in section 2.4.2 and (2.58),

$$\begin{aligned} (\operatorname{div} \tau, e_v^P) &:= (\operatorname{div} \tau, v - P_h v) = 0, \qquad \tau \in M_h, \\ (\operatorname{div} e_\sigma^P, w) &:= (\operatorname{div} \sigma - \operatorname{div} \tilde{\Pi}_h \sigma, w) = 0, \qquad w \in V_h. \end{aligned} \tag{3.24}$$

By the triangle inequality, Theorem 3.4 will follow from a priori estimates of the projection errors and the approximation errors, respectively. We state and prove these estimates in sections 3.3.3 and 3.3.4.

3.3.3 Projection error estimates for the AFW elements

A priori estimates of the $L^\infty L^2$ norms of the projection errors follow from the approximation property of $M_h \times V_h \times K_h$.

Theorem 3.5. *There exists a constant $c > 0$ independent of h and $t \in [0, T_0]$ such that*

$$\|e_\sigma^P\|_{L^\infty L^2} \le ch^m \|\sigma\|_{L^\infty H^m}, \qquad 1 \le m \le k+1, \tag{3.25}$$

$$\|e_v^P\|_{L^\infty L^2} \le ch^m \|v\|_{L^\infty H^m}, \qquad 0 \le m \le k, \tag{3.26}$$

$$\|e_r^P\|_{L^\infty L^2} \le ch^m \|r\|_{L^\infty H^m}, \qquad 0 \le m \le k. \tag{3.27}$$

Furthermore, similar inequalities hold for higher order time derivatives of σ, v, *and r, respectively.*

Proof. For any $t \in [0, T_0]$ and $1 \leq m \leq k+1$, by (2.57) and **(A1)**, we have

$$\|e_\sigma^P(t)\| = \|\sigma(t) - \tilde{\Pi}_h \sigma(t)\| \leq ch^m \|\sigma(t)\|_m,$$

and (3.25) is proved. Similarly, from definitions of e_v^P and e_r^P, we have

$$\|e_v^P(t)\| \leq ch^m \|v(t)\|_m, \qquad \|e_r^P(t)\| \leq ch^m \|r(t)\|_m,$$

for any $t \in [0, T_0]$, $0 \leq m \leq k$. The results for higher order time derivatives are obtained by applying same argument to higher order time derivatives of the projection errors because the time derivative and the maps $\tilde{\Pi}_h$, P_h, $\tilde{P}_h$ are commutative. □

3.3.4 Approximation error estimates for the AFW elements

We now turn to a priori estimates of the $L^\infty L^2$ norms of the approximation errors.

Theorem 3.6. *For the approximation error $(e_\sigma^h, e_v^h, e_r^h)$ and $1 \leq m \leq k$,*

$$\|e_\sigma^h, e_v^h, e_r^h\|_{L^\infty L^2} \leq ch^m \|\sigma, v, r\|_{W^{1,1}H^m}, \tag{3.28}$$

where c depends on ρ_0, ρ_1, and A but is independent of h.

Proof. The proof depends on two estimates which we shall prove below: There exist constants c depending on ρ_0, ρ_1, and A but is independent of h so that,

$$\|e_\sigma^h, e_v^h\|_{L^\infty L^2} \leq ch^m (\|\sigma(0), r(0)\|_m + \|\dot{\sigma}, \dot{v}, \dot{r}\|_{L^1 H^m}), \tag{3.29}$$

$$\|e_r^h\|_{L^\infty L^2} \leq c\|e_\sigma^h, e_\sigma^P, e_r^P\|_{L^\infty L^2}, \tag{3.30}$$

for $1 \leq m \leq k$. Assuming these estimates are true one can prove Theorem 3.6 easily from Theorem 3.5 because $\|\sigma(0), r(0)\|_m \leq c\|\sigma, r\|_{W^{1,1}H^m}$ by the Sobolev embedding.

Now we prove the estimates (3.29–3.30). We first remark that $e_\sigma^h(0) \perp K_h$ because $\sigma_h(0) \perp K_h$ from (3.18) and $\sigma_h^P(0) \perp K_h$ by the definition of $\sigma_h^P(0)$. From this fact, and the fact that $(\dot{e}_\sigma^h, q) = (\dot{e}_\sigma^P, q) = 0$ for all $q \in K_h$, one can see that $e_\sigma^h \perp K_h$ by the fundamental theorem of calculus. To show (3.29), we rewrite (3.20–3.21), using the notations in (3.23) and the reductions in (3.24), as

$$
\begin{aligned}
(A\dot e^h_\sigma, \tau) + (\operatorname{div}\tau, e^h_v) + (\dot e^h_r, \tau) &= -(A\dot e^P_\sigma, \tau) - (\dot e^P_r, \tau), && \tau \in M_h,\\
(\rho \dot e^h_v, w) - (\operatorname{div} e^h_\sigma, w) &= -(\rho \dot e^P_v, w), && w \in V_h.
\end{aligned}
$$

If we take $\tau = e^h_\sigma$, $w = e^h_v$ in the above two equations, add them, and use the fact $e^h_\sigma \perp \dot e^h_r$ from $\dot e^h_r \in K_h$, then we have

$$
\frac{1}{2}\frac{d}{dt}\|e^h_\sigma\|^2_A + \frac{1}{2}\frac{d}{dt}\|e^h_v\|^2_\rho = -(A\dot e^P_\sigma, e^h_\sigma) - (\dot e^P_r, e^h_\sigma) - (\rho \dot e^P_v, e^h_v),
$$

where $\|e^h_\sigma\|^2_A = (Ae^h_\sigma, e^h_\sigma)$ and $\|e^h_v\|^2_\rho = (\rho e^h_v, e^h_v)$. If we use a weighted Cauchy–Schwarz inequality on the right-hand side of this equality and the fact that $\|\dot e^P_\sigma\|_A \le c\|\dot e^P_\sigma\|$, then we get

$$
\frac{1}{2}\frac{d}{dt}(\|e^h_\sigma\|^2_A + \|e^h_v\|^2_\rho) \le c\|\dot e^P_\sigma, \dot e^P_r, \dot e^P_v\| \, (\|e^h_\sigma\|^2_A + \|e^h_v\|^2_\rho)^{\frac{1}{2}}. \qquad (3.31)
$$

If we apply Corollary 2.15 to (3.31), regarding $(\|e^h_\sigma(t)\|^2_A + \|e^h_v(t)\|^2_\rho)^{1/2}$ as $Q(t)$, then we have

$$
\big(\|e^h_\sigma(t)\|^2_A + \|e^h_v(t)\|^2_\rho\big)^{\frac{1}{2}} \le \big(\|e^h_\sigma(0)\|^2_A + \|e^h_v(0)\|^2_\rho\big)^{\frac{1}{2}} + c\int_0^t \|\dot e^P_\sigma, \dot e^P_r, \dot e^P_v\| \, ds.
$$

By the coercivity of A and the lower bound $\rho_0 > 0$ of ρ, it suffices to show that the right-hand side is bounded by $ch^m(\|\sigma(0), r(0)\|_m + \|\dot\sigma, \dot v, \dot r\|_{L^1 H^m})$ for (3.29). For the integral term, we can simply use $\|\dot e^P_\sigma(t)\| \le ch^m\|\dot\sigma(t)\|_m$, $\|\dot e^P_v\| \le ch^m\|\dot v\|_m$, and $\|\dot e^P_r\| \le ch^m\|\dot r\|_m$, which were proved in Theorem 3.5. Note that $e^h_v(0) = 0$ from the choice of $v_h(0)$. For $\|e^h_\sigma(0)\|_A$, we use the boundedness of A and ρ, the triangle inequality, (3.19), (2.57), (2.33), and get

$$
\|e^h_\sigma(0)\|_A \le c(\|\sigma_h(0) - \sigma(0)\| + \|\sigma(0) - \tilde\Pi_h\sigma(0)\|) \le ch^m\|\sigma(0), r(0)\|_m,
$$

for $1 \le m \le k$, so (3.29) is proved.

In order to prove (3.30), using $u(t) = u(0) + \int_0^t v(s)\,ds$, note that $(A\sigma, \tau) + (u, \operatorname{div}\tau) + (r, \tau) = 0$ for $\tau \in M$. From this observation and (3.18), for $\tau \in M_h$ satisfying $\operatorname{div}\tau = 0$,

$$
(A\sigma, \tau) + (r, \tau) = 0, \qquad (A\sigma_h(0), \tau) + (r_h(0), \tau) = 0,
$$

so $(Ae_\sigma(0), \tau) + (e_r(0), \tau) = 0$. If we consider (3.20) for $\tau \in M_h$ satisfying $\operatorname{div} \tau = 0$, and the fundamental theorem of calculus, then we have

$$(Ae_\sigma(t), \tau) + (e_r(t), \tau) = 0, \quad \tau \in M_h, \ \operatorname{div} \tau = 0,$$

for all $t \in [0, T_0]$. This is equivalent to

$$(e_r^h(t), \tau) = -(A(e_\sigma^h(t) + e_\sigma^P(t)), \tau) + (e_r^P(t), \tau), \quad \tau \in M_h, \ \operatorname{div} \tau = 0.$$

By **(A2)** in section 2.4.2, we can choose $\tau \in M_h$ such that $\operatorname{div} \tau = 0$, $(\tau, q) = \|e_r^h(t)\|^2$, and $\|\tau\| \le c\|e_r^h(t)\|$ for each $t \in [0, T_0]$. If we take such τ in the above equation, then we get $\|e_r^h(t)\| \le c\|e_\sigma^h(t), e_\sigma^P(t), e_r^P(t)\|$ by the Cauchy–Schwarz inequality, so (3.30) follows. □

3.4 Full discretization

In this section, we use the Crank–Nicolson scheme for full discretization and prove a priori error estimates of fully discrete solutions for the AFW elements. We use $\Delta t > 0$ to denote the time step interval. For given $T_0 > 0$, we assume that Δt, N, and t_j for $0 \le j \le N$ are defined as in section 2.1.

The fully discrete solution of (σ, v, r) at time t_j will be denoted by (Σ^j, V^j, R^j). Let the initial data (Σ^0, V^0, R^0) be given. In the Crank–Nicolson scheme, using notations in (2.3), the sequence $(\Sigma^{j+1}, V^{j+1}, R^{j+1})$ for $1 \le j \le N$ is defined inductively by

$$\left(A\bar{\partial}_t \Sigma^{j+\frac{1}{2}}, \tau\right) + (\hat{V}^{j+\frac{1}{2}}, \operatorname{div} \tau) + (\bar{\partial}_t R^{j+\frac{1}{2}}, \tau) = 0, \tag{3.32}$$

$$\left(\rho\bar{\partial}_t V^{j+\frac{1}{2}}, w\right) - (\operatorname{div} \hat{\Sigma}^{j+\frac{1}{2}}, w) = (\hat{f}^{j+\frac{1}{2}}, w), \tag{3.33}$$

$$(\bar{\partial}_t \Sigma^{j+\frac{1}{2}}, q) = 0, \tag{3.34}$$

for $(\tau, w, q) \in M_h \times V_h \times K_h$.

Theorem 3.7. *Let $M_h \times V_h \times K_h$ be the* AFW *elements of degree $k \ge 1$. For given initial data $(\Sigma^0, V^0, R^0) \in M_h \times V_h \times K_h$, the sequence of fully discrete solutions (Σ^j, V^j, R^j) for $1 \le j \le N$ is well-defined by* (3.32–3.34). *Suppose*

$$\sigma, v, r \in W^{1,1}([0, T_0]; H^m) \cap W^{3,1}([0, T_0]; L^2), \tag{3.35}$$

and the initial data (Σ^0, V^0, R^0) satisfies the conditions (3.17–3.19). *Then the* *fully discrete solution* (Σ^j, V^j, R^j) *satisfies*

$$\|\sigma^j - \Sigma^j, v^j - V^j, r^j - R^j\| \le c(\Delta t^2 + h^m)\|\sigma, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}, \qquad (3.36)$$

for $c > 0$ *which depends on* A, ρ_0, ρ_1, *but is independent of* h *and* Δt.

The proof of this theorem will be given in the rest of this section.

3.4.1 Well-definedness

First of all, we show the fully discrete solution is well-defined. We need to check that $(\Sigma^{j+1}, V^{j+1}, R^{j+1})$ is uniquely determined as a solution of the linear system (3.32–3.34) when $\Sigma^j, V^j, R^j, f^j, f^{j+1}$ are given. Rewriting (3.32–3.34), we have

$$\begin{aligned}
\big(A\Sigma^{j+1}, \tau\big) + \frac{\Delta t}{2}(V^{j+1}, \operatorname{div}\tau) + (R^{j+1}, \tau)& \\
= (A\Sigma^j, \tau) - \frac{\Delta t}{2}&(V^j, \operatorname{div}\tau) + (R^j, \tau), \\
(\rho V^{j+1}, w) - \frac{\Delta t}{2}(\operatorname{div}\Sigma^{j+1}, w) = (\rho V^j, w) + \frac{\Delta t}{2}&(\operatorname{div}\Sigma^j, w) + \Delta t(\hat{f}^{j+\frac{1}{2}}, w), \\
(\Sigma^{j+1}, q) = (\Sigma^j, q),&
\end{aligned}$$

for $(\tau, w, q) \in M_h \times V_h \times K_h$. Note that the above is a system of linear equations with same number of equations and unknowns. In order to show that its solution $(\Sigma^{j+1}, V^{j+1}, R^{j+1})$ exists uniquely, it suffices to show that $\Sigma^{j+1} = V^{j+1} = R^{j+1} = 0$ if all right-hand sides vanish. Suppose the right-hand sides vanish. If we take $\tau = \Sigma^{j+1}$, $w = V^{j+1}$, $q = -R^{j+1}$ in the above equations and add them, then we have $(A\Sigma^{j+1}, \Sigma^{j+1}) + (\rho V^{j+1}, V^{j+1}) = 0$ which yields $\Sigma^{j+1} = V^{j+1} = 0$. By Corollary 2.4, there is $\tau \in M_h$ so that $(R^{j+1}, \tau) = (R^{j+1}, R^{j+1})$. If we take such τ in the first equation of the system, we obtain $R^{j+1} = 0$ since $\Sigma^{j+1} = V^{j+1} = 0$. Hence the full discretization is well-defined.

3.4.2 Convergence

We now turn to the proof of the a priori estimate (3.36). Let us denote the errors $(\sigma^j - \Sigma^j, v^j - V^j, r^j - R^j)$ by

$$E_\sigma^j := \sigma^j - \Sigma^j = (\sigma^j - \sigma_h^{P,j}) + (\sigma_h^{P,j} - \Sigma^j) =: e_\sigma^{P,j} + \theta_\sigma^j, \tag{3.37}$$

$$E_v^j := v^j - V^j = (v^j - v_h^{P,j}) + (v_h^{P,j} - V^j) =: e_v^{P,j} + \theta_v^j, \tag{3.38}$$

$$E_r^j := r^j - R^j = (r^j - r_h^{P,j}) + (r_h^{P,j} - R^j) =: e_r^{P,j} + \theta_r^j. \tag{3.39}$$

In Theorem 3.5 of semidiscrete error analysis, we already obtained error bounds of the projection errors $(e_\sigma^P, e_v^P, e_r^P)$. Thus we only need to consider a priori estimates of $(\theta_\sigma^j, \theta_v^j, \theta_r^j)$ for Theorem 3.7. Here is a precise statement of our claim.

Theorem 3.8. *Suppose the assumptions of Theorem 3.7 hold and θ_σ^i, θ_v^i, θ_r^i are defined as in* (3.37–3.39). *Then there exists a constant $c > 0$ such that*

$$\|\theta_\sigma^i, \theta_v^i, \theta_r^i\| \le c(\Delta t^2 + h^m)\|\sigma, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}, \tag{3.40}$$

for $1 \le i \le N$, $1 \le m \le k$, where the constant c depends on A, ρ_0, ρ_1 but is independent of h and Δt.

Proof. In order to show (3.40), consider the arithmetic mean of equations (3.9–3.10) at $t = t_j$ and $t = t_{j+1}$, which are

$$\begin{aligned} (A\hat{\dot\sigma}^{j+\frac{1}{2}}, \tau) + (\operatorname{div}\tau, \hat{\dot v}^{j+\frac{1}{2}}) + (\hat{\dot r}^{j+\frac{1}{2}}, \tau) &= 0, && \tau \in M_h, \\ (\rho\hat{\dot v}^{j+\frac{1}{2}}, w) - (\operatorname{div}\hat{\sigma}^{j+\frac{1}{2}}, w) &= (\hat{f}^{j+\frac{1}{2}}, w), && w \in V_h. \end{aligned}$$

We subtract (3.32–3.33) from the above two equations and consider the difference of equations. If we rewrite the difference of equations using (E_σ, E_v, E_r) defined in (3.37–3.39),

$$\begin{aligned} (A(\hat{\dot\sigma}^{j+\frac{1}{2}} - \bar\partial_t \Sigma^{j+\frac{1}{2}}), \tau) + (\operatorname{div}\tau, \hat{E}_v^{j+\frac{1}{2}}) + (\hat{\dot r}^{j+\frac{1}{2}} - \bar\partial_t R^{j+\frac{1}{2}}, \tau) &= 0, \\ (\rho\hat{\dot v}^{j+\frac{1}{2}} - \rho\bar\partial_t V^{j+\frac{1}{2}}, w) - (\operatorname{div}\hat{E}_\sigma^{j+\frac{1}{2}}, w) &= 0, \end{aligned}$$

for $(\tau, w) \in M_h \times V_h$. If we do some algebraic manipulations on those equations regarding equalities

$$\hat{\sigma}^{j+\frac{1}{2}} - \bar{\partial}_t \Sigma^{j+\frac{1}{2}} = \hat{\sigma}^{j+\frac{1}{2}} - \bar{\partial}_t \sigma^{j+\frac{1}{2}} + \bar{\partial}_t E_\sigma^{j+\frac{1}{2}},$$
$$\hat{r}^{j+\frac{1}{2}} - \bar{\partial}_t R^{j+\frac{1}{2}} = \hat{r}^{j+\frac{1}{2}} - \bar{\partial}_t r^{j+\frac{1}{2}} + \bar{\partial}_t E_r^{j+\frac{1}{2}},$$
$$\hat{v}^{j+\frac{1}{2}} - \bar{\partial}_t V^{j+\frac{1}{2}} = \hat{v}^{j+\frac{1}{2}} - \bar{\partial}_t v^{j+\frac{1}{2}} + \bar{\partial}_t E_v^{j+\frac{1}{2}},$$

then one can have

$$\begin{aligned}(A\bar{\partial}_t E_\sigma^{j+\frac{1}{2}}, \tau) + (\operatorname{div} \tau, \hat{E}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t E_r^{j+\frac{1}{2}}, \tau) \\ = (A(\bar{\partial}_t \sigma^{j+\frac{1}{2}} - \hat{\sigma}^{j+\frac{1}{2}}), \tau) + (\bar{\partial}_t r^{j+\frac{1}{2}} - \hat{r}^{j+\frac{1}{2}}, \tau),\end{aligned} \tag{3.41}$$

$$(\rho\bar{\partial}_t E_v^{j+\frac{1}{2}}, w) - (\operatorname{div} \hat{E}_\sigma^{j+\frac{1}{2}}, w) = (\rho\bar{\partial}_t v^{j+\frac{1}{2}} - \rho\hat{v}^{j+\frac{1}{2}}, w). \tag{3.42}$$

Considering the decomposition of errors in (3.37–3.39) and using the reductions from (3.24), we have

$$\begin{aligned}(A\bar{\partial}_t \theta_\sigma^{j+\frac{1}{2}}, \tau) + (\operatorname{div} \tau, \hat{\theta}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t \theta_r^{j+\frac{1}{2}}, \tau) \\ = (A(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \tau),\end{aligned} \tag{3.43}$$

$$(\rho\bar{\partial}_t \theta_v^{j+\frac{1}{2}}, w) - (\operatorname{div} \hat{\theta}_\sigma^{j+\frac{1}{2}}, w) = (\omega_5^{j+\frac{1}{2}} + \omega_6^{j+\frac{1}{2}}, w), \tag{3.44}$$

where

$$\begin{aligned}\omega_1^{j+\frac{1}{2}} = \bar{\partial}_t \sigma^{j+\frac{1}{2}} - \hat{\sigma}^{j+\frac{1}{2}}, \quad \omega_2^{j+\frac{1}{2}} = -\bar{\partial}_t e_\sigma^{P,j+\frac{1}{2}}, \quad \omega_3^{j+\frac{1}{2}} = \bar{\partial}_t r^{j+\frac{1}{2}} - \hat{r}^{j+\frac{1}{2}}, \\ \omega_4^{j+\frac{1}{2}} = -\bar{\partial}_t e_r^{P,j+\frac{1}{2}}, \quad \omega_5^{j+\frac{1}{2}} = \rho(\bar{\partial}_t v^{j+\frac{1}{2}} - \hat{v}^{j+\frac{1}{2}}), \quad \omega_6^{j+\frac{1}{2}} = -\rho\bar{\partial}_t e_v^{P,j+\frac{1}{2}}.\end{aligned} \tag{3.45}$$

Letting $\tau = \hat{\theta}_\sigma^{j+1/2}$, $w = \hat{\theta}_v^{j+1/2}$ in (3.43) and (3.44), and adding those equations,

$$\begin{aligned}&(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2) - (\|\theta_\sigma^j\|_A^2 + \|\theta_v^j\|_\rho^2) \\ &= 2\Delta t(A(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \hat{\theta}_\sigma^{j+\frac{1}{2}}) + 2\Delta t(\omega_5^{j+\frac{1}{2}} + \omega_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).\end{aligned} \tag{3.46}$$

If we divide both sides by $(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{1/2} + (\|\theta_\sigma^j\|_A^2 + \|\theta_v^j\|_\rho^2)^{1/2}$ and apply a weighted Cauchy–Schwarz inequality, then we get

$$(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^j\|_A^2 + \|\theta_v^j\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\|, \tag{3.47}$$

for each $0 \le j \le N-1$, with c independent of h, Δt, and j. By using (3.47) inductively, we have

$$(\|\theta_\sigma^i\|_A^2 + \|\theta_v^i\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^0\|_A^2 + \|\theta_v^0\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{j=0}^{i-1} \sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\|. \tag{3.48}$$

Since A is coercive and $0 < \rho_0 \le \rho$ for a constant ρ_0, there exists $c > 0$ depending only on A and ρ_0 such that

$$\|\theta_\sigma^i\| + \|\theta_v^i\| \le c(\|\theta_\sigma^i\|_A^2 + \|\theta_v^i\|_\rho^2)^{\frac{1}{2}}. \tag{3.49}$$

We can see that $(\|\theta_\sigma^0\|_A^2 + \|\theta_v^0\|_\rho^2)^{\frac{1}{2}} = \|\theta_\sigma^0\|_A \le ch^m \|\sigma(0), r(0)\|_m$ holds by an argument similar to that of semidiscrete error analysis from the fact $\theta_v^0 = 0$, the assumption (3.19) of (Σ^0, V^0, R^0), and the triangle inequality. Hence, if we show

$$c\Delta t \sum_{j=0}^{i-1} \sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \quad 1 \le m \le k, \tag{3.50}$$

then the estimate (3.40) for $\|\theta_\sigma^i\| + \|\theta_v^i\|$ is proved by (3.48) and (3.49).

In order to show the above inequality, we recall Taylor expansions for $g \in C^4([-a, a])$,

$$|g(a) - g(-a) - 2ag'(0)| \le ca^2 \|g'''\|_{L^1(-a,a)}, \tag{3.51}$$

$$|g(a) + g(-a) - 2g(0)| \le ca\|g''\|_{L^1(-a,a)}, \tag{3.52}$$

$$|g(a) - g(-a) - a(g'(a) + g'(-a))| \le ca^2 \|g'''\|_{L^1(-a,a)}, \tag{3.53}$$

$$|2g(a) - 4g(0) + 2g(-a) - a(g'(a) - g'(-a))| \le ca^3 \|g^{(4)}\|_{L^1(-a,a)}, \tag{3.54}$$

where $g^{(4)}$ denotes the fourth derivative of g.

From the definitions of $\omega_l^{j+1/2}$, $l = 1, 3, 5$ in (3.45), we can use (3.53) by substituting a by $\Delta t/2$, 0 by t_j, and g by σ, v, r. Then we have

$$\Delta t\|\omega_1^{j+\frac{1}{2}}\| = \frac{1}{2}\|2\sigma^{j+1} - 2\sigma^j - \Delta t\dot{\sigma}^{j+1} - \Delta t\dot{\sigma}^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{\sigma}\|\, ds, \quad (3.55)$$

$$\Delta t\|\omega_3^{j+\frac{1}{2}}\| = \frac{1}{2}\|2r^{j+1} - 2r^j - \Delta t\dot{r}^{j+1} - \Delta t\dot{r}^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{r}\|\, ds, \quad (3.56)$$

$$\Delta t\|\omega_5^{j+\frac{1}{2}}\| = \frac{1}{2}\|\rho(2v^{j+1} - 2v^j - \Delta t\dot{v}^{j+1} - \Delta t\dot{v}^j)\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{v}\|\, ds. \quad (3.57)$$

By the definitions of $\omega_l^{j+1/2}$, $l = 2, 4, 6$ in (3.45), and Theorem 3.5, one can see

$$\Delta t\|\omega_2^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_\sigma^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot{e}_\sigma^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{\sigma}\|_m\, ds, \quad (3.58)$$

$$\Delta t\|\omega_4^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_r^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot{e}_r^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{r}\|_m\, ds, \quad (3.59)$$

$$\Delta t\|\omega_6^{j+\frac{1}{2}}\| = \Delta t\|\rho\bar{\partial}_t e_v^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \rho\dot{e}_v^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m\, ds, \quad (3.60)$$

for $1 \le m \le k$. The estimate (3.50) is obtained by combining (3.55–3.60) and as we remarked before, the estimate (3.40) is proved for $\|\theta_\sigma^i, \theta_v^i\|$.

To complete the proof of (3.40), we estimate $\|\theta_r^i\|$ following the outline of semidiscrete analysis. We first note that $(A\sigma^j, \tau) + (r^j, \tau) = 0$ for $0 \le j \le N$, for all $\tau \in M_h$ such that $\operatorname{div}\tau = 0$. From the condition (3.18) of initial data, we have $(A\Sigma^0, \tau) + (R^0, \tau) = 0$ and combining this equality with (3.32), we get

$$(A\Sigma^j, \tau) + (R^j, \tau) = 0, \quad \tau \in M_h,\ \operatorname{div}\tau = 0, \quad 0 \le j \le N.$$

Therefore we have an error equation $(AE_\sigma^j, \tau) + (E_r^j, \tau) = 0$ for $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, which is equivalent to

$$(\theta_r^j, \tau) = -(A(\theta_\sigma^j + e_\sigma^{P,j}), \tau) - (e_r^{P,j}, \tau), \quad \tau \in M_h,\ \operatorname{div}\tau = 0, \quad 0 \le j \le N.$$

If we take $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(\tau, q) = \|\theta_r^j\|^2$, and $\|\tau\| \le c\|\theta_r^j\|$ by **(A2)**, then we obtain

$$\|\theta_r^j\| \le c\|\theta_\sigma^j, e_\sigma^{P,j}, e_r^{P,j}\| \le c(\|\theta_\sigma^j\| + h^m\|\sigma, r\|_{L^\infty H^m}).$$

Combining this with the estimate of $\|\theta_\sigma^j\|$, then the estimate (3.40) for $\|\theta_r^j\|$ is proved. $\square$

In some applications, we need to compute numerical solutions of the displacement. Let $U^0 \in V_h$ be an approximation of initial displacement $u(0)$. Then we can compute a numerical displacement U^i from $\{V^j\}$, using the trapezoidal rule, as

$$U^i = U^0 + \Delta t \sum_{j=1}^{i} \frac{V^j + V^{j-1}}{2} = U^0 + \Delta t \sum_{j=1}^{i} \hat{V}^{j+\frac{1}{2}}. \tag{3.61}$$

Corollary 3.9 (Numerical solutions of displacement)**.** *Let* $U^0 \in V_h$ *be an approximation of initial displacement* $u(0)$ *with* $\|u(0) - U^0\| \le ch^m\|u(0)\|_m$, $1 \le m \le k$, *and* U^i *be defined by* (3.61). *Then, for* $0 \le i \le N$, $1 \le m \le k$,

$$\|u^i - U^i\| \le c(h^m + \Delta t^2)(\|u(0)\|_m + \|\sigma, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}).$$

Proof. Noting that $u^i = u^0 + \int_0^{t_i} v\, ds$ and using the triangle inequality,

$$\|u^i - U^i\| \le \|u(0) - U^0\| + \sum_{j=1}^{i} \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{V}^{j+\frac{1}{2}} \right\|. \tag{3.62}$$

Recall an estimate in the trapezoidal rule, which is

$$\left| \int_{-a}^{a} g(s)\, ds - a(g(a) + g(-a)) \right| \le ca^3 \|g''\|_{L^\infty(-a,a)}.$$

Combining the triangle inequality, the above estimate, and $\|v^j - V^j\| \le c(h^m + \Delta t^2)$ from Theorem 3.7 yields

$$\begin{aligned} \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{V}^{j+\frac{1}{2}} \right\| &\le \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{v}^{j+\frac{1}{2}} \right\| + \Delta t \|\hat{v}^{j+\frac{1}{2}} - \hat{V}^{j+\frac{1}{2}}\| \\ &\le c\Delta t^3 \|\ddot{v}\|_{L^\infty L^2} + c(h^m + \Delta t^2)\Delta t. \end{aligned} \tag{3.63}$$

The conclusion follows from the assumption of U^0, (3.62), and (3.63). $\square$

3.5 Error analysis for the GG elements

In this section we discuss the error analysis for the GG elements. We shall show that we can obtain better error bounds of σ and r for the GG elements than for the AFW elements when the degree k of elements are same. We also show that a postprocessing is eligible for the GG elements to obtain a better numerical solution of u.

3.5.1 A priori error estimates

Since many steps of the error analysis for the GG elements are similar to the one for the AFW elements, we do not repeat all details but focus on the steps that should be modified. Before we state a main result, we define $\|\rho\|_{W_h^{1,\infty}}$ as

$$\|\rho\|_{W_h^{1,\infty}} = \|\rho\|_{L^\infty} + \|\operatorname{grad}_h \rho\|_{L^\infty}, \tag{3.64}$$

where grad_h is the piecewise gradient operator adapted to the triangulation $\mathcal{T}_h$.

Theorem 3.10. *Let (M_h, V_h, K_h) be the* GG *elements of degree $k \geq 1$, m be an integer for which $1 \leq m \leq k+1$, and $m' = m - \delta_{m,k+1}$ where $\delta_{m,k+1}$ is the Kronecker delta. Suppose $\|\rho\|_{W_h^{1,\infty}} < \infty$ and*

$$\begin{aligned} \sigma, r &\in W^{1,1}([0,T_0]; H^m) \cap W^{3,1}([0,T_0]; L^2), \\ v &\in W^{1,1}([0,T_0]; H^{m'}) \cap W^{3,1}([0,T_0]; L^2). \end{aligned} \tag{3.65}$$

Assume the initial data satisfy (3.17–3.19) *for $1 \leq m \leq k+1$. Then the fully discrete solution (Σ^j, V^j, R^j) in* (3.32–3.34) *is well-defined and for all $0 \leq j \leq N$,*

$$\begin{aligned} \|\sigma^j - \Sigma^j\| + \|P_h v^j - V^j\| + \|r^j - R^j\| &\leq c(\Delta t^2 + h^m), \\ \|v^j - V^j\| &\leq c(\Delta t^2 + h^{m'}), \end{aligned} \tag{3.66}$$

where c depends on the quantities $\|\sigma, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}$, $\|v\|_{W^{1,1}H^{m'} \cap W^{3,1}L^2}$, A, ρ_0, and $\|\rho\|_{W_h^{1,\infty}}$ but is independent of h and Δt.

In comparison with Theorem 3.7, if k is same, the above theorem gives better accuracy of the errors of σ and r. In particular, we will show later that we can use the estimate of $\|P_h v^j - V^j\|$ for a postprocessing for the numerical solution of u.

In the previous section, the proof of Theorem 3.7 consists of the proof of well-definedness of (Σ^j, V^j, R^j) and the proof of error estimates. Since we can prove that (Σ^j, V^j, R^j) is well-defined with the same argument shown in section 3.4.1, we will only discuss the a priori error estimates in the proof of Theorem 3.10.

For the error estimates, we decompose the errors $(E_\sigma^j, E_v^j, E_r^j)$ into the projection errors $(e_\sigma^{P,j}, e_v^{P,j}, e_r^{P,j})$ and the approximation errors $(\theta_\sigma^j, \theta_v^j, \theta_r^j)$ as in (3.37–3.39). The error estimates of the projection errors are obtained from the approximation property of the GG elements.

Theorem 3.11. *Let (M_h, V_h, K_h) be the* GG *elements of degree $k \geq 1$. There exists a constant $c > 0$ independent of h so that the following inequalities hold.*

$$\|e_\sigma^P\|_{L^\infty L^2} \leq ch^m \|\sigma\|_{L^\infty H^m}, \qquad 1 \leq m \leq k+1, \tag{3.67}$$

$$\|e_v^P\|_{L^\infty L^2} \leq ch^m \|v\|_{L^\infty H^m}, \qquad 0 \leq m \leq k, \tag{3.68}$$

$$\|e_r^P\|_{L^\infty L^2} \leq ch^m \|r\|_{L^\infty H^m}, \qquad 0 \leq m \leq k+1. \tag{3.69}$$

Furthermore, similar inequalities hold for time derivatives of σ, v, r, respectively, as in Theorem 3.5.

The proof of Theorem 3.11 is similar to that of Theorem 3.5 with **(A1)**. Note that a better approximation (3.69) in K_h is obtained because the shape functions of K_h for the GG elements of degree k are one degree higher than the ones for the AFW elements of same degree k.

Now we state the a priori estimates of approximation errors $(\theta_\sigma^j, \theta_v^j, \theta_r^j)$.

Theorem 3.12. *Suppose the assumptions in Theorem 3.10 hold and θ_σ^i, θ_v^i, θ_r^i are defined as in* (3.37–3.39). *Then there exists a constant $c > 0$ such that*

$$\|\theta_\sigma^i, \theta_v^i, \theta_r^i\| \\ \leq c(\Delta t^2 + h^m)(\|\sigma, r\|_{W^{1,1}H^m \cap W^{3,1}L^2} + \|v\|_{W^{1,1}H^{m'} \cap W^{3,1}L^2}), \tag{3.70}$$

for $1 \leq i \leq N$, $1 \leq m \leq k+1$, where c depends on A, ρ_0, $\|\rho\|_{W_h^{1,\infty}}$ but is independent of h and Δt.

Proof. For the a priori estimates, we can follow same argument in the proof of Theorem 3.7 and obtain (3.46). Let ρ_c be the orthogonal L^2 projection of ρ into the space of piecewise constant functions associated to the triangulation $\mathcal{T}_h$. Define $\tilde{\omega}_6^{j+1/2} = (\rho - \rho_c)\bar{\partial}_t e_v^{P,j+1/2}$ and note that $\rho_c \bar{\partial}_t e_v^{P,j+1/2} \perp V_h$ because of

$e_v^{P,j}, e_v^{P,j+1} \perp V_h$ and the facts that ρ_c is a piecewise constant function and V_h is a space of piecewise polynomials without any interelement continuity. From the orthogonality $\rho_c \bar{\partial}_t e_v^{P,j+1/2} \perp V_h$, one can see that

$$(\omega_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\rho \bar{\partial}_t e_v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = ((\rho - \rho_c)\bar{\partial}_t e_v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\tilde{\omega}_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).$$

Therefore, by replacing $\omega_6^{j+1/2}$ by $\tilde{\omega}_6^{j+1/2}$ and repeating the steps in (3.46–3.48), we have an inequality analogous to (3.48),

$$(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^0\|_A^2 + \|\theta_v^0\|_\rho^2)^{\frac{1}{2}} + c\Delta t \left(\sum_{l=1}^{5} \|\omega_l^{j+\frac{1}{2}}\| + \|\tilde{\omega}_6^{j+\frac{1}{2}}\| \right).$$

The first term can be bounded by $ch^m \|\sigma(0), r(0)\|_m$ by (3.19) as we have seen before. To prove (3.70) for $\|\theta_\sigma^i\| + \|\theta_v^i\|$, we only need to show

$$c\Delta t \sum_{j=0}^{i-1} \left(\sum_{l=1}^{5} \|\omega_l^{j+\frac{1}{2}}\| + \|\tilde{\omega}_6^{j+\frac{1}{2}}\| \right) \le c(\Delta t^2 + h^m), \quad 1 \le m \le k+1.$$

For the estimates of $\omega_l^{j+1/2}$, $l = 1, 3, 5$, we use (3.55–3.57). For the estimates of $\omega_2^{j+1/2}$ and $\omega_4^{j+1/2}$, we use (3.58) and (3.59), respectively and note that the integer m ranges $1 \le m \le k+1$ for the GG elements. For $\tilde{\omega}_6^{j+1/2}$, we use

$$\begin{aligned} \Delta t \|\tilde{\omega}_6^{j+\frac{1}{2}}\| = \Delta t \|(\rho - \rho_c)\bar{\partial}_t v^{P,j}\| &= \left\| \int_{t_j}^{t_{j+1}} (\rho - \rho_c)(\dot{v}_h^P - \dot{v})\, ds \right\| \\ &\le ch^{m+1} \|\rho\|_{W_h^{1,\infty}} \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m \, ds, \quad 1 \le m \le k, \end{aligned}$$

and the proof of (3.70) for $\|\theta_\sigma^i\| + \|\theta_v^i\|$ is completed.

For the proof of (3.70) we omit details because the proof is same as that of $\|\theta_r^i\|$ estimate in Theorem 3.7. □

The proof of Theorem 3.10 follows from the above theorem and Theorem 3.11. Note that $P_h v^j - V^j = \theta_v^j$ and the $\|P_h v^j - V^j\|$ estimate in (3.66) is a consequence of (3.70). The other estimates in (3.66) are easily obtained from the estimates (3.67–3.69), (3.70), and the triangle inequality.

3.5.2 Postprocessing

We show that a postprocessing is eligible for the GG elements. For simplicity, we only consider the numerical solution of displacement u defined in Corollary 3.9. We shall show that a better numerical solution of u is obtained by a relatively simple computation.

For V_h in the GG elements, let V_h^* be the space of (possibly discontinuous) piecewise polynomials adapted to $\mathcal{T}_h$ of one degree higher than V_h, and $\tilde{V}_h$ be the orthogonal complement of V_h in V_h^*. The orthogonal L^2 projections onto V_h^* and $\tilde{V}_h$ are denoted by P_h^* and $\tilde{P}_h$, respectively. We claim the following postprocessing result.

Theorem 3.13. *Suppose that the assumptions of Theorem 3.10 holds with $m = k+1$ and (Σ^i, V^i, R^i) is the fully discrete solution. Suppose also that $\|v\|_{W^{1,1}H^{k+1}}$ is finite. Let U^i be the numerical solution of displacement in Corollary 3.9 with the assumption $\|P_h u(0) - U^0\| \le ch^{k+1}$. We define $U^{*,i} \in V_h^*$ by*

$$(\operatorname{grad}_h U^{*,i}, \operatorname{grad}_h w) = (A\Sigma^i + R^i, \operatorname{grad}_h w), \qquad w \in \tilde{V}_h, \tag{3.71}$$

$$(U^{*,i}, w) = (U^i, w), \qquad w \in V_h, \tag{3.72}$$

for each $0 \le i \le N$ where grad_h is the piecewise gradient operator adapted to the triangulation $\mathcal{T}_h$. Then $U^{*,i}$ is well-defined and there exists $c > 0$ independent of h and Δt such that

$$\|u^i - U^{*,i}\| \le c(\Delta t^2 + h^{k+1}) \|\sigma, v, r\|_{W^{1,1}H^{k+1} \cap W^{3,1}L^2}, \tag{3.73}$$

where c depends on A, ρ_0, $\|\rho\|_{W_h^{1,\infty}}$.

Proof. We can show that $U^{*,i}$ is well-defined with the same argument we used in (2.48), so we omit details. Note that $P_h U^{*,i} = U^i$ and let $\tilde{U}^i := U^{*,i} - U^i \in \tilde{V}_h$.

To prove (3.73), by the triangle inequality and the Bramble–Hilbert lemma, it suffices to show $\|P_h^* u^i - U^{*,i}\| \le c(\Delta t^2 + h^{k+1})$. To show this reduced estimate, consider

$$P_h^* u^i - U^{*,i} = (P_h u^i - U^i) + (\tilde{P}_h u^i - \tilde{U}^i) \in V_h \oplus \tilde{V}_h.$$

We first show that

$$\|P_h u^i - U^i\| \le c(\Delta t^2 + h^{k+1}) \|\sigma, v, r\|_{W^{1,1}H^{k+1} \cap W^{3,1}L^2}. \tag{3.74}$$

From the definition of U^i,

$$\begin{aligned}\|P_h u^i - U^i\| &= \left\| P_h u(0) - U^0 + \int_0^{t_i} P_h v\, ds - \Delta t \sum_{j=0}^{i-1} \hat{V}^{j+\frac{1}{2}} \right\| \\ &\le \|P_h u(0) - U^0\| + \sum_{j=0}^{i-1} \left\| \int_{t_j}^{t_{j+1}} P_h v\, ds - \Delta t P_h \hat{v}^{j+\frac{1}{2}} \right\| \\ &\quad + \Delta t \sum_{j=0}^{i-1} \|P_h \hat{v}^{j+\frac{1}{2}} - \hat{V}^{j+\frac{1}{2}}\| \\ &\le ch^{k+1} + \Delta t^2 \|P_h \ddot{v}\|_{L^\infty L^2} + c(\Delta t^2 + h^{k+1}),\end{aligned}$$

where the last inequality is due to the assumption of U^0, the error bound of trapezoidal rule, and the $\|P_h v^j - V^j\|$ estimate in (3.66). Thus the desired estimate (3.74) is proved.

Now we turn to the estimation of $\|\tilde{P}_h u^i - \tilde{U}^i\|$. Since $\operatorname{grad} u^i = A\sigma^i + r^i$, we have $(\operatorname{grad}_h u^i, \operatorname{grad}_h w) = (A\sigma^i + r^i, \operatorname{grad}_h w)$ for $w \in \tilde{V}_h$. By subtracting the equation (3.71) from this equation, we get, for $w \in \tilde{V}_h$,

$$(\operatorname{grad}_h(u^i - U^{*,i}), \operatorname{grad}_h w) = (A(\sigma^i - \Sigma^i) + r^i - R^i, \operatorname{grad}_h w). \tag{3.75}$$

Regarding equalities,

$$\begin{aligned}u^i - U^{*,i} &= (u^i - P_h^* u^i) + (P_h^* u^i - U^{*,i}) \\ &= (u^i - P_h^* u^i) + (\tilde{P}_h u^i - \tilde{U}^i) + (P_h u^i - U^i),\end{aligned}$$

and by replacing $u^i - U^{*,i}$ in (3.75), a direct computation gives

$$\begin{aligned}&(\operatorname{grad}_h(\tilde{P}_h u^i - \tilde{U}^i), \operatorname{grad}_h w) \\ &= -(\operatorname{grad}_h(u^i - P_h^* u^i), \operatorname{grad}_h w) - (\operatorname{grad}_h(P_h u^i - U^i), \operatorname{grad}_h w) \\ &\quad + (A(\sigma^i - \Sigma^i) + r^i - R^i, \operatorname{grad}_h w).\end{aligned}$$

Taking $w = \tilde{P}_h u^i - \tilde{U}^i$ in this equation, we have

$$\begin{aligned}&\|\operatorname{grad}_h(\tilde{P}_h u^i - \tilde{U}^i)\| \\ &\qquad \le c(\|\operatorname{grad}_h(u^i - P_h^* u^i)\| + \|\operatorname{grad}_h(P_h u^i - U^i)\| \\ &\qquad\qquad + \|A(\sigma^i - \Sigma^i) + r^i - R^i\|).\end{aligned} \tag{3.76}$$

By applying (2.53) to $\tilde{P}_h u^i - \tilde{U}^i$, we have $\|\tilde{P}_h u^i - \tilde{U}^i\| \le ch\| \operatorname{grad}_h(\tilde{P}_h u^i - \tilde{U}^i)\|$. From this inequality, by using the inequality that we multiply h to (3.76), we get

$$\begin{aligned}\|\tilde{P}_h u^i - \tilde{U}^i\| &\le ch(\| \operatorname{grad}_h(u^i - P_h^* u^i)\| + \| \operatorname{grad}_h(P_h u^i - U^i)\| \\ &\qquad + \|A(\sigma^i - \Sigma^i) + r^i - R^i\|) \\ &\le ch\| \operatorname{grad}_h(u^i - P_h^* u^i)\| + c\|P_h u^i - U^i\| \\ &\qquad + ch\|A(\sigma^i - \Sigma^i) + r^i - R^i\|,\end{aligned}$$

where the second term on the right-hand side of last inequality is due to (2.52). Now we only need to prove that the three quantities on the right-hand side of the above inequality are bounded by $c(\Delta t^2 + h^{k+1})$.

For the first term, one can see $h\| \operatorname{grad}_h(u^i - P_h^* u^i)\| \le ch^{k+1}\|u^i\|_{k+1}$, by the Bramble–Hilbert lemma. For the second, we use (3.74). For the third, we use the triangle inequality and (3.66). The proof is completed. □

Remark 3.14. This postprocessing does not need postprocessing of all the previous time steps.

3.6 Robustness for nearly incompressible materials

In this section, we assume that an elastic medium is homogeneous isotropic, i.e., the compliance tensor A has the form (2.7) with the Lamé coefficients μ and λ which are constants. We also assume that the boundary conditions are the homogeneous displacement boundary conditions. In nearly incompressible elastic materials, λ is very large, and, in the incompressible limit, $\lambda = +\infty$. Many standard discretization of elasticity suffer from locking, which means that the errors, while they decay with the mesh size, grow as λ increases. A robust or locking-free method is one in which the error estimates hold uniformly as $\lambda \to +\infty$. As we have seen in Theorem 2.5, stable mixed methods for stationary elasticity problems are typically locking-free (see [8, 19]). In this section, we will show that the locking-free advantage of mixed methods still holds in our numerical schemes for linear elastodynamics.

Theorem 3.15. *Let $M_h \times V_h \times K_h$ be the* AFW *elements of degree $k \ge 1$ and assume that A have the form of* (2.7) *with constants μ and λ. We assume that* *the exact solution (σ, v, r) is sufficiently regular for simplicity. Then there exists a $c > 0$ which is uniformly bounded as $\lambda \to +\infty$ such that for $0 \le i \le N-1$,*

$$\|\hat{\sigma}^{i-\frac{1}{2}} - \hat{\Sigma}^{i-\frac{1}{2}}\| + \|v^i - V^i\| \le c(\Delta t^2 + h^k)\|\sigma, v, r\|_{W^{2,1}H^k \cap W^{4,1}L^2}.$$

Proof. Since the projection error estimates do not depend on the form of A, we only need to consider a priori estimates of the approximation errors $\theta_\sigma^{i+1/2}$, θ_v^i, and θ_r^i.

In the proof of Theorem 3.7, the dependence of c in (3.36) on λ arises only from (3.49) when coercivity of A is used. To avoid this dependence of the constant c on λ, we show that $\|\theta_\sigma^{j+1/2}\| \le c(\Delta t^2 + h^m)$ holds without using the coercivity of A on $L^2(\Omega; \mathbb{M})$.

We first show that, for $1 \le i \le N$,

$$\|\theta_\sigma^i\|_A + \|\theta_v^i\| \le c(\Delta t^2 + h^k)\|\sigma, v, r\|_{W^{2,1}H^k \cap W^{4,1}L^2}, \tag{3.77}$$

holds with c which is uniformly bounded as $\lambda \to +\infty$. To prove it, we follow the proof of Theorem 3.8 and obtain (3.46) with no change. Note that if τ is skew-symmetric, then, from (2.7), we have

$$A\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda}\operatorname{tr}(\tau)I\right) = \frac{1}{2\mu}\tau.$$

Using this and the facts that $\omega_3^{j+1/2}$, $\omega_4^{j+1/2}$ in (3.45) are skew-symmetric, we can rewrite (3.46) as

$$\begin{aligned}
(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2) - (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2) \\
= 2\Delta t(A(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}} + 2\mu\omega_3^{j+\frac{1}{2}} + 2\mu\omega_4^{j+\frac{1}{2}}), \hat{\theta}_\sigma^{j+\frac{1}{2}}) \\
+ 2\Delta t(\omega_5^{j+\frac{1}{2}} + \omega_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).
\end{aligned} \tag{3.78}$$

If we use the fact that A is symmetric positive definite, then we have a weighted Cauchy–Schwarz inequality, $(A\sigma, \tau) = (A^{\frac{1}{2}}\sigma, A^{\frac{1}{2}}\tau) \le \|\sigma\|_A\|\tau\|_A$. We use this and the Cauchy–Schwarz inequality to (3.78), and divide both sides by $(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{1/2} + (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2)^{1/2}$. Then we can see

$$(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2)^{\frac{1}{2}} + c\Delta t\sum_{l=1}^{6}\|\omega_l^{j+\frac{1}{2}}\|.$$

By applying the estimates (3.55–3.56), (3.58–3.59), we obtain (3.77).

As a consequence of (3.77), it is easy to see that the error estimate for the error of v holds with a constant c which is uniformly bounded as $\lambda \to +\infty$. Now we consider the estimate for the error of σ.

From the form of A in (2.7), we have

$$(A\tau, I) = \frac{1}{2\mu + n\lambda} \int_\Omega \operatorname{tr}(\tau)\, dx. \tag{3.79}$$

We first note that $(A\dot{\sigma}, I) = 0$ by taking $\tau = I$ in (3.9). Moreover, $(A\sigma, I) = 0$ because $(A\sigma, I) = (\epsilon(u), I) = (\operatorname{grad} u, I) = (u, \operatorname{div} I) = 0$. By using these facts, we have $(A\bar{\partial}_t E_\sigma^{j+1/2}, I) = 0$ if we take $\tau = I$ in (3.41) because $\operatorname{div} I = 0$ and r, $\dot{r}$ are skew-symmetric. From the assumption (3.18), $(AE_\sigma^0, I) = 0$, whence $(AE_\sigma^j, I) = 0$ for all $0 \le j \le N$. Furthermore, $(Ae_\sigma^{P,j}, I) = (e_\sigma^{P,j}, AI) = 0$ from the definitions of e_σ^P and $\tilde{\Pi}_h$ in (2.57), and the facts that $\operatorname{div} AI = 0$, $AI \perp K_h$. Thus we have $(A\theta_\sigma^j, I) = 0$ for $0 \le j \le N$.

If we recall $AI = (2\mu + n\lambda)^{-1} I$, then $(\theta_\sigma^j, AI) = 0$ implies $\int_\Omega \operatorname{tr}(\theta_\sigma^j)\, dx = 0$ for $\lambda < +\infty$ and (2.43) can be applied to θ_σ^j. By Lemma 2.6 and Lemma 2.7,

$$\|\hat{\theta}_\sigma^{j+\frac{1}{2}}\|^2 \le c(\|\hat{\theta}_\sigma^{j+\frac{1}{2}}\|_A^2 + \|\operatorname{div} \hat{\theta}_\sigma^{j+\frac{1}{2}}\|_{-1}^2). \tag{3.80}$$

Note that $\|\hat{\theta}_\sigma^{j+\frac{1}{2}}\|_A^2$ can be estimated by (3.77) with a uniform constant c not growing as $\lambda \to +\infty$. To handle $\|\operatorname{div} \hat{\theta}_\sigma^{j+1/2}\|_{-1}^2$, we take $w = \operatorname{div} \hat{\theta}_\sigma^{j+1/2}$ in (3.44) and get $\|\operatorname{div} \hat{\theta}_\sigma^{j+1/2}\| \le c(\|\bar{\partial}_t \theta_v^{j+1/2}\| + \|\omega_5^{j+1/2}\| + \|\omega_6^{j+1/2}\|)$. For the estimate of $\|\bar{\partial}_t \theta_v^{j+1/2}\|$, we show a stronger estimate,

$$(\|\bar{\partial}_t \theta_\sigma^{j+\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t \theta_v^{j+\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \le c(\Delta t^2 + h^m), \quad 1 \le m \le k. \tag{3.81}$$

In order to show it, consider the differences of (3.43–3.44) with indices j and $j-1$, which are

$$\begin{aligned}(A(\bar{\partial}_t \theta_\sigma^{j+\frac{1}{2}} - \bar{\partial}_t \theta_\sigma^{j-\frac{1}{2}}), \tau) + (\operatorname{div} \tau, \hat{\theta}_v^{j+\frac{1}{2}} - \hat{\theta}_v^{j-\frac{1}{2}}) + (\bar{\partial}_t \theta_r^{j+\frac{1}{2}} - \bar{\partial}_t \theta_r^{j-\frac{1}{2}}, \tau)& \\ = (A(\eta_1^j + \eta_2^j) + \eta_3^j + \eta_4^j, \tau)&, \\ (\rho(\bar{\partial}_t \theta_v^{j+\frac{1}{2}} - \bar{\partial}_t \theta_v^{j-\frac{1}{2}}), w) - (\operatorname{div}(\hat{\theta}_\sigma^{j+\frac{1}{2}} - \hat{\theta}_\sigma^{j-\frac{1}{2}}), w) = (\eta_5^j + \eta_6^j, w)&,\end{aligned}$$

for $\tau \in M_h$, $w \in V_h$, where $\eta_l^j = \omega_l^{j+1/2} - \omega_l^{j-1/2}$, $l = 1, \cdots, 6$, $1 \le j \le N-1$.

Note that

$$\bar{\partial}_t Y^{j+1/2} + \bar{\partial}_t Y^{j-1/2} = (\hat{Y}^{j+1/2} - \hat{Y}^{j-1/2})/\Delta t, \tag{3.82}$$

for any sequence $\{Y^j\}_{j\in\mathbb{N}}$. If we take $\tau = \bar{\partial}_t\theta_\sigma^{j+1/2} + \bar{\partial}_t\theta_\sigma^{j-1/2}$, $w = \bar{\partial}_t\theta_v^{j+1/2} + \bar{\partial}_t\theta_v^{j-1/2}$ in the above two equations, add them, and consider cancellations due to (3.82), then we have

$$\begin{aligned}
&(\|\bar{\partial}_t\theta_\sigma^{j+\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t\theta_v^{j+\frac{1}{2}}\|_\rho^2) - (\|\bar{\partial}_t\theta_\sigma^{j-\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t\theta_v^{j-\frac{1}{2}}\|_\rho^2) \\
&= (A(\eta_1^j + \eta_2^j) + \eta_3^j + \eta_4^j, \bar{\partial}_t\theta_\sigma^{j+\frac{1}{2}} + \bar{\partial}_t\theta_\sigma^{j-\frac{1}{2}}) + (\eta_5^j + \eta_6^j, \bar{\partial}_t\theta_v^{j+\frac{1}{2}} + \bar{\partial}_t\theta_v^{j-\frac{1}{2}}).
\end{aligned}$$

By repeating an argument used in (3.46–3.48), we get

$$(\|\bar{\partial}_t\theta_\sigma^{i+\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t\theta_v^{i+\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \le (\|\bar{\partial}_t\theta_\sigma^{\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t\theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} + c\sum_{j=1}^{i-1}\sum_{l=1}^{6}\|\eta_l^j\|. \tag{3.83}$$

To prove (3.81), we first estimate $(\|\bar{\partial}_t\theta_\sigma^{1/2}\|_A^2 + \|\bar{\partial}_t\theta_v^{1/2}\|_\rho^2)^{1/2}$ in (3.83). We take $\tau = \bar{\partial}_t\theta_\sigma^{1/2}$ and $w = \bar{\partial}_t\theta_v^{1/2}$ in (3.43–3.44) and add them. Recall that $\operatorname{div}\theta^0 = 0 = \theta_v^0$ from the choice of initial data by (3.17), so $(\operatorname{div}\bar{\partial}_t\theta_\sigma^{1/2}, \hat{\theta}_v^{1/2}) = (1/2\Delta t)(\operatorname{div}\theta_\sigma^1, \theta_v^1) = (\operatorname{div}\hat{\theta}_\sigma^{1/2}, \bar{\partial}_t\theta_v^{1/2})$ gives a cancellation in the sum. Therefore we have

$$\begin{aligned}
&(A\bar{\partial}_t\theta_\sigma^{\frac{1}{2}}, \bar{\partial}_t\theta_\sigma^{\frac{1}{2}}) + (\rho\bar{\partial}_t\theta_v^{\frac{1}{2}}, \bar{\partial}_t\theta_v^{\frac{1}{2}}) \\
&\qquad = (A(\omega_1^{\frac{1}{2}} + \omega_2^{\frac{1}{2}}) + \omega_3^{\frac{1}{2}} + \omega_4^{\frac{1}{2}}, \partial_t\theta_\sigma^{\frac{1}{2}}) + (\omega_5^{\frac{1}{2}} + \omega_6^{\frac{1}{2}}, \bar{\partial}_t\theta_v^{\frac{1}{2}}),
\end{aligned}$$

which yields

$$(\|\bar{\partial}_t\theta_\sigma^{\frac{1}{2}}\|_A^2 + \|\bar{\partial}_t\theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \le c\sum_{l=1}^{6}\|\omega_l^{\frac{1}{2}}\|.$$

For $\omega_1^{1/2}$ and $\omega_2^{1/2}$, from (3.55) and (3.58), one can have

$$\begin{aligned}
\|\omega_1^{\frac{1}{2}}\| &\le c\Delta t\int_0^{t_1}\|\dddot{\sigma}\|ds \le c\Delta t^2\|\dddot{\sigma}\|_{L^\infty L^2}, \\
\|\omega_2^{\frac{1}{2}}\| &\le \frac{ch^m}{\Delta t}\int_0^{t_1}\|\dot{\sigma}\|_m ds \le ch^m\|\dot{\sigma}\|_{L^\infty H^m}.
\end{aligned}$$

Similarly, using (3.56–3.57) and (3.59–3.60), we can show $\|\omega_l^{1/2}\| \le c(\Delta t^2 + h^m)$

for $l = 3, 4, 5, 6$.

Now it remains to show $\sum_{j=0}^{i-1} \|\eta_l^i\| \le c(h^m + \Delta t^2)$ in (3.83) for each $l = 1, \cdots, 6$. Note that

$$\begin{aligned}\eta_1^j &= \omega_1^{j+\frac{1}{2}} - \omega_1^{j-\frac{1}{2}} = \bar{\partial}_t \sigma^j - \hat{\sigma}^{j+\frac{1}{2}} - \bar{\partial}_t \sigma^{j-1} + \hat{\sigma}^{j-\frac{1}{2}} \\ &= \frac{1}{2\Delta t}(2\sigma^{j+1} - 4\sigma^j + 2\sigma^{j-1} - \Delta t(\dot{\sigma}^{j+1} - \dot{\sigma}^{j-1})).\end{aligned}$$

If we apply (3.54) to the last formula of the above with $0 = t_j$, $a = \Delta t$, $g = \sigma$, then we obtain $\sum_{j=0}^{i-1} \|\eta_1^j\| \le c\Delta t^2$ with c depending on $\|\sigma^{(4)}\|_{L^1 L^2}$. Applying a similar argument to η_3^j and η_5^j yields $\sum_{j=0}^{i-1}(\|\eta_3^j\| + \|\eta_5^j\|) \le c\Delta t^2$ with c depending on $\|r^{(4)}\|_{L^1 L^2}$, $\|v^{(4)}\|_{L^1 L^2}$. We also see, by using (3.52), that

$$\|\eta_2^j\| = \frac{1}{\Delta t}\|e_\sigma^{P,j+1} + e_\sigma^{P,j-1} - 2e_\sigma^{P,j}\| \le c\int_{t_{j-1}}^{t_{j+1}} \|\ddot{e}_\sigma^P\|\, ds \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\ddot{\sigma}\|_m\, ds,$$

$$\|\eta_4^j\| = \frac{1}{\Delta t}\|e_r^{P,j+1} + e_r^{P,j-1} - 2e_r^{P,j}\| \le c\int_{t_{j-1}}^{t_{j+1}} \|\ddot{e}_r^P\|\, ds \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\ddot{r}\|_m\, ds,$$

$$\|\eta_6^j\| = \frac{1}{\Delta t}\|\rho(e_v^{P,j+1} + e_v^{P,j-1} - 2e_v^{P,j})\| \le c\int_{t_{j-1}}^{t_{j+1}} \|\ddot{e}_v^P\|\, ds \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\ddot{v}\|_m\, ds,$$

and obtain $\sum_{j=0}^{m-1}(\|\eta_2^j\| + \|\eta_4^j\| + \|\eta_6^j\|) \le ch^m$. By the estimate (3.81), and the estimates (3.57), (3.60), we have

$$\begin{aligned}\|\operatorname{div}\hat{\theta}_\sigma^{j+\frac{1}{2}}\|_{-1} \le \|\operatorname{div}\hat{\theta}_\sigma^{j+\frac{1}{2}}\| &\le c(\|\bar{\partial}_t \theta_v^{j+\frac{1}{2}}\| + \|\omega_5^{j+\frac{1}{2}}\| + \|\omega_6^{j+\frac{1}{2}}\|) \\ &\le c(\Delta t^2 + h^m)\|\sigma, v, r\|_{W^{2,1}H^k}.\end{aligned}$$

Thus the inequality (3.80) yields $\|\hat{\theta}_\sigma^{j+1/2}\| \le c(\Delta t^2 + h^m)$ as desired. □

Remark 3.16. A similar result holds for the GG elements with appropriate modifications of proof.

3.7 Numerical results

In this section, we present numerical results. We used $\Omega = [0,1] \times [0,1]$ and the AFW elements of degree $k = 2$ in most experiments except the one with RadauIIA time discretization in Example 3.20. We assume the medium is homogeneous with density $\rho = 1$ and isotropic with constant Lamé coefficients λ and μ. For each mesh size h, we take $\Delta t = h$ for the time step Δt and the expected order of convergence from our analysis is 2. All numerical results present the errors at time $T_0 = 1$. The implementations are carried out using the Dolfin Python module [1] of the FEniCS project [2, 38].

Example 3.17. Consider the displacement field

$$u(t, x, y) = \begin{pmatrix} \sin(\pi x)\sin(\pi y)\sin t \\ x(1-x)y(1-y)\sin t \end{pmatrix}, \tag{3.84}$$

with homogeneous displacement boundary conditions. The numerical results for (3.84) are shown in Table 3.1. One can see that the convergence rates are same as that we expect in our error analysis.

Table 3.1: Order of convergence for the exact solution with displacement as in (3.84) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|v - v_h\|$		$\|u - u_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	5.73e-02	–	1.03e-02	–	1.61e-02	–	2.42e-02	–
8	1.19e-02	1.99	2.62e-03	1.98	4.06e-03	1.99	6.09e-03	1.99
16	2.78e-03	2.00	6.57e-04	2.00	1.02e-03	2.00	1.52e-03	2.00
32	6.77e-04	2.00	1.64e-04	2.00	2.54e-04	2.00	3.80e-04	2.00
64	1.67e-04	2.00	4.10e-05	2.00	6.35e-05	2.00	9.51e-05	2.00

Example 3.18. For an example of inhomogeneous displacement boundary condition, we use

$$u(t, x, y) = \begin{pmatrix} e^{-y}\sin x\cos t \\ e^{t+x} \end{pmatrix}, \tag{3.85}$$

and the variational form proposed in (3.13). The numerical results are shown in Table 3.2.

Example 3.19. We consider an example that the solution is not smooth. Let

$$u(t, x, y) = \begin{pmatrix} (1+t^2)x^{\frac{17}{8}}y \\ (1+t)y^{\frac{14}{5}} \end{pmatrix}. \tag{3.86}$$

The corresponding stress tensor with $\lambda = \mu = 1$ is

$$\sigma = \begin{pmatrix} \frac{51}{8}(1+t^2)x^{\frac{9}{8}}y + \frac{14}{5}(1+t)y^{\frac{9}{5}} & (1+t^2)x^{\frac{17}{8}} \\ (1+t^2)x^{\frac{17}{8}} & \frac{17}{8}(1+t^2)x^{\frac{9}{8}}y + \frac{42}{5}(1+t)y^{\frac{9}{5}} \end{pmatrix}.$$

Table 3.2: Order of convergence for the exact solution with displacement as in (3.85) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|v - v_h\|$		$\|u - u_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	2.36e-02	–	8.42e-03	–	2.75e-02	–	9.00e-03	–
8	5.82e-03	2.02	2.08e-03	2.01	6.87e-03	2.00	2.25e-03	2.00
16	1.45e-03	2.00	5.17e-04	2.01	1.72e-03	2.00	5.63e-04	2.00
32	3.62e-04	2.00	1.29e-04	2.00	4.30e-04	2.00	1.41e-04	2.00
64	9.05e-05	2.00	3.22e-05	2.00	1.07e-04	2.00	3.52e-05	2.00

Note that σ is not in H^2 but in $H^{13/8-\delta}$ for any $\delta > 0$. The numerical results are shown in Table 3.3 and the order of convergence for σ error is limited by $13/8 = 1.625$. However, we can see the order of convergences for the other errors are 2. From these numerical results, we expect that there is a way to obtain error bounds of v and r errors, which are better than that of σ error. This will be studied in the future.

Table 3.3: Order of convergence for the exact solution with displacement as in (3.86) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|v - v_h\|$		$\|u - u_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	3.50e-02	–	1.09e-02	–	2.58e-02	–	3.88e-03	–
8	1.17e-02	1.59	2.70e-03	2.02	6.46e-03	2.00	9.53e-04	2.02
16	3.84e-03	1.60	6.68e-04	2.01	1.62e-03	2.00	2.37e-04	2.01
32	1.25e-03	1.61	1.66e-04	2.01	4.04e-04	2.00	5.98e-05	1.99
64	4.08e-04	1.62	4.15e-05	2.00	1.01e-04	2.00	1.53e-05	1.96

Example 3.20. For higher order time discretization, we consider the implicit Runge–Kutta methods. For an evolution equation $\dot{y} = f(t, y)$, general Runge–Kutta schemes are described by the Butcher's table 3.4. When the i-th numerical solution y_i at time t_i is given, the next numerical solution y_{i+1} with time step interval Δt is defined as

$$y_{i+1} = y_i + \Delta t \sum_{l=1}^{s} b_l f(t_i + c_l \Delta t, Y_l), \tag{3.87}$$

Table 3.4: The Butcher's table for general Runge–Kutta schemes.

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

where $\{Y_l\}_{l=1}^s$ are obtained by solving

$$Y_j = y_i + \Delta t \sum_{l=1}^{s} a_{jl} f(t_i + c_l \Delta t, Y_l), \quad 1 \leq j \leq s. \tag{3.88}$$

A key idea of this numerical scheme is that the Y_l is an approximation of $\dot{y}$ at $t_i + c_l \Delta t$ because (3.88) is analogous to the equation $\dot{y} = f(t, y)$, and a linear combination of them with appropriate coefficient can be a good approximation of $\dot{y}$ at t_i with high accuracy. For more details, we refer [12].

We present numerical results for (3.84) with the 2-stage RadauIIA method whose Butcher's table is as in Table 3.5.

Table 3.5: The Butcher's table for the 2-stage RadauIIA Runge–Kutta scheme.

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

For the s-stage RadauIIA method, the order of convergence in time is $2s-1$.

Table 3.6: Order of convergence for the exact solution with displacement in (3.84) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$ and $T_0 = 1$). The AFW elements of degree 3 and the 2-stage RadauIIA time discretization are used.

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|v - v_h\|$		$\|u - u_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	1.31e-02	–	1.38e-03	–	2.10e-03	–	3.77e-03	–
8	1.02e-03	3.68	1.78e-04	2.96	2.58e-04	3.02	4.18e-04	3.17
16	9.88e-05	3.37	2.23e-05	3.00	3.25e-05	2.99	5.05e-05	3.05
32	1.14e-05	3.12	2.78e-06	3.00	4.09e-06	2.99	6.28e-06	3.01
64	1.40e-06	3.03	3.46e-07	3.00	5.13e-07	2.99	7.85e-07	3.00

Hence the expected order of convergence is 3 if the AFW element of degree 3 is used. In order to reconstruct numerical displacement, we use the numerical approximations of $\dot{v}$ which correspond to Y_l in (3.88) in the Runge–Kutta method. These numerical approximations are byproducts of the Runge–Kutta scheme. If we denote those terms as $V_t^{i+1/3}$ and V_t^{i+1} in the 2-stage RadauIIA method, we have

$$V_t^{i+\frac{1}{3}} \approx \dot{v}(t_i + \Delta t/3), \qquad V_t^{i+1} \approx \dot{v}(t_i + \Delta t),$$

in the time step at t_i. By Taylor expansions, one can check that

$$g(a) = g(0) + ag'(0) + \frac{a^2}{2} g''(a/3) + o(a^3),$$

so we define

$$U^{i+1} = U^i + \Delta t V^i + \frac{\Delta t^2}{2} V_t^{i+\frac{1}{3}}.$$

The numerical results in Table 3.6 show that the expected convergence rates are obtained for all errors.

Chapter 4

Mixed methods for the Kelvin–Voigt model of viscoelasticity

4.1 Introduction

In this chapter, we consider mixed methods for the Kelvin–Voigt model which is the simplest model of linear viscoelastic solids. The Kelvin–Voigt model itself is used to model soft tissues in biomedical engineering and is also an important basis to make more complicated viscoelastic models. Therefore developing a numerical scheme for the Kelvin–Voigt model is meaningful not only for its direct applications but also as a landmark when more complicated models of viscoelastic materials are studied.

There are not many results on mixed methods for the Kelvin–Voigt model. Quasistatic problems of the Kelvin–Voigt model, meaning that the mass density in equations is vanishing, were studied by Rognes and Winther [44] but, to our knowledge, no one has studied fully dynamic problems of it. In the work of Rognes and Winther, they suggest a mixed method framework for the Kelvin–Voigt model using weak symmetry of stress and show a priori error analysis for the AFW elements and a variant of the lowest order AFW element modified by Falk (see [10, 30]).

In this chapter, using the mixed method framework in [44], we study mixed methods for the Kelvin–Voigt model with a positive mass density. We prove a priori error estimates for the AFW and GG elements. The rest of this chapter is organized as follows. In section 4.2, we establish a velocity-stress weak formulation of the Kelvin–Voigt model with weak symmetry of stress. In section 4.3, we show an error analysis of semidiscrete solutions for the AFW elements. In section 4.4, we show an error analysis of fully discrete solutions for the AFW elements. In section 4.5, we show an error analysis for the GG elements and present a postprocessing technique to improve accuracy of numerical solutions of displacement. In section 4.6, we present some numerical results which support our error analysis.

4.2 Weak formulations with weak symmetry

Let u be the displacement vector. As we have seen in (2.8), there are two stresses related to spring and dashpot units in the Kelvin–Voigt model, and they are called elastic and viscous stresses, respectively. By (2.8), the elastic and viscous stresses have the relations

$$\sigma_0 = C_0\epsilon(u), \qquad \sigma_1 = C_1\epsilon(\dot{u}),$$

with rank 4 tensors C_0 and C_1 which satisfy (2.6) and are uniformly bounded above and below. The total stress is $\sigma_0 + \sigma_1 = C_0\epsilon(u) + C_1\epsilon(\dot{u})$, and by the balance law of linear momentum, we have

$$\rho\ddot{u} - \operatorname{div}(C_0\epsilon(u) + C_1\epsilon(\dot{u})) = f, \tag{4.1}$$

for an external body force f. Throughout the discussion in this chapter, we assume that the mass density ρ satisfies $0 < \rho_0 \leq \rho \leq \rho_1 < \infty$ for constants ρ_0, ρ_1.

To have a velocity-stress weak formulation, we use $v = \dot{u}$, $\sigma_0 = C_0\epsilon(u)$, $\sigma_1 = C_1\epsilon(\dot{u})$ as unknowns. Then we have equations of the Kelvin–Voigt model as

$$A_0\dot{\sigma}_0 = A_1\sigma_1 = \epsilon(v), \qquad \rho\dot{v} - \operatorname{div}(\sigma_0 + \sigma_1) = f, \tag{4.2}$$

where $A_0 = C_0^{-1}$, $A_1 = C_1^{-1}$. We assume the homogeneous displacement boundary conditions $u \equiv 0$, so $v \equiv 0$, on $\partial\Omega$ for simplicity.

To rewrite the equations as a mixed form with weak symmetry of stress, we also use a new unknown $p = \operatorname{skw}\operatorname{grad} v$. For $f \in W^{1,1}([0,T_0];V)$, a weak formulation of (4.1) with weak symmetry of stress is to seek $(\sigma_0, \sigma_1, v, p)$ such that

$$\sigma_0 + \sigma_1 \in C^0([0,T_0];M), \qquad \sigma_0 \in C^1([0,T_0];L^2(\Omega;\mathbb{M})), \tag{4.3}$$

$$v \in C^1([0,T_0];V), \qquad p \in C^0([0,T_0];K), \tag{4.4}$$

satisfying

$$(A_0\dot{\sigma}_0, \tau_0) + (v, \operatorname{div}\tau_0) + (p, \tau_0) = 0, \qquad \tau_0 \in M, \tag{4.5}$$

$$(A_1\sigma_1, \tau_1) + (v, \operatorname{div}\tau_1) + (p, \tau_1) = 0, \qquad \tau_1 \in M, \tag{4.6}$$

$$(\rho\dot{v}, w) - (\operatorname{div}(\sigma_0 + \sigma_1), w) = (f, w), \qquad w \in V, \tag{4.7}$$

$$(\sigma_0 + \sigma_1, q) = 0, \qquad q \in K, \tag{4.8}$$

with initial data $(\sigma_0(0), v(0)) \in L^2(\Omega, \mathbb{S}) \times \mathring{H}^1(\Omega; \mathbb{V})$.

Theorem 4.1. *For given initial data $\sigma_0(0) \in S$, and $v(0) \in \mathring{H}^1(\Omega;\mathbb{V})$ such that $C_1\epsilon(v(0)) \in S$, there is a unique solution $(\sigma_0, \sigma_1, v, p)$ of (4.5–4.8) satisfying (4.3–4.4).*

Proof. For existence, we use the Hille–Yosida theorem. Using $\sigma_1 = C_1\epsilon(v)$ in (4.2), we rewrite (4.2) as

$$\dot{\sigma}_0 = C_0\epsilon(v), \qquad \dot{v} = \rho^{-1}(\operatorname{div}(\sigma_0 + C_1\epsilon(v)) + f).$$

Let $\mathcal{X} = L^2(\Omega;\mathbb{S}) \times V$ with the inner product

$$((\sigma, v), (\tau, w))_{\mathcal{X}} = (A_0\sigma, \tau) + (\rho v, w)_\rho.$$

We define an unbounded operator $\mathcal{L}$ on $\mathcal{X}$ as $\mathcal{L}(\tau, w) = (C_0\epsilon(w), \rho^{-1}(\operatorname{div}(\tau + C_1\epsilon(w))$ with its domain,

$$D(\mathcal{L}) = \{(\tau, w) \in L^2(\Omega;\mathbb{S}) \times \mathring{H}^1(\Omega;\mathbb{V}) \,|\, \tau + C_1\epsilon(w) \in S\}. \tag{4.9}$$

To apply the Hille–Yosida theorem, we need to check that the domain of $\mathcal{L}$ is dense in $\mathcal{X}$, and $\mathcal{L}$ is an m-dissipative operator. To see the domain of $\mathcal{L}$ is dense, let us define

$$\mathring{H}^1(\mathrm{div}) = \{w \in \mathring{H}^1(\Omega;\mathbb{V}) \,|\, C_1\epsilon(w) \in S\}.$$

It is obvious that $S \times \mathring{H}^1(\mathrm{div})$ is a subset of the domain of $\mathcal{L}$, so it suffices to show that $S \times \mathring{H}^1(\mathrm{div})$ is dense in $\mathcal{X}$. Since S is dense in $L^2(\Omega;\mathbb{S})$, we only need to show that $\mathring{H}^1(\mathrm{div})$ is dense in V.

For given $g \in H^{-1}(\Omega;\mathbb{V})$, consider the equation

$$(C_1\epsilon(v), \epsilon(w)) = (g, w), \qquad w \in \mathring{H}^1(\Omega;\mathbb{V}). \tag{4.10}$$

By the coercivity of C_1, Korn's inequality, and the Lax–Milgram lemma, this equation gives a bijective map $L : \mathring{H}^1(\Omega;\mathbb{V}) \to H^{-1}(\Omega;\mathbb{V})$. Since L and L^{-1} are bounded by the open mapping theorem, a dense set in $\mathring{H}^1(\Omega;\mathbb{V})$ is mapped to a dense set in $H^{-1}(\Omega;\mathbb{V})$ and vice versa. For $g \in V$, consider the solution v of (4.10) and one sees that $\mathrm{div}\, C_1\epsilon(v) = g$ in the sense of distributions and $v \in \mathring{H}^1(\mathrm{div})$. Therefore the L^{-1} operator maps V to a subset of $\mathring{H}^1(\mathrm{div})$. As V is dense in $H^{-1}(\Omega;\mathbb{V})$, so is $\mathring{H}^1(\mathrm{div})$ in $\mathring{H}^1(\Omega;\mathbb{V})$.

The closedness of the operator $\mathcal{L}$ is not difficult to check, so we omit details. To show $\mathcal{L}$ is m-dissipative, we first check that for $\lambda > 0$,

$$\begin{aligned}((I-\lambda\mathcal{L})(\sigma, v), (\sigma, v))_{\mathcal{X}} &= \|\sigma\|_{A_0}^2 + \|v\|_\rho^2 - \lambda(\epsilon(v), \sigma) - \lambda(\mathrm{div}(\sigma + C_1\epsilon(v)), v) \\ &= \|\sigma\|_{A_0}^2 + \|v\|_\rho^2 + \lambda(C_1\epsilon(v), \epsilon(v)) \\ &\geq \|(\sigma, v)\|_{\mathcal{X}}^2,\end{aligned}$$

so $\|(I-\lambda\mathcal{L})(\sigma, v)\|_{\mathcal{X}} \geq \|(\sigma, v)\|_{\mathcal{X}}$. Now we check that $I - \lambda\mathcal{L} : D(\mathcal{L}) \to \mathcal{X}$ is surjective. For given $(\eta, z) \in \mathcal{X}$, we find $(\sigma, v) \in D(\mathcal{L})$ satisfying

$$\sigma - \lambda C_0\epsilon(v) = \eta, \qquad v - \rho^{-1}(\mathrm{div}\,\sigma + C_1\epsilon(v)) = z.$$

In the first equation, we get a constraint $\sigma = \lambda C_0\epsilon(v) + \eta$. We can obtain $v - \rho^{-1}(\mathrm{div}(\lambda C_0\epsilon(v) + \eta) + C_1\epsilon(v)) = z$ by substituting σ in the second equation using the aforementioned constraint. Rewriting in a weak form, we first find $v \in \mathring{H}^1(\Omega;\mathbb{V})$ such that

$$(v, w)_\rho + (\lambda C_0\epsilon(v) + \eta + C_1\epsilon(v), \epsilon(w)) = (z, w)_\rho, \quad w \in \mathring{H}^1(\Omega;\mathbb{V}).$$

Existence of solutions of this equation is a consequence of Korn's inequality, coercivity of C_0 and C_1, and the Lax–Milgram lemma. For the solution v, $\sigma = \lambda C_0\epsilon(v) + \eta$ from the previous constraint, and the above equation implies that $\sigma + C_1\epsilon(v) \in S$, i.e., $(\sigma, v) \in D(\mathcal{L})$. Thus $I - \lambda\mathcal{L}$ is surjective.

By the Hille–Yosida theorem, for initial data $(\sigma_0(0), v(0))$ in $D(\mathcal{L})$, there is a solution of (4.2) such that

$$(\sigma_0, v) \in C^0([0, T_0]; D(\mathcal{L})) \cap C^1([0, T_0]; \mathcal{X}),$$

and σ_1, p are defined to be $\sigma_1 = C_1\epsilon(v)$, $p = \operatorname{skw}\operatorname{grad} v$. From these definitions and Lemma 2.20, we can see that $(\sigma_0, \sigma_1, v, p)$ satisfies (4.5–4.8), so existence is proved.

For uniqueness, suppose that there are two solutions for same initial data, and we use $(\sigma_0^d, \sigma_1^d, v^d, p^d)$ to denote the difference of the solutions. Then $(\sigma_0^d, \sigma_1^d, v^d, p^d)$ satisfies (4.5–4.8) and $\sigma_0^d(0) = 0 = v^d(0)$. Taking $\tau_0 = \sigma_0$, $\tau_1 = \sigma_1$, $w = v$, $q = -p$ in (4.5–4.8), and adding all of them, we get

$$\frac{1}{2}\frac{d}{dt}\|\sigma_0^d\|_{A_0}^2 + \|\sigma_1^d\|_{A_1}^2 + \frac{1}{2}\frac{d}{dt}\|v^d\|_\rho^2 = 0.$$

By integrating and using the fundamental theorem of calculus,

$$\|\sigma_0^d(t)\|_{A_0}^2 + \|v^d(t)\|_\rho^2 + \int_0^t \|\sigma_1^d(s)\|_{A_1}^2 \, ds = 0,$$

so $\sigma_0^d = v^d = \sigma_1^d \equiv 0$. Since $p^d = \operatorname{skw}\operatorname{grad} v^d$, $p^d \equiv 0$ as well. Thus uniqueness is proved and the proof is completed. □

Remark 4.2. The homogeneous displacement boundary conditions $v = 0$ can be generalized to inhomogeneous boundary conditions $v = g$ on $\partial\Omega$. Then (4.5) in the weak formulation is changed to be

$$(A_0\dot{\sigma}_0, \tau_0) + (A_1\sigma_1, \tau_1) + (\operatorname{div}(\tau_0 + \tau_1), v) + (p, \tau_0 + \tau_1) = \int_{\partial\Omega} ((\tau_0 + \tau_1)\nu, g) \, dS, \quad (4.11)$$

where ν is the outward unit normal vector field on $\partial\Omega$.

4.3 Semidiscrete problems

We show that semidiscrete problems of the Kelvin–Voigt model is well-posed for the AFW elements. We also state and prove a priori error estimates.

4.3.1 Existence and uniqueness of semidiscrete solutions

Let (M_h, V_h, K_h) be the AFW elements of degree $k \geq 1$. We seek $\sigma_{0,h} \in C^1([0,T_0]; M_h)$, $\sigma_{1,h} \in C^0([0,T_0]; M_h)$, $v_h \in C^1([0,T_0]; V_h)$, $p_h \in C^0([0,T_0]; K_h)$ in the semidiscrete problem of Kelvin–Voigt model such that

$$(A_0 \dot{\sigma}_{0,h}, \tau_0) + (\operatorname{div} \tau_0, v_h) + (p_h, \tau_0) = 0, \tag{4.12}$$

$$(A_1 \sigma_{1,h}, \tau_1) + (\operatorname{div} \tau_1, v_h) + (p_h, \tau_1) = 0, \tag{4.13}$$

$$(\rho \dot{v}_h, w) - (\operatorname{div}(\sigma_{0,h} + \sigma_{1,h}), w) = (f, w), \tag{4.14}$$

$$(\sigma_{0,h} + \sigma_{1,h}, q) = 0, \tag{4.15}$$

for all $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$ with given initial data $\sigma_{0,h}(0) \in M_h$, $v_h(0) \in V_h$.

Theorem 4.3. *For given $\sigma_{0,h}(0) \in M_h$, $v_h(0) \in V_h$, the system* (4.12–4.15) *has a unique solution.*

Proof. Let $\{\phi_i\}$, $\{\psi_i\}$, $\{\chi_i\}$ be bases of M_h, V_h, and K_h, respectively. We use $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$, $\mathscr{D}$ to denote the matrices whose (i,j)-entries are

$$(A_0 \phi_j, \phi_i), \quad (A_1 \phi_j, \phi_i), \quad (\operatorname{div} \phi_j, \psi_i), \quad (\phi_j, \chi_i), \quad (\rho \psi_j, \psi_i), \quad (\psi_j, \psi_i),$$

respectively. We write $\sigma_{0,h} = \sum_i \alpha_i \phi_i$, $\sigma_{1,h} = \sum_i \beta_i \phi_i$, $v_h = \sum_i \gamma_i \psi_i$, $p_h = \sum_i \zeta_i \chi_i$, $P_h f = \xi_i \psi_i$, and use α, β, γ, ζ, ξ to denote the coefficient vectors. Then, based on a new order of unknowns $(\sigma_{0,h}, v_h, \sigma_{1,h}, p_h)$, we may rewrite (4.12–4.15) as a matrix equation of the form

$$\begin{pmatrix} \mathscr{A}_0 & 0 & 0 & 0 \\ 0 & \mathscr{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\gamma} \\ \dot{\beta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & -\mathscr{B}^T & 0 & -\mathscr{C}^T \\ \mathscr{B} & 0 & \mathscr{B} & 0 \\ 0 & -\mathscr{B}^T & \mathscr{A}_1 & -\mathscr{C}^T \\ -\mathscr{C} & 0 & -\mathscr{C} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \\ \beta \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ \mathscr{D}\xi \\ 0 \\ 0 \end{pmatrix}. \tag{4.16}$$

We use Lemma 2.17 to show existence and uniqueness of solutions of this system.

If we regard the above system in the context of Lemma 2.17, the block matrices E_0, F_{22} in Lemma 2.17 are

$$E_0 = \begin{pmatrix} \mathscr{A}_0 & 0 \\ 0 & \mathscr{M} \end{pmatrix}, \quad F_{22} = \begin{pmatrix} \mathscr{A}_1 & -\mathscr{C}^T \\ -\mathscr{C} & 0 \end{pmatrix}.$$

It is obvious that E_0 is nonsingular because $\mathscr{A}_0$ and $\mathscr{M}$ are symmetric positive definite. Since $\mathscr{A}_1$ is symmetric positive definite and $\mathscr{C}^T$ is injective from the inf-sup condition **(A2)**, F_{22} is nonsingular as well. By Lemma 2.17, there is a unique solution of (4.12–4.15) for initial data $(\sigma_{0,h}(0), v_h(0))$. □

Remark 4.4. If we let $r = \operatorname{skw}\operatorname{grad} u$ and use $\dot{r}_h$ instead of p_h for a weak formulation of the Kelvin–Voigt model with weak symmetry, we have a matrix equation with different coefficient matrices. In this case, the coefficient matrix of the left-hand side is

$$\begin{pmatrix} \mathscr{A}_0 & 0 & 0 & 0 \\ 0 & \mathscr{M} & 0 & 0 \\ 0 & 0 & 0 & \mathscr{C}^T \\ 0 & 0 & \mathscr{C} & 0 \end{pmatrix},$$

and it is not obvious to make a block matrix form that Lemma 2.17 can be applied.

4.3.2 Decomposition of semidiscrete errors

In this section, we state the main theorem in the semidiscrete error analysis and discuss a decomposition of errors for the proof of the theorem.

In the semidiscrete problem of Kelvin–Voigt model, the initial data of $\sigma_{0,h}$ and v_h are given and $\sigma_{1,h}(0)$, $p_h(0)$ are determined by the equations. For our error analysis, the initial data of $\sigma_{0,h}$, v_h for semidiscrete problems should be chosen that the corresponding $\sigma_{1,h}(0)$, $p_h(0)$ are good approximations of $\sigma_1(0)$, $p(0)$. More precisely, when $\sigma_{0,h}(0) \in M_h$, $v_h(0) \in V_h$ are given to be

$$\|\sigma_0(0) - \sigma_{0,h}(0), v(0) - v_h(0)\| \le ch^m,$$

for some $c > 0$ and $m \ge 1$ independent of h, we want that $\sigma_{1,h}(0) \in M_h$,

$p_h(0) \in K_h$ corresponding to them satisfy

$$\|\sigma_1(0) - \sigma_{1,h}(0), p(0) - p_h(0)\| \le c' h^m,$$

for some $c' > 0$ independent of h. We claim that there exists such initial data $(\sigma_{0,h}(0), v_h(0))$ which is computable for implementations. Recall that $(\sigma_1(0), v(0), p(0))$ satisfies

$$(A_1\sigma_1(0), \tau) + (\operatorname{div}\tau, v(0)) + (\tau, p(0)) = 0, \qquad (\sigma_1(0), q) = 0, \tag{4.17}$$

for $(\tau, q) \in M \times K$ from (4.6) at $t = 0$ and the symmetry of $\sigma_1(0)$. We consider a solution (σ_h', v_h', p_h') of the system

$$\begin{aligned} (A_1\sigma_h', \tau) + (\operatorname{div}\tau, v_h') + (p_h', \tau) &= 0, \\ (\operatorname{div}\sigma_h', w) &= (\operatorname{div}\sigma_1(0), w), \\ (\sigma_h', q) &= 0, \end{aligned} \tag{4.18}$$

for $(\tau, w, q) \in M_h \times V_h \times K_h$. From (2.32–2.33), we have

$$\|\sigma_1(0) - \sigma_h'\| + \|v(0) - v_h'\| + \|p(0) - p_h'\| \le ch^m \|\sigma_1(0), v(0), p(0)\|_m, \tag{4.19}$$

for $1 \le m \le k$ and $P_h \operatorname{div}\sigma_1(0) = \operatorname{div}\sigma_h'$. Note that σ_h', v_h', p_h' are computable quantities by (4.18) when $\operatorname{div}\sigma_1(0) = \operatorname{div} C_1\epsilon(v(0))$ is known.

Theorem 4.5. *Let (M_h, V_h, K_h) be the* AFW *elements of degree $k \ge 1$. Suppose that $\sigma_0, v \in W^{2,1}([0, T_0]; H^m)$, $\sigma_1, p \in W^{1,1}([0, T_0]; H^m)$ and suppose also that $(\sigma_{0,h}, \sigma_{1,h}, v_h, p_h)$ is the semidiscrete solution of* (4.12–4.15) *with initial data*

$$\sigma_{0,h}(0) = \tilde{\Pi}_h\sigma_0(0), \quad v_h(0) = v_h',$$

for which v_h' defined by (4.18)*. Then we have*

$$\begin{aligned} &\|\sigma_0 - \sigma_{0,h}, \sigma_1 - \sigma_{1,h}, v - v_h, p - p_h\|_{L^\infty L^2} \\ &\qquad\qquad \le ch^m(\|\sigma_0, v\|_{W^{2,1}H^m} + \|\sigma_1, p\|_{W^{1,2}H^m}), \end{aligned}$$

for $1 \le m \le k$, where c depends on the time interval T_0, elastic and viscous compliance tensors A_0, A_1, and the lower and upper bound of the mass density ρ_0, ρ_1 but is independent of h.

In order to prove Theorem 4.5, we decompose errors into the projection and approximation errors. We use e_{σ_0}, e_{σ_1}, e_v, and e_p to denote the semidiscrete errors $\sigma_0 - \sigma_{0,h}$, $\sigma_1 - \sigma_{1,h}$, $v - v_h$, and $p - p_h$, respectively. The error equations are

$$(A_0 \dot{e}_{\sigma_0}, \tau_0) + (\operatorname{div} \tau_0, e_v) + (e_p, \tau_0) = 0, \tag{4.20}$$

$$(A_1 e_{\sigma_1}, \tau_1) + (\operatorname{div} \tau_1, e_v) + (e_p, \tau_1) = 0, \tag{4.21}$$

$$(\rho \dot{e}_v, w) - (\operatorname{div}(e_{\sigma_0} + e_{\sigma_1}), w) = 0, \tag{4.22}$$

$$(e_{\sigma_0} + e_{\sigma_1}, q) = 0, \tag{4.23}$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$. We define $(\sigma^P_{0,h}, \sigma^P_{1,h}, v^P_h, p^P_h)$ by

$$\sigma^P_{0,h} = \tilde{\Pi}_h \sigma_0, \quad \sigma^P_{1,h} = \tilde{\Pi}_h \sigma_1, \quad v^P_h = P_h v, \quad p^P_h = P'_h p, \tag{4.24}$$

and split the semidiscrete errors into

$$\begin{aligned} e_{\sigma_i} &= e^P_{\sigma_i} + e^h_{\sigma_i} := (\sigma_i - \sigma^P_{i,h}) + (\sigma^P_{i,h} - \sigma_{i,h}), \quad i = 0, 1, \\ e_v &= e^P_v + e^h_v := (v - v^P_h) + (v^P_h - v_h), \\ e_p &= e^P_p + e^h_p := (p - p^P_h) + (p^P_h - p_h), \end{aligned} \tag{4.25}$$

where e^P and e^h terms are called the projection and approximation errors, respectively.

4.3.3 Projection error estimates for the AFW elements

We state the projection error estimates but we omit the proof of this theorem because it is almost same as the one of Theorem 3.5.

Theorem 4.6. *Let* (M_h, V_h, K_h) *be the* AFW *elements of degree* $k \geq 1$ *and suppose that* $(\sigma_{0,h}, \sigma_{1,h}, v_h, p_h)$ *is a semidiscrete solution of* (4.12–4.15)*. For the projection errors* $e^P_{\sigma_0}$, $e^P_{\sigma_1}$, e^P_v, e^P_p*, defined in* (4.25)*, we get*

$$\begin{aligned} \|e^P_{\sigma_0}, e^P_{\sigma_1}\|_{L^\infty L^2} &\leq c h^m \|\sigma_0, \sigma_1\|_{L^\infty H^m}, \quad 1 \leq m \leq k+1, \\ \|e^P_v, e^P_p\|_{L^\infty L^2} &\leq c h^m \|v, p\|_{L^\infty H^m}, \quad 0 \leq m \leq k. \end{aligned}$$

Similar results hold for time derivatives of σ_i*,* v*, and* p *for* $i = 0, 1$.

4.3.4 Approximation error estimates for the AFW elements

We show a priori estimates of the approximation errors.

Theorem 4.7. *Suppose that* $(\sigma_0, \sigma_1, v, p)$ *is a solution of* (4.5–4.8) *with the regularity assumptions in Theorem 4.5 and let* $e^h_{\sigma_0}, e^h_{\sigma_1}, e^h_v, e^h_p$ *be the approximation errors defined in* (4.25)*. Then, for* $1 \le m \le k$*,*

$$\|e^h_{\sigma_0}, e^h_{\sigma_1}, e^h_v, e^h_p\|_{L^\infty L^2} \le ch^m(\|\sigma_0, v\|_{W^{2,1}H^m} + \|\sigma_1, p\|_{W^{1,2}H^m}), \tag{4.26}$$

where c *depends on* T_0*,* ρ_1*,* A_0*,* A_1 *but is independent of* h*.*

Proof. We claim that, for $1 \le m \le k$, it is enough to prove

$$\|e^h_{\sigma_0}, e^h_v\|_{L^\infty L^2} \le ch^m(\|v(0)\|_m + \max\{\|\sigma_1, p\|_{L^2H^m}, \|\dot{\sigma}_0, \dot{v}, p\|_{L^1H^m}\}), \tag{4.27}$$

$$\|\dot{e}^h_{\sigma_0}\|_{L^\infty L^2} \le c(\|\dot{e}^h_{\sigma_0}(0)\| + \|\dot{e}^h_v(0)\|_\rho + h^m \max\{\|\dot{\sigma}_1, \dot{p}\|_{L^2H^m}, \|\ddot{\sigma}_0, \ddot{v}, \dot{p}\|_{L^1H^m}\}), \tag{4.28}$$

$$\|e^h_{\sigma_1}, e^h_p\|_{L^\infty L^2} \le c\|\dot{e}^h_{\sigma_0}, \dot{e}^P_{\sigma_0}, e^P_{\sigma_1}, e^P_p\|_{L^\infty L^2}, \tag{4.29}$$

$$\|\dot{e}^h_{\sigma_0}(0)\| + \|\dot{e}^h_v(0)\|_\rho \le ch^m\|\dot{\sigma}_0(0), \sigma_1(0), \dot{v}(0)\|_m. \tag{4.30}$$

Note that (4.27) gives (4.26) for $e^h_{\sigma_0}$ and e^h_v. From the estimates (4.28) and (4.30), we have $\|\dot{e}^h_{\sigma_0}\|_{L^\infty L^2} \le ch^m(\|\sigma_0, v\|_{W^{2,1}H^m} + \|\sigma_1, p\|_{W^{1,2}H^m})$. If we combine this estimate with (4.29) and use Theorem 4.6, then (4.26) for $e^h_{\sigma_1}$, e^h_p follows.

Now we begin proving (4.27–4.30). We first derive reduced error equations. Rewriting (4.20–4.22), using the error decomposition in (4.25), we get

$$\begin{aligned}
(A_0(\dot{e}^h_{\sigma_0} + \dot{e}^P_{\sigma_0}), \tau_0) + (\operatorname{div}\tau_0, e^h_v + e^P_v) + (\tau_0, e^h_p + e^P_p) &= 0,\\
(A_1(e^h_{\sigma_1} + e^P_{\sigma_1}), \tau_1) + (\operatorname{div}\tau_1, e^h_v + e^P_v) + (\tau_1, e^h_p + e^P_p) &= 0,\\
(\rho(\dot{e}^h_v + \dot{e}^P_v), w) - (\operatorname{div}(e^h_{\sigma_0} + e^P_{\sigma_0} + e^h_{\sigma_1} + e^P_{\sigma_1}), w) &= 0,\\
(e_{\sigma_0} + e_{\sigma_1}, q) &= 0,
\end{aligned}$$

for $(\tau_0, \tau_1, w) \in M_h \times M_h \times V_h$. Note that $(\operatorname{div}(e^P_{\sigma_0} + e^P_{\sigma_1}), w)$, $(\operatorname{div}\tau_0, e^P_v)$, $(\operatorname{div}\tau_1, e^P_v)$, and $(e^P_{\sigma_0} + e^P_{\sigma_1}, q)$ vanish from the same argument as in (3.24).

Thus

$$(A_0\dot{e}^h_{\sigma_0},\tau_0)+(\operatorname{div}\tau_0,e^h_v)+(\tau_0,e^h_p)=-(A_0\dot{e}^P_{\sigma_0},\tau_0)-(\tau_0,e^P_p), \tag{4.31}$$

$$(A_1e^h_{\sigma_1},\tau_1)+(\operatorname{div}\tau_1,e^h_v)+(\tau_1,e^h_p)=-(A_1e^P_{\sigma_1},\tau_1)-(\tau_1,e^P_p), \tag{4.32}$$

$$(\rho\dot{e}^h_v,w)-(\operatorname{div}(e^h_{\sigma_0}+e^h_{\sigma_1}),w)=-(\rho\dot{e}^P_v,w), \tag{4.33}$$

$$(e^h_{\sigma_0}+e^h_{\sigma_1},q)=0, \tag{4.34}$$

for $(\tau_0,\tau_1,w,q)\in M_h\times M_h\times V_h\times K_h$.

For (4.27), we take $\tau_0=e^h_{\sigma_0}$, $\tau_1=e^h_{\sigma_1}$, $w=e^h_v$, $q=-e^h_p$ in (4.31–4.34) and add all equations, then,

$$\frac{1}{2}\frac{d}{dt}\|e^h_{\sigma_0}\|^2_{A_0}+\|e^h_{\sigma_1}\|^2_{A_1}+\frac{1}{2}\frac{d}{dt}\|e^h_v\|^2_\rho$$
$$=-(A_0\dot{e}^P_{\sigma_0},e^h_{\sigma_0})-(A_1e^P_{\sigma_1},e^h_{\sigma_1})-(e^h_{\sigma_0}+e^h_{\sigma_1},e^P_p)-(\rho\dot{e}^P_v,e^h_v).$$

Using Young's inequality to $(A_1e^P_{\sigma_1},e^h_{\sigma_1})$ and $(e^h_{\sigma_1},e^P_p)$, one can have

$$-(A_1e^P_{\sigma_1}+e^P_p,e^h_{\sigma_1})-\|e^h_{\sigma_1}\|^2_{A_1}\le c(\|e^P_{\sigma_1}\|^2+\|e^P_p\|^2),$$

for $c>0$ depending only on A_0, A_1. If we use this inequality to the previous equality, then one gets

$$\frac{1}{2}\frac{d}{dt}\|e^h_{\sigma_0}\|^2_{A_0}+\frac{1}{2}\frac{d}{dt}\|e^h_v\|^2_\rho$$
$$\le c(\|e^P_{\sigma_1}\|^2+\|e^P_p\|^2)-(A_0\dot{e}^P_{\sigma_0},e^h_{\sigma_0})-(e^h_{\sigma_0},e^P_p)-(\rho\dot{e}^P_v,e^h_v). \tag{4.35}$$

By using a weighted Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|e^h_{\sigma_0}\|^2_{A_0}+\|e^h_v\|^2_\rho)\le c(\|e^P_{\sigma_1}\|^2+\|e^P_p\|^2)$$
$$+c(\|\dot{e}^P_{\sigma_0}\|+\|e^P_p\|+\|\dot{e}^P_v\|)(\|e^h_{\sigma_0}\|^2_{A_0}+\|e^h_v\|^2_\rho)^{\frac{1}{2}}.$$

We apply Lemma 2.14 to this inequality regarding $Q=(\|e^h_{\sigma_0}\|^2_{A_0}+\|e^h_v\|^2_\rho)^{1/2}$, $F=c(\|\dot{e}^P_{\sigma_0}\|+\|e^P_p\|+\|\dot{e}^P_v\|)$, and $G=c(\|e^P_{\sigma_1}\|^2+\|e^P_p\|^2)$. Then, using the facts $e^h_{\sigma_0}(0)=0$, $e^h_v(0)=v^P_h(0)-v'_h$ and the coercivity of A_0, we get

$$\|e^h_{\sigma_0},e^h_v\|_{L^\infty L^2}\le c\|v^P_h(0)-v'_h\|+c\max\left\{\|e^P_{\sigma_1},e^P_p\|_{L^2L^2},\|\dot{e}^P_{\sigma_0},\dot{e}^P_v,e^P_p\|_{L^1L^2}\right\}.$$

If we use the fact $\|v^P_h(0)-v(0)\|\le ch^m\|v(0)\|_m$, (4.19), the projection error estimates in Theorem 4.6, and the triangle inequality, then we have

$$\|e^h_{\sigma_0}, e^h_v\|_{L^\infty L^2} \le ch^m(\|v(0)\| + \max\{\|\sigma_1, p\|_{L^2H^m}, \|\dot\sigma_0, \dot v, p\|_{L^1H^m}\}),$$

for $1 \le m \le k$, so (4.27) is proved.

For (4.28), we take time derivatives of (4.31–4.34) and use a similar argument by taking $\tau_0 = \dot e^h_{\sigma_0}$, $\tau_1 = \dot e^h_{\sigma_1}$, $w = \dot e^h_v$, and $q = -\dot e^h_p$. Following same steps in the proof of (4.27), we have (4.28).

Now we turn to the proof of (4.29). We first show

$$\|e^h_p\|_{L^\infty L^2} \le c\|e^h_{\sigma_1}, e^P_{\sigma_1}, e^P_p\|_{L^\infty L^2}. \tag{4.36}$$

To prove (4.36), by using **(A2)**, we take τ_1 in (4.31) such that $\|\tau_1\| \le c\|e^h_p\|$, $\operatorname{div}\tau_1 = 0$ and $(\tau_1, q) = (e^h_p, q)$ for all $q \in K_h$. Then,

$$\|e^h_p\|^2 = (\tau_1, e^h_p) = -(A_1\dot e^h_{\sigma_1}, \tau_1) - (A_1\dot e^P_{\sigma_1}, \tau_1) - (\tau_1, e^P_p),$$

and (4.36) is proved by the Cauchy–Schwarz inequality. To complete the proof of (4.29), now it is enough to show $\|e^h_{\sigma_1}\|_{L^\infty L^2} \le c\|\dot e^h_{\sigma_0}, \dot e^P_{\sigma_0}, e^P_{\sigma_1}\|_{L^\infty L^2}$. For this, take $-\tau_0 = e^h_{\sigma_1} = \tau_1$ in (4.31–4.32) and add them. Then we get

$$(A_1e^h_{\sigma_1}, e^h_{\sigma_1}) = -(A_0(\dot e^h_{\sigma_0} + \dot e^P_{\sigma_0}), e^h_{\sigma_1}) - (A_1e^P_{\sigma_1}, e^h_{\sigma_1}),$$

and therefore $\|e^h_{\sigma_1}\| \le c\|\dot e^h_{\sigma_0}, \dot e^P_{\sigma_0}, e^P_{\sigma_1}\|$, which proves (4.29).

Finally, for (4.30), recall that $e^h_{\sigma_0}(0) = 0$, so $\operatorname{div} e^h_{\sigma_0}(0) = 0$ and

$$\operatorname{div} e^h_{\sigma_1}(0) = \operatorname{div}\tilde\Pi_h\sigma_1(0) - \operatorname{div}\sigma'_h = 0.$$

If we take $w = \dot e^h_v(0)$ in (4.33), then we have

$$\|\dot e^h_v(0)\|^2_\rho = -(\rho\dot e^P_v(0), \dot e^h_v(0)) \le \|\dot e^P_v(0)\|_\rho\|\dot e^h_v(0)\|_\rho,$$

so $\|\dot e^h_v(0)\|_\rho \le \|\dot e^P_v(0)\|_\rho \le ch^m\|\dot v(0)\|_m$ for $1 \le m \le k$.

If we take $\tau_0 = -\tau_1 = \dot e^h_{\sigma_0}(0)$ in (4.31–4.32) and add them, then we get

$$\|\dot e^h_{\sigma_0}(0)\|^2_{A_0} = -(A_0\dot e^P_{\sigma_0}(0) + A_1(e^h_{\sigma_1}(0) + e^P_{\sigma_1}(0)), \dot e^h_{\sigma_0}(0)).$$

By the coercivity of A_0, the triangle inequality, and the estimate (4.19) with the projection error estimates, we get

$$\|\dot{e}^h_{\sigma_0}(0)\| \le c\|\dot{e}^P_{\sigma_0}(0), e^h_{\sigma_1}(0), e^P_{\sigma_1}(0)\| \le ch^m\|\dot{\sigma}_0(0), \sigma_1(0)\|_m,$$

for $1 \le m \le k+1$. □

4.4 Full discretization

In this section, we consider the error analysis of full discretization with the Crank–Nicolson scheme.

In semidiscrete problems, we only need initial data of σ_0, v. However, if we use the Crank–Nicolson scheme for full discretization, we need initial data of σ_1 and p as well. Thus we take $\Sigma_0^0 = \tilde{\Pi}_h\sigma_0(0)$, $\Sigma_1^0 = \tilde{\Pi}_h\sigma_1(0)$, $V^0 = P_h v(0)$, and $P^0 = P'_h p(0)$ as initial data. In full discretization with the Crank–Nicolson scheme for the equations of Kelvin–Voigt model, initial data of σ_1 and v are not necessarily same as the ones in the semidiscrete problem. In fact, the algebraic equation of the system involves the arithmetic mean of numerical solutions at times 0 and Δt, so the restriction for initial data is not as strong as that in semidiscrete problems.

In the Crank–Nicolson scheme, $(\Sigma_0^{j+1}, \Sigma_1^{j+1}, V^{j+1}, P^{j+1})$, the numerical solution at time t_{j+1} is defined inductively by

$$(A_0\bar{\partial}_t\Sigma_0^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div}\tau_0, \hat{V}^{j+\frac{1}{2}}) + (\hat{P}^{j+\frac{1}{2}}, \tau_0) = 0, \tag{4.37}$$

$$(A_1\hat{\Sigma}_1^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div}\tau_1, \hat{V}^{j+\frac{1}{2}}) + (\hat{P}^{j+\frac{1}{2}}, \tau_0) = 0, \tag{4.38}$$

$$\left(\bar{\partial}_t V^{j+\frac{1}{2}}, w\right) - (\operatorname{div}(\hat{\Sigma}_0^{j+\frac{1}{2}} + \hat{\Sigma}_1^{j+\frac{1}{2}}), w) = (\hat{f}^{j+\frac{1}{2}}, w), \tag{4.39}$$

$$(\hat{\Sigma}_0^{j+\frac{1}{2}} + \hat{\Sigma}_1^{j+\frac{1}{2}}, q) = 0, \tag{4.40}$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$ and $j \ge 0$.

Theorem 4.8. *Suppose $\sigma_0(0), \sigma_1(0), v(0), p(0) \in H^m$ for $1 \le m \le k$ and*

$$\begin{aligned} &\sigma_0, v \in W^{2,1}([0,T_0]; H^m) \cap W^{4,1}([0,T_0]; L^2), \\ &\sigma_1, p \in W^{1,2}([0,T_0]; H^m) \cap W^{3,2}([0,T_0]; L^2). \end{aligned} \tag{4.41}$$

For initial data $\Sigma_0^0 = \tilde{\Pi}_h\sigma_0(0)$, $\Sigma_1^0 = \tilde{\Pi}_h\sigma_1(0)$, $V^0 = P_h v(0)$, $P^0 = P'_h p(0)$, the fully discrete solution $(\Sigma_0^j, \Sigma_1^j, V^j, P^j)$ in (4.37–4.40) is well-defined and for

$1 \le j \le N$,

$$\|\sigma_0^j - \Sigma_0^j\| + \|v^j - V^j\| + \|\hat{\sigma}_1^{j-\frac{1}{2}} - \hat{\Sigma}_1^{j-\frac{1}{2}}\| + \|\hat{p}^{j-\frac{1}{2}} - \hat{P}^{j-\frac{1}{2}}\|$$
$$\le c(\Delta t^2 + h^m)(\|\sigma_0, v\|_{W^{2,1}H^m \cap W^{4,1}L^2} + \|\sigma_1, p\|_{W^{1,2}H^m \cap W^{3,2}L^2}),$$

for $c > 0$ depending on A_0, A_1, ρ_1, T_0 but independent of h and Δt.

The proof will be given in the rest of this section.

4.4.1 Well-definedness

First of all, we show that the full discretization scheme is well-defined. We need to check when $\Sigma_0^j, \Sigma_1^j, V^j, P^j, f^j, f^{j+1}$ are given, a solution of the linear system (4.37–4.40), which is $(\Sigma_0^{j+1}, \Sigma_1^{j+1}, V^{j+1}, P^{j+1})$, is well-defined. If we rewrite (4.37–4.40),

$$\begin{aligned}
&(A_0\Sigma_0^{j+1}, \tau_0) + \frac{\Delta t}{2}(V^{j+1}, \operatorname{div}\tau_0) + \frac{\Delta t}{2}(P^{j+1}, \tau_0) \\
&\qquad = (A_0\Sigma_0^j, \tau_0) - \frac{\Delta t}{2}(V^j, \operatorname{div}\tau_0) - \frac{\Delta t}{2}(P^j, \tau_0), \\
&(A_1\Sigma_1^{j+1}, \tau_1) + (V^{j+1}, \operatorname{div}\tau_1) + (P^{j+1}, \tau_1) \\
&\qquad = -(A_1\Sigma_1^j, \tau_1) - (V^j, \operatorname{div}\tau_1) - (P^j, \tau_1), \\
&(V^{j+1}, w) - \frac{\Delta t}{2}(\operatorname{div}(\Sigma_0^{j+1} + \Sigma_1^{j+1}), w) \\
&\qquad = (V^j, w) + \frac{\Delta t}{2}(\operatorname{div}(\Sigma_0^j + \Sigma_1^j), w) + \Delta t(\hat{f}^{j+\frac{1}{2}}, w), \\
&-\frac{\Delta t}{2}(\Sigma_0^{j+1} + \Sigma_1^{j+1}, q) = \frac{\Delta t}{2}(\Sigma_0^j + \Sigma_1^j, q).
\end{aligned}$$

Regarding $\Sigma_0^{j+1}, \Sigma_1^{j+1}, V^{j+1}, P^{j+1}$ as unknowns, the above is a system of linear equations with same number of equations and unknowns. In order to show it is well-defined, it is enough to show that $\Sigma_0^{j+1} = \Sigma_1^{j+1} = V^{j+1} = P^{j+1} = 0$ when all of the right-hand side vanish. Suppose all right-hand side terms vanish. If we take $\tau_0 = \Sigma_0^{j+1}$, $\tau_1 = \Sigma_1^{j+1}$, $w = V^{j+1}$, $q = P^{j+1}$ and add all equations, we have $|\Sigma_0^{j+1}|_{A_0}^2 + (\Delta t/2)|\Sigma_1^{j+1}|_{A_1}^2 + |V^{j+1}|^2 = 0$ which implies $\Sigma_0^{j+1} = \Sigma_1^{j+1} = V^{j+1} = 0$. For P^{j+1}, take τ_0 in (4.37) such that $(\tau_0, q) = (P^{j+1}, q)$ for all $q \in K_h$. Then the first equation yields $|P^{j+1}|^2 = 0$, so $P^{j+1} = 0$ and the fully discrete solution is well-defined.

4.4.2 Convergence

Let us denote the error $(\sigma_0^j - \Sigma_0^j, \sigma_1^j - \Sigma_1^j, v^j - V^j, p^j - P^j)$ by

$$E_{\sigma_i}^j = \sigma_i^j - \Sigma_i^j = (\sigma_i^j - \sigma_{i,h}^{P,j}) + (\sigma_{i,h}^{P,j} - \Sigma_i^j) =: e_{\sigma_i}^{P,j} + \theta_{\sigma_i}^j, \quad i = 0, 1, \tag{4.42}$$

$$E_v^j = v^j - V^j = (v^j - v_h^{P,j}) + (v_h^{P,j} - V^j) =: e_v^{P,j} + \theta_v^j, \tag{4.43}$$

$$E_p^j = p^j - P^j = (p^j - p_h^{P,j}) + (p_h^{P,j} - P^j) =: e_p^{P,j} + \theta_p^j. \tag{4.44}$$

In the semidiscrete error analysis of the previous section, we already obtained error bounds of $e_{\sigma_0}^{P,j}$, $e_{\sigma_1}^{P,j}$, $e_v^{P,j}$, and $e_p^{P,j}$ in Theorem 4.6. By the triangle inequality, we only need to consider the a priori estimates of $\theta_{\sigma_0}^j$, $\theta_{\sigma_1}^j$, θ_v^j, and θ_p^j.

Theorem 4.9. *Suppose the assumptions of Theorem 4.8 hold. Then there exists $c > 0$ which is independent of h and Δt such that*

$$\|\theta_{\sigma_0}^i\| + \|\hat{\theta}_{\sigma_1}^{i-\frac{1}{2}}\| + \|\theta_v^i\| + \|\hat{\theta}_p^{i-\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \tag{4.45}$$

for $1 \le i \le N$ and $1 \le m \le k+1$.

Proof. The arithmetic mean of equations (4.5–4.7) at $t = t_j, t_{j+1}$ gives

$$\begin{aligned}(A_0\hat{\sigma}_0^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div}\tau_0, \hat{v}^{j+\frac{1}{2}}) + (\hat{p}^{j+\frac{1}{2}}, \tau_0) &= 0,\\ (A_1\hat{\sigma}_1^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div}\tau_1, \hat{v}^{j+\frac{1}{2}}) + (\hat{p}^{j+\frac{1}{2}}, \tau_1) &= 0,\\ (\rho\hat{\dot{v}}^{j+\frac{1}{2}}, w) - (\operatorname{div}(\hat{\sigma}_0^{j+\frac{1}{2}} + \hat{\sigma}_1^{j+\frac{1}{2}}), w) &= (\hat{f}^{j+\frac{1}{2}}, w).\end{aligned}$$

By subtracting (4.37–4.39) from the above three equations and using definitions of $(E_{\sigma_0}, E_{\sigma_1}, E_v, E_p)$ in (4.42–4.44), we have

$$\begin{aligned}(A_0\bar{\partial}_t E_{\sigma_0}^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div}\tau_0, \hat{E}_v^{j+\frac{1}{2}}) + (\hat{E}_p^{j+\frac{1}{2}}, \tau_0) &= (A_0(\bar{\partial}_t\sigma_0^{j+\frac{1}{2}} - \hat{\dot{\sigma}}_0^{j+\frac{1}{2}}), \tau_0),\\ (A_1\hat{E}_{\sigma_1}^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div}\tau_1, \hat{E}_v^{j+\frac{1}{2}}) + (\hat{E}_p^{j+\frac{1}{2}}, \tau_1) &= 0,\\ (\rho\bar{\partial}_t E_v^{j+\frac{1}{2}}, w) - (\operatorname{div}(\hat{E}_{\sigma_0}^{j+\frac{1}{2}} + \hat{E}_{\sigma_1}^{j+\frac{1}{2}}), w) &= (\rho(\bar{\partial}_t v^j - \hat{\dot{v}}^{j+\frac{1}{2}}), w).\end{aligned}$$

If we consider (4.42–4.44), and the facts $(\operatorname{div}\tau, e_v^P) = (\operatorname{div} e_{\sigma_i}^P, w) = 0$, $i = 0, 1$ for $\tau \in M_h$, $w \in V_h$, and adding the first two equations for simplicity, then we get

$$(A_0\bar{\partial}_t\theta_{\sigma_0}^{j+\frac{1}{2}},\tau_0)+(A_1\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}},\tau_1)+(\operatorname{div}(\tau_0+\tau_1),\hat{\theta}_v^{j+\frac{1}{2}})+(\hat{\theta}_p^{j+\frac{1}{2}},\tau_0+\tau_1) \quad (4.46)$$
$$=(A_0(\omega_1^{j+\frac{1}{2}}+\omega_2^{j+\frac{1}{2}})+\omega_4^{j+\frac{1}{2}},\tau_0)+(A_1\omega_3^{j+\frac{1}{2}}+\omega_4^{j+\frac{1}{2}},\tau_1),$$
$$(\rho\bar{\partial}_t\theta_v^{j+\frac{1}{2}},w)-(\operatorname{div}(\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}+\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}),w)=(\omega_5^{j+\frac{1}{2}}+\omega_6^{j+\frac{1}{2}},w), \quad (4.47)$$

where

$$\begin{aligned}&\omega_1^{j+\frac{1}{2}}=\bar{\partial}_t\sigma_0^{j+\frac{1}{2}}-\hat{\sigma}_0^{j+\frac{1}{2}},\quad \omega_2^{j+\frac{1}{2}}=-\bar{\partial}_te_{\sigma_0}^{P,j+\frac{1}{2}},\quad \omega_3^{j+\frac{1}{2}}=-\hat{e}_{\sigma_1}^{P,j+\frac{1}{2}},\\&\omega_4^{j+\frac{1}{2}}=-\hat{e}_p^{P,j+\frac{1}{2}},\quad \omega_5^{j+\frac{1}{2}}=\rho(\bar{\partial}_tv^{j+\frac{1}{2}}-\hat{v}^{j+\frac{1}{2}}),\quad \omega_6^{j+\frac{1}{2}}=-\rho\bar{\partial}_te_v^{P,j+\frac{1}{2}}.\end{aligned} \quad (4.48)$$

Letting $\tau_0=\hat{\theta}_{\sigma_0}^{j+1/2}$, $\tau_1=\hat{\theta}_{\sigma_1}^{j+1/2}$ in (4.46), $w=\hat{\theta}_v^{j+1/2}$ in (4.47), and adding these equations,

$$\begin{aligned}&\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2+2\Delta t\|\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}\|_{A_1}^2+\|\theta_v^{j+1}\|_\rho^2-(\|\theta_{\sigma_0}^j\|_{A_0}^2+\|\theta_v^j\|_\rho^2)\\&=2\Delta t(A_0(\omega_1^{j+\frac{1}{2}}+\omega_2^{j+\frac{1}{2}})+\omega_4^{j+\frac{1}{2}},\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}})\\&\quad+2\Delta t(A_1\omega_3^{j+\frac{1}{2}}+\omega_4^{j+\frac{1}{2}},\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}})+2\Delta t(\omega_5^{j+\frac{1}{2}}+\omega_6^{j+\frac{1}{2}},\hat{\theta}_v^{j+\frac{1}{2}}).\end{aligned} \quad (4.49)$$

By Young's inequality and the coercivity of A_1, there is $c>0$ independent of h and Δt such that

$$2\Delta t(A_1\omega_3^{j+\frac{1}{2}}+\omega_4^{j+\frac{1}{2}},\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}})\le 2\Delta t\|\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}\|_{A_1}^2+c\Delta t(\|\omega_3^{j+\frac{1}{2}}\|^2+\|\omega_4^{j+\frac{1}{2}}\|^2).$$

If we use this inequality to (4.49), then we can get

$$\begin{aligned}&(\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2+\|\theta_v^{j+1}\|_\rho^2)-(\|\theta_{\sigma_0}^j\|_{A_0}^2+\|\theta_v^j\|_\rho^2)\\&\le 2\Delta t(A_0(\omega_1^{j+\frac{1}{2}}+\omega_2^{j+\frac{1}{2}})+\omega_4^{j+\frac{1}{2}},\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}})+2\Delta t(\omega_5^{j+\frac{1}{2}}+\omega_6^{j+\frac{1}{2}},\hat{\theta}_v^{j+\frac{1}{2}})\\&\quad+c\Delta t(\|\omega_3^{j+\frac{1}{2}}\|^2+\|\omega_4^{j+\frac{1}{2}}\|^2).\end{aligned} \quad (4.50)$$

By the Cauchy–Schwarz inequality, regarding $\|\theta_{\sigma_0}^j\|_{A_0}^2+\|\theta_v^j\|_\rho^2$ as Q_i^2, we can apply Lemma 2.16 and have one of the followings:

$$(\|\theta^i_{\sigma_0}\|^2_{A_0} + \|\theta^i_v\|^2_\rho)^{\frac{1}{2}} \leq c\Delta t \sum_{j=0}^{i-1} \|\omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}, \omega_5^{j+\frac{1}{2}}, \omega_6^{j+\frac{1}{2}}\|, \tag{4.51}$$

$$(\|\theta^i_{\sigma_0}\|^2_{A_0} + \|\theta^i_v\|^2_\rho)^{\frac{1}{2}} \leq c(\Delta t)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1} (\|\omega_3^{j+\frac{1}{2}}\|^2 + \|\omega_4^{j+\frac{1}{2}}\|^2) \right)^{\frac{1}{2}}. \tag{4.52}$$

We claim that

$$\Delta t \sum_{j=0}^{i-1} \|\omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}, \omega_5^{j+\frac{1}{2}}, \omega_6^{j+\frac{1}{2}}\| \leq c(\Delta t^2 + h^m), \tag{4.53}$$

$$(\Delta t)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1} (\|\omega_3^{j+\frac{1}{2}}\|^2 + \|\omega_4^{j+\frac{1}{2}}\|^2) \right)^{\frac{1}{2}} \leq c(\Delta t^2 + h^m), \tag{4.54}$$

hold and note that these estimates imply (4.45) for $\theta^i_{\sigma_0}$ by the coercivity of A_0.

By applying the Taylor expansion (3.53), definitions of ω terms in (4.48), and the projection error estimates, the estimates (4.53–4.54) are obtained from the estimates,

$$\Delta t\|\omega_1^{j+\frac{1}{2}}\| = \frac{1}{2}\|2\sigma_0^{j+1} - 2\sigma_0^j - \Delta t\dot{\sigma}_0^{j+1} - \Delta t\dot{\sigma}_0^j\| \leq c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{\sigma}_0\|\, ds, \tag{4.55}$$

$$\Delta t\|\omega_5^{j+\frac{1}{2}}\| = \frac{1}{2}\|\rho(2v^{j+1} - 2v^j - \Delta t\dot{v}^{j+1} - \Delta t\dot{v}^j)\| \leq c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{v}\|\, ds, \tag{4.56}$$

$$\Delta t\|\omega_2^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_{\sigma_0}^{P,j+\frac{1}{2}}\| = \left\| \int_{t_j}^{t_{j+1}} \dot{e}^P_{\sigma_0}\, ds \right\| \leq ch^m \int_{t_j}^{t_{j+1}} \|\dot{\sigma}_0\|_m\, ds, \tag{4.57}$$

$$\Delta t\|\omega_6^{j+\frac{1}{2}}\| = \Delta t\|\rho\bar{\partial}_t e_v^{P,j+\frac{1}{2}}\| = \left\| \int_{t_j}^{t_{j+1}} \rho\dot{e}^P_v\, ds \right\| \leq ch^m \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m\, ds, \tag{4.58}$$

$$\|\omega_3^{j+\frac{1}{2}}\| = ch^m\|\sigma_1\|_{L^\infty H^m}, \qquad \|\omega_4^{j+\frac{1}{2}}\| = ch^m\|p\|_{L^\infty H^m}. \tag{4.59}$$

It is easy to see that (4.55–4.59) imply (4.53). The estimate (4.54) comes from

(4.59) with

$$\left(\sum_{j=0}^{i-1}\|\omega_3^{j+\frac{1}{2}}\|^2\right)^{\frac{1}{2}} \le cN^{\frac{1}{2}}h^m\|\sigma_1\|_{L^\infty H^m} \le cT_0(\Delta t)^{-\frac{1}{2}}h^m\|\sigma_1\|_{L^\infty H^m}, \quad (4.60)$$

$$\left(\sum_{j=0}^{i-1}\|\omega_4^{j+\frac{1}{2}}\|^2\right)^{\frac{1}{2}} \le cN^{\frac{1}{2}}h^m\|p\|_{L^\infty H^m} \le cT_0(\Delta t)^{-\frac{1}{2}}h^m\|p\|_{L^\infty H^m}. \quad (4.61)$$

Hence the estimate (4.45) for $\theta^i_{\sigma_0}$, θ^i_v is proved.

Now we consider estimates in (4.45) for $\hat{\theta}^{i+1/2}_{\sigma_1}$, and $\hat{\theta}^{i+1/2}_p$. For the estimate of $\hat{\theta}^{j+1/2}_{\sigma_1}$, let $-\tau_0 = \tau_1 = \hat{\theta}^{j+1/2}_{\sigma_1}$ in (4.46). Then

$$\begin{aligned}
\|\hat{\theta}^{j+\frac{1}{2}}_{\sigma_1}\|^2 &\le c\|\hat{\theta}^{j+\frac{1}{2}}_{\sigma_1}\|^2_{A_1} \\
&= c(A_0\bar{\partial}_t\theta^{j+\frac{1}{2}}_{\sigma_0} - A_0(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + A_1\omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \hat{\theta}^{j+\frac{1}{2}}_{\sigma_1}) \\
&\le c\|\bar{\partial}_t\theta^{j+\frac{1}{2}}_{\sigma_0}, \omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_3^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}\|\|\hat{\theta}^{j+\frac{1}{2}}_{\sigma_1}\|.
\end{aligned}$$

For $\hat{\theta}^{j+1/2}_p$, in (4.46), let $\tau_1 = 0$ and take τ_0 to be $\operatorname{div}\tau_0 = 0$, $(\tau_0, \hat{\theta}^{j+1/2}_p) = \|\hat{\theta}^{j+1/2}_p\|^2$ so that $\|\tau_0\| \le c\|\hat{\theta}^{j+1/2}_p\|$ by Corollary 2.4. Then

$$\begin{aligned}
\|\hat{\theta}^{j+\frac{1}{2}}_p\|^2 &= (-A_0\bar{\partial}_t\theta^{j+\frac{1}{2}}_{\sigma_0} + \omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}} + \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \tau_0) \\
&\le c\|\bar{\partial}_t\theta^{j+\frac{1}{2}}_{\sigma_0}, \omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_3^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}\|\|\hat{\theta}^{j+\frac{1}{2}}_p\|.
\end{aligned}$$

Consequently, we get $\|\hat{\theta}^{j+\frac{1}{2}}_{\sigma_1}, \hat{\theta}^{j+\frac{1}{2}}_p\| \le \|\bar{\partial}_t\theta^{j+\frac{1}{2}}_{\sigma_0}, \omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_3^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}\|$.

From the estimates (4.55), (4.57), and (4.59), it suffices to show $\|\bar{\partial}_t\theta^{j+1/2}_{\sigma_0}\| \le c(\Delta t^2 + h^m)$ to complete the proof of (4.45).

To estimate $\|\bar{\partial}_t\theta^{j+1/2}_{\sigma_0}\|$, we consider the differences of (4.46), (4.47) with indices j and $j-1$, which are

$$\begin{aligned}
&\Delta t(A_0\bar{\partial}^2_t\theta^j_{\sigma_0}, \tau_0) + \frac{1}{2}(A_1(\theta^{j+1}_{\sigma_1} - \theta^{j-1}_{\sigma_1}), \tau_1) + \frac{1}{2}(\operatorname{div}(\tau_0 + \tau_1), \theta^{j+1}_v - \theta^{j-1}_v) \\
&\quad + \frac{1}{2}(\theta^{j+1}_p - \theta^{j-1}_p, \tau_0 + \tau_1) = (A_0(\eta^j_1 + \eta^j_2) + \eta^j_4, \tau_0) + (A_1\eta^j_3 + \eta^j_4, \tau_1), \quad (4.62)
\end{aligned}$$

$$\Delta t(\bar{\partial}^2_t\theta^j_v, w) - \frac{1}{2}(\operatorname{div}(\theta^{j+1}_{\sigma_0} - \theta^{j-1}_{\sigma_0} + \theta^{j+1}_{\sigma_1} - \theta^{j-1}_{\sigma_1}), w) = (\eta^j_5 + \eta^j_6, w), \quad (4.63)$$

where $\eta_l^j = \omega_l^{j+1/2} - \omega_l^{j-1/2}$, $l = 1, \cdots, 6$. Let

$$\tau_0 = \frac{\theta_{\sigma_0}^{j+1} - \theta_{\sigma_0}^{j-1}}{\Delta t}, \quad \tau_1 = \frac{\theta_{\sigma_1}^{j+1} - \theta_{\sigma_1}^{j-1}}{\Delta t}, \quad w = \frac{\theta_v^{j+1} - \theta_v^{j-1}}{\Delta t},$$

in (4.62) and (4.63), and add them. Then, after cancellations from the facts

$$\frac{Y^{j+1} - Y^{j-1}}{\Delta t} = \bar{\partial}_t Y^{j+\frac{1}{2}} + \bar{\partial}_t Y^{j-\frac{1}{2}}, \quad \Delta t \bar{\partial}_t^2 Y^j = \bar{\partial}_t Y^{j+\frac{1}{2}} - \bar{\partial}_t Y^{j-\frac{1}{2}},$$

for a sequence $\{Y^j\}$, we can get

$$\begin{aligned} &\|\bar{\partial}_t \theta_{\sigma_0}^{j+\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{j+\frac{1}{2}}\|_\rho^2 + \Delta t \|\bar{\partial}_t \theta_{\sigma_1}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_1}^{j-\frac{1}{2}}\|_{A_1}^2 - (\|\bar{\partial}_t \theta_{\sigma_0}^{j-\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{j-\frac{1}{2}}\|_\rho^2) \\ &\quad = (A_0(\eta_1^j + \eta_2^j) + \eta_4^j, \bar{\partial}_t \theta_{\sigma_0}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_0}^{j-\frac{1}{2}}) + (A_1 \eta_3^j + \eta_4^j, \bar{\partial}_t \theta_{\sigma_1}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_1}^{j-\frac{1}{2}}). \end{aligned}$$

We apply the following inequality, which is obtained from Young's inequality, to the above equality,

$$\begin{aligned} &(A_1 \eta_3^j + \eta_4^j, \bar{\partial}_t \theta_{\sigma_1}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_1}^{j-\frac{1}{2}}) \\ &\qquad \le \Delta t \|\bar{\partial}_t \theta_{\sigma_1}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_1}^{j-\frac{1}{2}}\|_{A_1}^2 + (c/\Delta t)(\|\eta_3^j\|^2 + \|\eta_4^j\|^2), \end{aligned}$$

then we have

$$\begin{aligned} &(\|\bar{\partial}_t \theta_{\sigma_0}^{j+\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{j+\frac{1}{2}}\|_\rho^2) - (\|\bar{\partial}_t \theta_{\sigma_0}^{j-\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{j-\frac{1}{2}}\|_\rho^2) \\ &\qquad \le (A_0(\eta_1^j + \eta_2^j) + \eta_4^j, \bar{\partial}_t \theta_{\sigma_0}^{j+\frac{1}{2}} + \bar{\partial}_t \theta_{\sigma_0}^{j-\frac{1}{2}}) + (c/\Delta t)(\|\eta_3^j\|^2 + \|\eta_4^j\|^2). \end{aligned}$$

By Lemma 2.16, we have either

$$\begin{aligned} &(\|\bar{\partial}_t \theta_{\sigma_0}^{i+\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{i+\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \\ &\quad \le (\|\bar{\partial}_t \theta_{\sigma_0}^{\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} + \sum_{j=0}^{i-1} \|\eta_1^j, \eta_2^j, \eta_4^j, \eta_5^j, \eta_6^j\|, \end{aligned} \tag{4.64}$$

or

$$\begin{aligned} &(\|\bar{\partial}_t \theta_{\sigma_0}^{i+\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{i+\frac{1}{2}}\|^2)^{\frac{1}{2}} \\ &\quad \le (\|\bar{\partial}_t \theta_{\sigma_0}^{\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} + c(\Delta t)^{-1/2} \left(\sum_{j=0}^{i-1} (\|\eta_3^j\|^2 + \|\eta_4^j\|^2) \right)^{1/2}. \end{aligned} \tag{4.65}$$

Now our goal, the estimate $\|\bar{\partial}_t \theta_{\sigma_0}^{i+1/2}\| \le c(\Delta t^2 + h^m)$, is reduced to proving

$$(\|\bar{\partial}_t \theta_{\sigma_0}^{\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \le c(\Delta t^2 + h^m), \tag{4.66}$$

$$\sum_{j=1}^{i-1} \|\eta_1^j, \eta_2^j, \eta_4^j, \eta_5^j, \eta_6^j\| \le c(\Delta t^2 + h^m), \tag{4.67}$$

$$\left(\sum_{j=1}^{i-1}(\|\eta_3^j\|^2 + \|\eta_4^j\|^2)\right)^{\frac{1}{2}} \le c(\Delta t)^{\frac{1}{2}} h^m. \tag{4.68}$$

If we use the facts $\bar{\partial}_t \theta_{\sigma_0}^{1/2} = \theta_{\sigma_0}^1/\Delta t$, $\bar{\partial}_t \theta_v^{1/2} = \theta_v^1/\Delta t$, and the inequality (4.51), then we get

$$(\|\bar{\partial}_t \theta_{\sigma_0}^{\frac{1}{2}}\|_{A_0}^2 + \|\bar{\partial}_t \theta_v^{\frac{1}{2}}\|_\rho^2)^{\frac{1}{2}} \le \|\omega_1^{\frac{1}{2}}, \omega_2^{\frac{1}{2}}, \omega_4^{\frac{1}{2}}, \omega_5^{\frac{1}{2}}, \omega_6^{\frac{1}{2}}\|.$$

By (4.55–4.59), we have

$$\|\omega_1^{\frac{1}{2}}\| \le c\Delta t^2 \|\dddot{\sigma}_0\|_{L^\infty L^2}, \quad \|\omega_5^{\frac{1}{2}}\| \le c\Delta t^2 \|\dddot{v}\|_{L^\infty L^2},$$
$$\|\omega_2^{\frac{1}{2}}\| \le ch^m \|\dot{\sigma}_0\|_{L^\infty H^m}, \quad \|\omega_4^{\frac{1}{2}}\| \le ch^m \|p\|_{L^\infty H^m}, \quad \|\omega_6^{\frac{1}{2}}\| \le ch^m \|\dot{v}\|_{L^\infty H^m},$$

which completes the proof of (4.66).

Now we turn to the proof of (4.67). First, we obtain $\sum_{j=0}^{i-1} \|\eta_1^j\| \le c\Delta t^2$ with c depending on $\|\sigma^{(4)}\|_{L^1 L^2}$ from the Taylor expansion formula (3.54). Similarly, we have $\sum_{j=0}^{i-1} \|\eta_5^j\| \le c\Delta t^2$ with c depending on $\|v^{(4)}\|_{L^1 L^2}$. We also see

$$\|\eta_2^j\| = \frac{1}{\Delta t}\|e_{\sigma_0}^{P,j+1} + e_{\sigma_0}^{P,j-1} - 2e_{\sigma_0}^{P,j}\| \le \int_{t_{j-1}}^{t_{j+1}} \|\ddot{e}_{\sigma_0}^P\|\, ds \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\ddot{\sigma}_0\|_m\, ds,$$
$$\|\eta_6^j\| = \frac{1}{\Delta t}\|\rho(e_v^{P,j+1} + e_v^{P,j-1} - 2e_v^{P,j})\| \le \int_{t_{j-1}}^{t_{j+1}} \|\rho \ddot{e}_v^P\|\, ds \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\ddot{v}\|_m\, ds,$$

and conclude that $\sum_{j=0}^{m-1}(\|\eta_2^j\| + \|\eta_6^j\|) \le ch^m$. Note also that

$$\|\eta_4^j\| = \|e_p^{P,j+1} - e_p^{P,j-1}\| = \left\|\int_{t_{j-1}}^{t_{j+1}} \dot{e}_p^P\, ds\right\| \le ch^m \int_{t_{j-1}}^{t_{j+1}} \|\dot{p}\|_m\, ds,$$

so combining all the above estimates of η terms, (4.67) is proved.

For (4.68), by the Cauchy–Schwarz inequality, we get

$$\|\eta_3^j, \eta_4^j\|^2 \le ch^{2m}\left(\int_{t_{j-1}}^{t_{j+1}} \|\dot\sigma_1, \dot p\|_m ds\right)^2 \le c\Delta t h^{2m}\int_{t_{j-1}}^{t_{j+1}} \|\dot\sigma_1, \dot p\|_m^2\, ds,$$

for each $1 \le j \le N-1$ and we have

$$\left(\sum_{j=1}^{i-1}(\|\eta_3^j\|^2 + \|\eta_4^j\|^2)\right)^{\frac{1}{2}} \le cT_0(\Delta t)^{\frac{1}{2}} h^m \|\dot\sigma_1, \dot p\|_{L^2H^m}.$$

Thus we proved $(\|\bar\partial_t \sigma_0^{i+1/2}\|_{A_0}^2 + \|\bar\partial_t v^{i+1/2}\|_\rho^2)^{1/2} \le c(\Delta t^2 + h^m)$ and therefore the proof of (4.45) is completed. □

4.5 Error analysis for the GG elements

In this section we discuss the error analysis for the GG elements.

Theorem 4.10. *Let* (M_h, V_h, K_h) *be the* GG *elements of degree* $k \ge 1$*,* m *be an integer of which* $1 \le m \le k+1$*, and* $m' = m - \delta_{m,k+1}$ *where* $\delta_{m,k+1}$ *is the Kronecker delta. Suppose* $\|\rho\|_{W_h^{1,\infty}} < \infty$ *for* $\|\rho\|_{W_h^{1,\infty}}$ *defined in* (3.64) *and*

$$\begin{aligned} \sigma_0 &\in W^{2,1}([0,T_0]; H^m) \cap W^{4,1}([0,T_0]; L^2), \\ v &\in W^{2,1}([0,T_0]; H^{m'}) \cap W^{4,1}([0,T_0]; L^2), \\ \sigma_1, p &\in W^{1,1}([0,T_0]; H^m) \cap W^{3,1}([0,T_0]; L^2). \end{aligned} \tag{4.69}$$

Assume the initial data are given to be $\Sigma_0^0 = \tilde\Pi_h \sigma_0(0)$*,* $\Sigma_1^0 = \tilde\Pi_h\sigma_1(0)$*,* $V^0 = P_h v(0)$*,* $P^0 = P_h' p(0)$*. Then the fully discrete solution* $(\Sigma_0^j, \Sigma_1^j, V^j, P^j)$ *in* (4.37–4.39) *is well-defined and for all* $1 \le j \le N$*,*

$$\|\sigma_0^j - \Sigma_0^j, P_h v^j - V^j, \hat\sigma_1^{j-\frac{1}{2}} - \hat\Sigma_1^{j-\frac{1}{2}}, \hat p^{j-\frac{1}{2}} - \hat P^{j-\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \tag{4.70}$$

$$\|v^j - V^j\| \le c(\Delta t^2 + h^{m'}), \tag{4.71}$$

where the constants c *depend on*

$$\|\sigma_0\|_{W^{2,1}H^m \cap W^{4,1}L^2}, \|\sigma_1, p\|_{W^{1,2}H^m\cap W^{3,1}L^2}, \|v\|_{W^{2,1}H^{m'}\cap W^{4,1}L^2},$$

and A_0, A_1, $\|\rho\|_{W_h^{1,\infty}}$, T_0 but is independent of h and Δt.

In the previous section, the proof of Theorem 4.8 consists of the proof of well-definedness of $(\Sigma_0^j, \Sigma_1^j, V^j, P^j)$ and the proof of error estimates. Since we can prove that $(\Sigma_0^j, \Sigma_1^j, V^j, P^j)$ is well-defined with the same argument as before, we only discuss the a priori error estimates.

For the error estimates, we consider the projection errors $(e_{\sigma_0}^{P,j}, e_{\sigma_1}^{P,j}, e_v^{P,j}, e_p^{P,j})$ and the approximation errors $(\theta_{\sigma_0}^j, \theta_{\sigma_1}^j, \theta_v^j, \theta_p^j)$ as in (4.42–4.44).

Lemma 4.11. *There exists a constant $c > 0$ independent of s and h so that the following inequalities hold.*

$$\|e_{\sigma_i}^P\| \leq ch^m \|\sigma_i\|_m, \qquad 1 \leq m \leq k+1, \quad i = 0, 1, \tag{4.72}$$

$$\|e_v^P\| \leq ch^m \|v\|_m, \qquad 0 \leq m \leq k, \tag{4.73}$$

$$\|e_p^P\| \leq ch^m \|p\|_m, \qquad 0 \leq m \leq k+1. \tag{4.74}$$

Similar estimates hold for the time derivatives of σ_i, v, p, respectively, as in Theorem 4.6.

Since the proof of this theorem is almost same as the one of Theorem 3.11, we omit details. For Theorem 4.10 we only need to consider a priori estimates of the approximation errors $(\theta_{\sigma_0}^j, \theta_{\sigma_1}^j, \theta_v^j, \theta_p^j)$.

Theorem 4.12. *Suppose the assumptions in Theorem 4.10 hold and $\theta_{\sigma_0}^i$, $\theta_{\sigma_1}^i$, θ_v^i, θ_r^i are defined as in (4.42–4.44). Then there exists a constant $c > 0$ independent of h and Δt so that*

$$\|\theta_{\sigma_0}^i, \hat{\theta}_{\sigma_1}^{i-\frac{1}{2}}, \theta_v^i, \hat{\theta}_p^{i-\frac{1}{2}}\| \leq c(\Delta t^2 + h^m), \qquad 1 \leq i \leq N, \quad 1 \leq m \leq k+1. \tag{4.75}$$

Proof. Since the proof is almost same as the one of Theorem 4.9, we only sketch it with some explanations of the steps which need to be modified.

For the a priori estimates, we can follow same argument in the proof of Theorem 4.8 and obtain (4.49). Let ρ_c be the orthogonal L^2 projection of ρ into the space of piecewise constant functions associated to $\mathcal{T}_h$. Define $\tilde{\omega}_6^{j+1/2} = (\rho - \rho_c)\bar{\partial}_t e_v^{P,j+1/2}$ and note that

$$(\omega_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\rho\bar{\partial}_t e_v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = ((\rho - \rho_c)\bar{\partial}_t e_v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\tilde{\omega}_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}),$$

because $\rho_c\bar{\partial}_t e_v^{P,j+1/2} \perp V_h$ from the definition of $\bar{\partial}_t e_v^{P,j+1/2}$ and the fact that $e_v^{P,j} \perp V_h$ for all $0 \leq j \leq N$. Therefore we have an equality analogous to (4.50) as

$$
\begin{aligned}
&\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2 + 2\Delta t\|\hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}\|_{A_1}^2 + \|\theta_v^{j+1}\|_\rho^2 - (\|\theta_{\sigma_0}^j\|_{A_0}^2 + \|\theta_v^j\|_\rho^2) \\
&= 2\Delta t(A_0(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + \omega_4^{j+\frac{1}{2}}, \hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}) \\
&\quad + 2\Delta t(A_1\omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}) + 2\Delta t(\omega_5^{j+\frac{1}{2}} + \tilde{\omega}_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).
\end{aligned}
$$

Repeating the steps from (4.50) to (4.52), we need to show

$$
c\Delta t \sum_{j=0}^{i-1} \|\omega_1^{j+\frac{1}{2}}, \omega_2^{j+\frac{1}{2}}, \omega_4^{j+\frac{1}{2}}, \omega_5^{j+\frac{1}{2}}, \tilde{\omega}_6^{j+\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \tag{4.76}
$$

$$
(\Delta t)^{1/2} \left(\sum_{j=0}^{i-1} (\|\omega_3^{j+\frac{1}{2}}\|^2 + \|\omega_4^{j+\frac{1}{2}}\|^2) \right)^{\frac{1}{2}} \le c(\Delta t^2 + h^m), \tag{4.77}
$$

for $1 \le m \le k+1$, to prove (4.75) for $\|\theta_{\sigma_0}^i\|$ and $\|\theta_v^i\|$.

In order to show (4.76), we use (4.55–4.57) and (4.59) for the estimates of $\omega_l^{j+1/2}$, $l = 1, \cdots 5$. For $\tilde{\omega}_6^{j+1/2}$, we use

$$
\begin{aligned}
\Delta t\|\tilde{\omega}_6^{j+\frac{1}{2}}\| = \Delta t\|(\rho - \rho_c)\bar{\partial}_t v^{P,j}\| &= \left\| \int_{t_j}^{t_{j+1}} (\rho - \rho_c)(\dot{v}_h^P - \dot{v})\, ds \right\| \\
&\le ch^{m+1}\|\rho\|_{W_h^{1,\infty}} \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m \, ds, \quad 1 \le m \le k.
\end{aligned}
$$

In order to show (4.77), we can use (4.60–4.61).

For the proof of (4.75) for $\|\hat{\theta}_{\sigma_1}^{i-1/2}, \hat{\theta}_p^{i-1/2}\|$, we repeat the argument in the proof of Theorem 4.9 with the only difference that $\omega_6^{j+1/2}$ is replaced by $\tilde{\omega}_6^{j+1/2}$. Since the proof is same, we omit details. □

We also show that a simple postprocessing can be used to find a more accurate numerical solution of v. As we have done in elastodynamics problem in the previous chapter, for V_h in the GG elements, let V_h^* be the space of (possibly discontinuous) piecewise polynomials of one degree higher than V_h and $\tilde{V}_h$ be the orthogonal complement of V_h in V_h^*. The orthogonal L^2 projections onto V_h^* and $\tilde{V}_h$ are denoted by P_h^* and $\tilde{P}_h$, respectively.

Theorem 4.13. *For simplicity, we assume that Theorem 4.10 holds with* $m =$

$k+1$ and $\|v\|_{W^{2,1}H^{k+1}\cap W^{4,1}L^2} < \infty$. We define $v_h^{*,i+1/2} \in V_h^*$ to be

$$(\operatorname{grad}_h v_h^{*,i+\frac{1}{2}}, \operatorname{grad}_h w) = (A_1\hat{\Sigma}_1^{i+\frac{1}{2}} + \hat{P}^{i+\frac{1}{2}}, \operatorname{grad}_h w), \quad w \in \tilde{V}_h, \tag{4.78}$$

$$(v_h^{*,i+\frac{1}{2}}, w) = (\hat{V}^{i+\frac{1}{2}}, w), \quad w \in V_h, \tag{4.79}$$

where grad_h is the piecewise gradient operator adapted to the triangulation $\mathcal{T}_h$ for each $0 \le i \le N-1$. Then for each $0 \le i \le N-1$,

$$\begin{aligned}\|v_h^{*,i+\frac{1}{2}} - \hat{v}^{i+\frac{1}{2}}\| \\ \le c(\Delta t^2 + h^{k+1})(\|\sigma_0, v\|_{W^{2,1}H^{k+1}\cap W^{4,1}L^2} + \|\sigma_1, p\|_{W^{1,1}H^{k+1}\cap W^{3,1}L^2}).\end{aligned}$$

We do not show its proof here because it is almost same as that of Theorem 3.13 with minor modifications.

4.6 Numerical results

We present numerical results in this section. For numerical computations, we use $\Omega = [0,1]\times[0,1]$ and the AFW elements of degree $k=2$ in all of our numerical computations. We assume that the medium is homogeneous with density $\rho = 1$, and the constitutive equations for A_0 and A_1 are the forms of isotropic materials introduced in (2.7) with constants λ_0, λ_1, μ_0, and μ_1. For each mesh size h, we take $\Delta t = h$ for time step Δt and the expected order of convergence from our analysis is 2. All numerical results are about the L^2 errors at time $T_0 = 1$.

Although we only proved that error estimates of σ_1, and p at the midpoints of time steps are $O(\Delta t^2 + h^m)$, it is not difficult to show, using trapezoidal rule, that error bounds of same order of accuracy can be obtained at t_j, $1 \le j \le N$.

The implementations are done using Dolfin Python module [1] of FEniCS project [2, 38].

Example 4.14. Consider the displacement field

$$u(t,x,y) = \begin{pmatrix} \sin t \sin(\pi x)\sin(\pi y) \\ \sin t x(1-x)y(1-y)\end{pmatrix}, \tag{4.80}$$

and boundary conditions are the homogeneous displacement boundary conditions. The numerical result for (4.81) is in Table 4.1.

Example 4.15. As an example of inhomogeneous displacement boundary conditions, we use

$$u(t,x,y) = \begin{pmatrix} e^{-y}\sin x \cos t \\ e^{t+x} \end{pmatrix}, \tag{4.81}$$

and the variational form proposed in (4.11) is used. The numerical results are shown in Table 4.2. We see that the order of convergence is same as that in our error analysis.

Example 4.16. For a nonsmooth solution with inhomogeneous displacement boundary conditions, we consider a solution with displacement

$$u(t,x,y) = \begin{pmatrix} x^{\frac{7}{3}}\cos t \\ x^{\frac{7}{3}}(1+t^2) \end{pmatrix}. \tag{4.82}$$

Then the corresponding σ_0, σ_1, p are

$$\begin{aligned}
\sigma_0 &= \begin{pmatrix} 0 & \frac{7}{3}(x^{\frac{4}{3}}(1+t^2) + y^{\frac{4}{3}}\cos t) \\ \frac{7}{3}(x^{\frac{4}{3}}(1+t^2) + y^{\frac{4}{3}}\cos t) & 0 \end{pmatrix}, \\
\sigma_1 &= \begin{pmatrix} 0 & \frac{70}{3}(2tx^{\frac{4}{3}} - y^{\frac{4}{3}}\sin t) \\ \frac{70}{3}(2tx^{\frac{4}{3}} - y^{\frac{4}{3}}\sin t) & 0 \end{pmatrix}, \\
p &= \begin{pmatrix} 0 & -\frac{7}{6}(2tx^{\frac{4}{3}} + y^{\frac{4}{3}}\sin t) \\ \frac{7}{6}(2tx^{\frac{4}{3}} + y^{\frac{4}{3}}\sin t) & 0 \end{pmatrix},
\end{aligned}$$

so they are in $H^{11/6-\delta}$ in space for any $\delta > 0$. We again use the variational form proposed in (4.11), and the numerical results are shown in Table 4.3. We can see that the orders of convergence of σ_0, σ_1, p errors are limited by $11/6 \approx 1.83$ but the order of convergence of v error is still 2. As in the nonsmooth solution

Table 4.1: Order of convergence for displacement as in (4.80) ($\lambda_0 = \mu_0 = 1$, $\lambda_1 = \mu_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|p - p_h\|$	
	error	order	error	order	error	order	error	order
4	6.76e-02	–	4.05e-01	–	1.04e-02	–	1.59e-02	–
8	1.38e-02	2.29	7.19e-02	2.49	2.67e-03	1.96	4.15e-03	1.94
16	3.19e-03	2.11	1.57e-02	2.19	6.74e-04	1.98	1.06e-03	1.97
32	7.78e-04	2.04	3.76e-03	2.06	1.69e-04	1.99	2.67e-04	1.99
64	1.93e-04	2.01	9.28e-04	2.02	4.24e-05	2.00	6.69e-05	2.00

Table 4.2: Order of convergence for displacement as in (4.81) ($\lambda_0 = \mu_0 = 1$, $\lambda_1 = \mu_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|p - p_h\|$	
	error	order	error	order	error	order	error	order
4	2.27e-02	–	6.65e-02	–	8.25e-03	–	4.58e-03	–
8	5.64e-03	2.01	1.55e-02	2.10	2.04e-03	2.02	1.12e-03	2.03
16	1.41e-03	2.00	3.73e-03	2.06	5.06e-04	2.01	2.78e-04	2.01
32	3.52e-04	2.00	9.14e-04	2.03	1.26e-04	2.01	6.92e-05	2.01
64	8.79e-05	2.00	2.26e-04	2.02	3.14e-05	2.00	1.72e-05	2.00

Table 4.3: Order of convergence for displacement as in (4.82) ($\lambda_0 = \mu_0 = 1$, $\lambda_1 = \mu_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|p - p_h\|$	
	error	order	error	order	error	order	error	order
4	5.34e-02	–	6.42e-01	–	3.69e-02	–	2.02e-02	–
8	1.58e-02	1.76	1.85e-01	1.80	9.04e-03	2.03	6.20e-03	1.70
16	4.54e-03	1.80	5.25e-02	1.82	2.23e-03	2.02	1.83e-03	1.76
32	1.28e-03	1.82	1.48e-02	1.82	5.55e-04	2.01	5.25e-04	1.80
64	3.61e-04	1.83	4.18e-03	1.83	1.38e-04	2.00	1.50e-04	1.81

example for elastodynamics, this result gives a motivation to find a better error analysis for v error.

Chapter 5

Mixed methods for the Maxwell and generalized Zener models

5.1 Introduction

In this chapter, we consider mixed methods for the Maxwell and generalized Zener models. We use a unified framework such that the Maxwell model is a special case of the Zener model, so we shall show a complete analysis only for the Zener model. Since the generalized Zener model is a parallel connection of multiple Zener components, it is straightforward to extend our analysis to the generalized Zener model.

Although equations of the Maxwell and Zener models are similar in our framework, they represent very different physical features. The Maxwell model is the simplest model for viscoelastic fluids, but the Zener model, which is also called the standard linear solid model, is a model one for viscoelastic solids.

There are several previous works for these two models using mixed methods. In [16], Bécache, Joly, and Tsogka studied a priori error analysis and PML implementation of Zener model with strongly symmetric rectangular elements developed in [15]. They used a generalized Kelvin–Voigt form of the Zener model and the difference of elastic and viscoelastic stresses is used as an internal variable for mixed formulation. In [44], Rognes and Winther studied quasistatic problems, i.e., mass density is vanishing, of the Maxwell model with the Arnold–Falk–Winther elements and a modified Arnold–Falk–Winther elements by Falk [30]. They suggested a unified framework for the Maxwell and Zener models that we adopt here but they did not show an analysis of the Zener model and did not cover problems with nonvanishing mass densities.

In this chapter, we show the a priori error analysis of the Zener model for the AFW and GG elements. We point out that our numerical scheme, compared to the previous study in [16], is advantages from several viewpoints. First, we use triangular finite elements, so our method is easy to apply to problems on domains with more general geometry. Secondly, we use the Crank–Nicolson scheme for time discretization which is absolutely stable whereas the leap-frog type scheme in [16] needs a CFL condition for stability. The CFL condition in [16] depends on material parameters and the constant of CFL condition may not be easily obtained in anisotropic materials. In particular, the constant of CFL condition in [16] is very small when the Lamé coefficient λ of material is large. However, the time discretization in our numerical scheme is unconditionally stable and can be used without concerning the range of material parameters.

The organization of this chapter is the following. In section 5.2, we derive the governing equations of the models we are interested in and rewrite them in velocity-stress weak forms with weak symmetry of stress. In sections 5.3 and 5.4, we show the a priori error analysis of the semidiscrete and fully discrete solutions for the AFW elements. In section 5.5, we carry out the a priori error analysis for the GG elements. Finally, in section 5.6, we present numerical results which support our error analysis.

5.2 Weak formulations with weak symmetry

In this section, we only discuss a weak formulation of the Zener model with weak symmetry of stresses in detail. Since the Maxwell model is a special case of the Zener one, we will only briefly explain it after we derived a velocity-stress mixed form of the Zener model.

We recall the description of Zener model in Figure 2.1 as a mechanical model. For the displacement u, let $u = u_0 + u_1$ where u_0, u_1 are the displacement parts involved in the spring and dashpot of the Maxwell component in the Zener model. The stresses from the Maxwell component and from the other spring unit, are denoted by σ_0 and σ_1. Then the total stress is $\sigma := \sigma_0 + \sigma_1$.

By (2.8), $\sigma_1 = C_1\epsilon(u)$ for the spring unit. In the Maxwell component, $\sigma_0 = C_0\epsilon(\dot{u}_0) = C_0'\epsilon(\dot{u}_1)$ where C_0, C_0' are the stiffness tensor given by the spring and dashpot units in the Maxwell component. Let A_0, A_0', A_1 be the inverses of C_0, C_0', C_1. Then we have $A_1\sigma_1 = \epsilon(u)$, $A_0\dot{\sigma}_0 = \epsilon(\dot{u}_0)$, and $A_0'\sigma_0 = \epsilon(\dot{u}_1)$. By adding the second and third equations and using $u = u_0 + u_1$,

$$A_0\dot{\sigma}_0 + A_0'\sigma_0 = \epsilon(\dot{u}), \qquad A_1\sigma_1 = \epsilon(u),$$

and from the balance of linear momentum, $\rho\ddot{u} - \operatorname{div}(\sigma_0 + \sigma_1) = f$ for an external body force f. We assume that $0 < \rho_0 \leq \rho \leq \rho_1 < \infty$ for constants ρ_0, ρ_1. Thus the equations of the Zener model are

$$A_0\dot{\sigma}_0 + A_0'\sigma_0 = \epsilon(\dot{u}), \quad A_1\sigma_1 = \epsilon(u), \quad \rho\ddot{u} - \operatorname{div}(\sigma_0 + \sigma_1) = f. \tag{5.1}$$

We only consider the problems with homogeneous displacement boundary conditions $u \equiv 0$ on $\partial\Omega$, and $u(0)$, $\dot{u}(0)$, $\sigma_0(0)$ are given as initial data. Note that $\sigma_1(0)$ is determined by $u(0)$. If we take $v = \dot{u}$, then we have a velocity-stress formulation

$$A_0\dot{\sigma}_0 + A_0'\sigma_0 = \epsilon(v), \quad A_1\dot{\sigma}_1 = \epsilon(v), \quad \rho\dot{v} - \operatorname{div}(\sigma_0 + \sigma_1) = f, \tag{5.2}$$

with $v \equiv 0$ on $\partial\Omega$ and initial data $v(0) = \dot{u}(0)$, $\sigma_0(0)$, and $\sigma_1(0)$.

In the Maxwell model, there is no spring unit which is related to σ_1, so $C_1 \equiv 0$, $\sigma_1 \equiv 0$ and we only get two equations $A_0\dot{\sigma}_0 + A_0'\sigma_0 = \epsilon(v)$, $\rho\dot{v} - \operatorname{div}\sigma_0 = f$.

For a weak formulation, consider the problem to seek $(\sigma_0, \sigma_1, v, r)$ such that

$$(\sigma_0, \sigma_1, v, r) \in C^1([0,T_0]; M) \times C^1([0,T_0]; M) \times C^1([0,T_0]; V) \times C^1([0,T_0]; K), \tag{5.3}$$

and

$$(A_0\dot{\sigma}_0 + A_0'\sigma_0, \tau_0) + (\operatorname{div}\tau_0, v) + (\dot{r}, \tau_0) = 0, \qquad \tau_0 \in M, \tag{5.4}$$

$$(A_1\dot{\sigma}_1, \tau_1) + (\operatorname{div}\tau_1, v) + (\dot{r}, \tau_1) = 0 \qquad \tau_1 \in M, \tag{5.5}$$

$$(\rho\dot{v}, w) - (\operatorname{div}(\sigma_0 + \sigma_1), w) = (f, w), \qquad w \in V, \tag{5.6}$$

$$(\dot{\sigma}_0 + \dot{\sigma}_1, q) = 0, \qquad q \in K, \tag{5.7}$$

with initial data $(\sigma_0(0), \sigma_1(0), v(0), r(0)) \in S \times S \times V \times K$.

Theorem 5.1. *For given initial data* $(\sigma_0(0), \sigma_1(0), v(0), r(0)) \in S \times S \times V \times K$*, satisfying* $\sigma_1(0) = C_1\epsilon(u(0))$ *and* $r(0) = \operatorname{skw}\operatorname{grad} u(0)$ *for some* $u(0) \in \mathring{H}^1(\Omega; \mathbb{V})$*, there is a unique solution* $(\sigma_0, \sigma_1, v, r)$ *of* (5.4–5.7) *satisfying* (5.3).

Proof. For existence, we use the Hille–Yosida theorem. Let $\mathcal{X} = L^2(\Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{S}) \times V$ with the inner product

$$((\sigma_0, \sigma_1, v), (\tau_0, \tau_1, w))_{\mathcal{X}} = (\sigma_0, \tau_0)_{A_0} + (\sigma_1, \tau_1)_{A_1} + (v, w)_\rho.$$

We define an unbounded operator $\mathcal{L}$ on $\mathcal{X}$ as

$$\mathcal{L}(\tau_0, \tau_1, w) = (C_0\epsilon(w) - C_0A_0'\tau_0, C_1\epsilon(w), \rho^{-1}(\operatorname{div}(\tau_0 + \tau_1))),$$

with domain

$$D(\mathcal{L}) = \{(\tau_0, \tau_1, w) \in \mathcal{X} \mid \tau_0 + \tau_1 \in S,\ w \in \mathring{H}^1(\Omega, \mathbb{V})\}.$$

It is obvious that $D(\mathcal{L})$ is dense in $\mathcal{X}$ because $S \times S \times \mathring{H}^1(\Omega, \mathbb{V}) \subset D(\mathcal{L})$ is dense in $\mathcal{X}$.

We check that $\mathcal{L}$ is an m-dissipative operator. For $(\tau_0, \tau_1, w) \in D(\mathcal{L})$, it is not difficult to check that $(\mathcal{L}(\tau_0, \tau_1, w), (\tau_0, \tau_1, w))_{\mathcal{X}} = -(A_0'\tau_0, \tau_0)$, so for some $\lambda > 0$ and $(\tau_0, \tau_1, w) \in D(\mathcal{L})$, $\|(I - \lambda\mathcal{L})(\tau_0, \tau_1, w)\|_{\mathcal{X}} \geq \|(\tau_0, \tau_1, w)\|_{\mathcal{X}}$. Now we check that $I - \lambda\mathcal{L} : D(\mathcal{L}) \to \mathcal{X}$ is surjective. Let $(\eta_0, \eta_1, z) \in \mathcal{X}$ be given. Rewriting the equation $(I - \lambda\mathcal{L})(\sigma_0, \sigma_1, v) = (\eta_0, \eta_1, z)$,

$$\sigma_0 - \lambda C_0\epsilon(v) + \lambda C_0A_0'\sigma_0 = \eta_0, \qquad \sigma_1 - \lambda C_1\epsilon(v) = \eta_1,$$
$$v - \lambda\rho^{-1}(\operatorname{div}(\sigma_0 + \sigma_1)) = z.$$

Since $I + \lambda C_0A_0'$ is coercive, so is invertible and the first equation gives a constraint $\sigma_0 = (I + \lambda C_0A_0')^{-1}(\lambda C_0\epsilon(v) + \eta_0)$. From the second equation, we get another constraint $\sigma_1 = \lambda C_1\epsilon(v) + \eta_1$. Substituting σ_0, σ_1 in the third equation, using these constraints, we have

$$v - \lambda\rho^{-1}(\operatorname{div}((I + \lambda C_0A_0')^{-1}(\lambda C_0\epsilon(v) + \eta_0) + \lambda C_1\epsilon(v) + \eta_1)) = z.$$

Rewriting in a weak formulation, we find $v \in \mathring{H}^1(\Omega; \mathbb{V})$ such that

$$(v, w)_\rho + \lambda(((I + \lambda C_0A_0')^{-1}(\lambda C_0\epsilon(v) + \eta_0) + \lambda C_1\epsilon(v) + \eta_1), \epsilon(w)) = (z, w)_\rho,$$

for $w \in \mathring{H}^1(\Omega; \mathbb{V})$. Since $(I + \lambda C_0 A_0')^{-1}$ is also coercive, existence of a unique solution of this equation follows from Korn's inequality and the Lax–Milgram lemma. Moreover, the equation above and the constraints of σ_0, σ_1 give $\sigma_0 + \sigma_1 \in S$, so $(\sigma_0, \sigma_1, v) \in D(\mathcal{L})$ and $\mathcal{L}$ is an m-dissipative operator.

By the Hille–Yosida theorem, for initial data $(\sigma_0(0), \sigma_1(0), v(0)) \in S \times S \times \mathring{H}^1(\Omega; \mathbb{V})$, (5.2) has a unique solution (σ_0, σ_1, v) such that

$$\sigma_0, \sigma_1 \in C^1([0, T_0]; L^2(\Omega; \mathbb{S})), \quad \sigma_0 + \sigma_1 \in C^0([0, T_0]; S), \quad v \in C^1([0, T_0]; V).$$

For $\sigma_1(0)$, there is a unique $u(0) \in \mathring{H}^1(\Omega; \mathbb{V})$ such that $\sigma_1(0) = C_1 \epsilon(u(0))$. Let $u(t) = u(0) + \int_0^t v(s)\, ds$. Then $u(t) \in \mathring{H}^1(\Omega; \mathbb{V})$ and $r(t) := \operatorname{skw} \operatorname{grad} u(t)$ is well-defined. Now it is clear that $\dot{r} = \operatorname{skw} \operatorname{grad} \dot{u}$. By Corollary 2.20, we can check that $(\sigma_0, \sigma_1, v, r)$ satisfies (5.4–5.5). Moreover, (5.6–5.7) are satisfied from the third equation in (5.1) and the symmetry of σ_0, σ_1, so existence is proved.

For uniqueness, we assume that there are two solutions for the same initial data. Then their difference, denoted by $(\sigma_0^d, \sigma_1^d, v^d, r^d)$ satisfies (5.4–5.7) with vanishing initial data and $f \equiv 0$. From (5.7) and the fundamental theorem of calculus, $(\sigma_0^d + \sigma_1^d, q) = 0$ for all $q \in K$. Taking $\tau_0^d = \sigma_0^d$, $\tau_1 = \sigma_1^d$, $w = v^d$ in (5.4–5.6) and adding them, we get

$$\frac{1}{2}\frac{d}{dt}\|\sigma_0^d\|_{A_0}^2 + \|\sigma_0^d\|_{A_0'}^2 + \frac{1}{2}\frac{d}{dt}\|\sigma_1^d\|_{A_1}^2 + \frac{1}{2}\frac{d}{dt}\|v^d\|_\rho^2 = 0.$$

From the vanishing initial data and the Gronwall inequality, we see $\sigma_0^d = \sigma_1^d = v^d \equiv 0$. We also see $r^d \equiv 0$ from $r^d = \operatorname{skw} \operatorname{grad} v^d$. Hence uniqueness is proved. □

Remark 5.2. As in the Kelvin–Voigt model, we may consider inhomogeneous displacement boundary conditions $v = g$ on $\partial\Omega$. Then, (5.4) and (5.5) in the weak formulation (5.4–5.7), are replaced by

$$(A_0 \dot{\sigma}_0 + A_0' \sigma_0, \tau_0) + (\operatorname{div} \tau_0, v) + (\dot{r}, \tau_0) = \int_{\partial\Omega} (\tau_0 \nu, g)\, dS, \qquad \tau_0 \in M, \tag{5.8}$$

$$(A_1 \dot{\sigma}_1, \tau_1) + (\operatorname{div} \tau_1, v) + (\dot{r}, \tau_1) = \int_{\partial\Omega} (\tau_1 \nu, g)\, dS, \qquad \tau_1 \in M, \tag{5.9}$$

where ν is the outward unit normal vector field on $\partial\Omega$.

5.3 Semidiscrete problems

5.3.1 Existence and uniqueness of semidiscrete solutions

Suppose that $\sigma_{0,h}(0), \sigma_{1,h}(0) \in M_h$, $v_h(0) \in V_h$, $r_h(0) \in K_h$ are given. Then the semidiscrete problem of Zener model is to seek $(\sigma_{0,h}, \sigma_{1,h}, v_h, r_h)$ such that

$$\sigma_{0,h}, \sigma_{1,h} \in C^1([0, T_0]; M_h), \quad v_h \in C^1([0, T_0]; V_h), \quad r_h \in C^1([0, T_0]; K_h),$$

and

$$(A_0 \dot{\sigma}_{0,h}, \tau_0) + (A_0' \sigma_{0,h}, \tau_0) + (\operatorname{div} \tau_0, v_h) + (\dot{r}_h, \tau_0) = 0, \tag{5.10}$$

$$(A_1 \dot{\sigma}_{1,h}, \tau_1) + (\operatorname{div} \tau_1, v_h) + (\dot{r}_h, \tau_1) = 0, \tag{5.11}$$

$$(\rho \dot{v}_h, w) - (\operatorname{div}(\sigma_{0,h} + \sigma_{1,h}), w) = (f, w), \tag{5.12}$$

$$(\dot{\sigma}_{0,h} + \dot{\sigma}_{1,h}, q) = 0, \tag{5.13}$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$, for all time $t \in [0, T_0]$.

Theorem 5.3. *For any given $\sigma_{0,h}(0), \sigma_{1,h}(0) \in M_h$, $v_h(0) \in V_h$, $r_h(0) \in K_h$, there is a unique solution of* (5.10–5.13).

Proof. Let $\{\phi_i\}$, $\{\psi_i\}$, $\{\chi_i\}$ be bases of M_h, V_h, and K_h, respectively. We use $\mathscr{A}_0$, $\mathscr{A}_0'$, $\mathscr{A}_1$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$, $\mathscr{D}$ to denote the matrices whose (i,j)-entries are

$$(A_0 \phi_j, \phi_i), \quad (A_0' \phi_j, \phi_i), \quad (A_1 \phi_j, \phi_i), \quad (\operatorname{div} \phi_j, \psi_i),$$
$$(\phi_j, \chi_i), \quad (\rho \psi_j, \psi_i), \quad (\psi_j, \psi_i),$$

respectively. We write $\sigma_{0,h} = \sum_i \alpha_i \phi_i$, $\sigma_{1,h} = \sum_i \beta_i \phi_i$, $v_h = \sum_i \gamma_i \psi_i$, $r_h = \sum_i \zeta_i \chi_i$, $P_h f = \xi_i \psi_i$, and use α, β, γ, ζ, ξ to denote the coefficient vectors of them. Then we may rewrite (5.10–5.13) as a matrix equation of the form

$$\begin{pmatrix} \mathscr{A}_0 & 0 & 0 & \mathscr{C}^T \\ 0 & \mathscr{A}_1 & 0 & \mathscr{C}^T \\ 0 & 0 & \mathscr{M} & 0 \\ \mathscr{C} & \mathscr{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} -\mathscr{A}_0' & 0 & -\mathscr{B}^T & 0 \\ 0 & 0 & -\mathscr{B}^T & 0 \\ \mathscr{B} & \mathscr{B} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathscr{D}\xi \\ 0 \end{pmatrix}.$$

The coefficient matrix of the left-hand side is nonsingular because $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{M}$ are symmetric positive definite and $\mathscr{C}^T$ is injective from the inf-sup condition **(A2)**. Hence it is a well-posed system of ordinary differential equations and a solution exists uniquely for given initial data. □

5.3.2 Decomposition of semidiscrete errors

Now we state the main theorem in semidiscrete error analysis.

Theorem 5.4. *Let* (M_h, V_h, K_h) *be the* AFW *elements of degree* $k \geq 1$ *and let* m *be an integer of* $1 \leq m \leq k$. *Suppose* $\sigma_0, \sigma_1, v, r \in W^{1,1}([0, T_0]; H^m)$ *and let* $(\sigma_{0,h}, \sigma_{1,h}, v_h, r_h)$ *be the semidiscrete solution of* (5.10–5.13) *with initial data satisfying* $\sigma_{0,h}(0) + \sigma_{1,h}(0) \perp K_h$ *and*

$$\|\sigma_0(0) - \sigma_{0,h}(0), \sigma_1(0) - \sigma_{1,h}(0), v(0) - v_h(0), r(0) - r_h(0)\| \\ \leq ch^m \|\sigma_0(0), \sigma_1(0), v(0), r(0)\|_m, \tag{5.14}$$

$$(\sigma_{1,h}(0), \tau) + (r_h(0), \tau) = 0, \qquad \tau \in M_h,\ \operatorname{div} \tau = 0. \tag{5.15}$$

Then we have

$$\|\sigma_0 - \sigma_{0,h}, \sigma_1 - \sigma_{1,h}, v - v_h, r - r_h\|_{L^\infty L^2} \leq ch^m \|\sigma_0, \sigma_1, v, r\|_{W^{1,1} H^m},$$

where c *depends on* A_0, A_0', A_1, ρ_0, *and* ρ_1 *but is independent of* h. *Moreover, the constant* c *in the estimate is uniformly bounded above as* A_0' *decays.*

As in our analysis of elastodynamics and the Kelvin–Voigt model, we decompose errors into the projection and approximation errors. We use e_{σ_0}, e_{σ_1}, e_v, and e_r to denote the semidiscrete errors $\sigma_0 - \sigma_{0,h}$, $\sigma_1 - \sigma_{1,h}$, $v - v_h$, and $r - r_h$, respectively. Then the error equations are

$$(A_0 \dot{e}_{\sigma_0} + A_0' e_{\sigma_0}, \tau_0) + (\operatorname{div} \tau_0, e_v) + (\dot{e}_r, \tau_0) = 0, \tag{5.16}$$

$$(A_1 \dot{e}_{\sigma_1}, \tau_1) + (\operatorname{div} \tau_1, e_v) + (\dot{e}_r, \tau_1) = 0, \tag{5.17}$$

$$(\rho \dot{e}_v, w) - (\operatorname{div}(e_{\sigma_0} + e_{\sigma_1}), w) = 0, \tag{5.18}$$

$$(\dot{e}_{\sigma_0} + \dot{e}_{\sigma_1}, q) = 0, \tag{5.19}$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$. We define $(\sigma_{0,h}^P, \sigma_{1,h}^P, v_h^P, r_h^P)$ by

$$\sigma_{0,h}^P = \tilde{\Pi}_h \sigma_0, \quad \sigma_{1,h}^P = \tilde{\Pi}_h \sigma_1, \quad v_h^P = P_h v, \quad r_h^P = P_h' r, \tag{5.20}$$

and split the semidiscrete errors into the projection errors and the approximation errors as

$$\begin{aligned} e_{\sigma_i} &= e^P_{\sigma_i} + e^h_{\sigma_i} := (\sigma_i - \sigma^P_{i,h}) + (\sigma^P_{i,h} - \sigma_{i,h}), \quad i = 0, 1, \\ e_v &= e^P_v + e^h_v := (v - v^P_h) + (v^P_h - v_h), \\ e_r &= e^P_r + e^h_r := (r - r^P_h) + (r^P_h - r_h). \end{aligned} \tag{5.21}$$

5.3.3 Projection error estimates for the AFW elements

We state the projection error estimates but omit its proof because it is same as that of Theorem 3.5.

Theorem 5.5. *Let* (M_h, V_h, K_h) *be the* AFW *elements of degree* $k \geq 1$*. Suppose* $e^P_{\sigma_0}$, $e^P_{\sigma_1}$, e^P_v, e^P_r *are defined as in* (5.21)*. Then we have*

$$\begin{aligned} \|e^P_{\sigma_i}\|_{L^\infty L^2} &\leq ch^m \|\sigma_i\|_{L^\infty H^m}, \quad 1 \leq m \leq k+1, \quad i = 0, 1, \\ \|e^P_v, e^P_r\|_{L^\infty L^2} &\leq ch^m \|v, r\|_{L^\infty H^m}, \quad 0 \leq m \leq k, \end{aligned}$$

where c *depends on* $\|\sigma_0, \sigma_1, v, r\|_{L^\infty H^m}$*. Similar estimates hold for the time derivatives of* σ_i*,* v*, and* r*.*

5.3.4 Approximation error estimates for the AFW elements

In this section, we will discuss the a priori estimates of approximation errors.

Theorem 5.6. *Let* $e^h_{\sigma_0}$, $e^h_{\sigma_1}$, e^h_v, e^h_r *be the approximation errors defined in* (5.21) *and the assumptions of Theorem 5.4 hold. Then, for* $1 \leq m \leq k$*,*

$$\|e^h_{\sigma_0}, e^h_{\sigma_1}, e^h_v, e^h_r\|_{L^\infty L^2} \leq ch^m \|\sigma_0, \sigma_1, v, r\|_{W^{1,1} H^m}, \tag{5.22}$$

where c *depends on* ρ_0*,* ρ_1*,* A_0*,* A_0'*, and* A_1 *but is independent of* h *and is uniformly bounded as* A_0' *decays.*

Proof. It is enough to show, for $1 \leq m \leq k$,

$$\|e^h_{\sigma_0}, e^h_{\sigma_1}, e^h_v\|_{L^\infty L^2} \leq ch^m \|\sigma_0, \dot{\sigma}_0, \dot{\sigma}_1, \dot{v}, \dot{r}\|_{L^1 H^m}, \tag{5.23}$$

$$\|e^h_r\|_{L^\infty L^2} \leq \|e^h_{\sigma_1}, e^P_{\sigma_1}, e^P_r\|_{L^\infty L^2}. \tag{5.24}$$

Suppose that these estimates are proved. Then the estimates (5.22) for $e^h_{\sigma_0}$, $e^h_{\sigma_1}$, and e_v^h follow immediately from (5.23). The estimate (5.22) for e_r^h is obtained by combining (5.24) and Theorem 5.5.

To show (5.23), we first rewrite (5.16–5.18), using the notions in (5.21), as

$$(A_0(\dot{e}_{\sigma_0}^h + \dot{e}_{\sigma_0}^P), \tau_0) + (A_0'(e_{\sigma_0}^h + e_{\sigma_0}^P), \tau_0) + (\operatorname{div} \tau_0, e_v^h + e_v^P) + (\dot{e}_r^h + \dot{e}_r^P, \tau_0) = 0,$$
$$(A_1(\dot{e}_{\sigma_1}^h + \dot{e}_{\sigma_1}^P), \tau_1) + (\operatorname{div} \tau_1, e_v^h + e_v^P) + (\dot{e}_r^h + \dot{e}_r^P, \tau_1) = 0,$$
$$(\rho(\dot{e}_v^h + \dot{e}_v^P), w) - (\operatorname{div}(e_{\sigma_0}^h + e_{\sigma_0}^P + e_{\sigma_1}^h + e_{\sigma_1}^P), w) = 0.$$

From (3.24), one can see $(\operatorname{div} \tau, e_v^P) = (\operatorname{div} e_{\sigma_i}^P, w) = 0$ for $\tau \in M_h$, $w \in V_h$ and we get simplified error equations

$$\begin{aligned}(A_0\dot{e}_{\sigma_0}^h, \tau_0) + (A_0' e_{\sigma_0}^h, \tau_0) + (\operatorname{div} \tau_0, e_v^h) + (\dot{e}_r^h, \tau_0) \\ = -(A_0\dot{e}_{\sigma_0}^P, \tau_0) - (A_0' e_{\sigma_0}^P, \tau_0) - (\dot{e}_r^P, \tau_0),\end{aligned} \tag{5.25}$$

$$(A_1\dot{e}_{\sigma_1}^h, \tau_1) + (\operatorname{div} \tau_1, e_v^h) + (\dot{e}_r^h, \tau_1) = -(A\dot{e}_{\sigma_1}^P, \tau_1) - (\dot{e}_r^P, \tau_1), \tag{5.26}$$

$$(\rho\dot{e}_v^h, w) - (\operatorname{div}(e_{\sigma_0}^h + e_{\sigma_1}^h), w) = -(\rho\dot{e}_v^P, w), \tag{5.27}$$

for $\tau_0, \tau_1 \in M_h$ and $w \in V_h$. Note that $e_{\sigma_0} + e_{\sigma_1} \perp K_h$ from (5.19) because $e_{\sigma_0}(0) + e_{\sigma_1}(0) \perp K_h$ and the fundamental theorem of calculus. Moreover, $e_{\sigma_0}^h + e_{\sigma_1}^h \perp K_h$ because $e_{\sigma_0}^P + e_{\sigma_1}^P \perp K_h$ from the definitions of $e_{\sigma_0}^P$ and $e_{\sigma_1}^P$. If we take $\tau_0 = e_{\sigma_0}^h$, $\tau_1 = e_{\sigma_1}^h$, $w = e_v^h$ in (5.25–5.27) and add the three equations, then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e_{\sigma_0}^h\|_{A_0}^2 + \|e_{\sigma_0}^h\|_{A_0'}^2 + \frac{1}{2}\frac{d}{dt}\|e_{\sigma_1}^h\|_{A_1}^2 + \frac{1}{2}\frac{d}{dt}\|e_v^h\|_\rho^2 \\ &= -(A_0\dot{e}_{\sigma_0}^P, e_{\sigma_0}^h) - (A_0' e_{\sigma_0}^P, e_{\sigma_0}^h) - (A_1\dot{e}_{\sigma_1}^P, e_{\sigma_1}^h) - (\dot{e}_r^P, e_{\sigma_0}^h + e_{\sigma_1}^h) - (\rho\dot{e}_v^P, e_v^h),\end{aligned}$$

where $\|e_\sigma^h\|_A^2 = (Ae_\sigma^h, e_\sigma^h)$ and $\|e_v^h\|_\rho^2 = (\rho e_v^h, e_v^h)$. By a weighted Cauchy–Schwarz inequality, and dropping the nonnegative term $\|e_{\sigma_0}^h\|_{A_0'}^2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e_{\sigma_0}^h\|_{A_0}^2 + \frac{1}{2}\frac{d}{dt}\|e_{\sigma_1}^h\|_{A_1}^2 + \frac{1}{2}\frac{d}{dt}\|e_v^h\|_\rho^2 \\ &\qquad \le c\|\dot{e}_{\sigma_0}^P, e_{\sigma_0}^P, \dot{e}_{\sigma_1}^P, \dot{e}_r^P, \dot{e}_v^P\|(\|e_{\sigma_0}^h\|_{A_0}^2 + \|e_{\sigma_1}^h\|_{A_1}^2 + \|e_v^h\|_\rho^2)^{\frac{1}{2}}.\end{aligned}$$

Regarding $(\|e_{\sigma_0}^h(t)\|_{A_0}^2 + \|e_{\sigma_1}^h(t)\|_{A_1}^2 + \|e_v^h(t)\|_\rho^2)^{1/2}$ as the $Q(t)$ in Lemma 2.14, the lemma gives

$$
\begin{aligned}
&(\|e^h_{\sigma_0}(t)\|^2_{A_0} + \|e^h_{\sigma_1}(t)\|^2_{A_1} + \|e^h_v(t)\|^2_\rho)^{\frac{1}{2}} \\
&\quad \le (\|e^h_{\sigma_0}(0)\|^2_{A_0} + \|e^h_{\sigma_1}(0)\|^2_{A_1} + \|e^h_v(0)\|^2_\rho)^{\frac{1}{2}} + c\int_0^t \|e^P_{\sigma_0}, \dot{e}^P_{\sigma_0}, \dot{e}^P_{\sigma_1}, \dot{e}^P_r, \dot{e}^P_v\|\, ds.
\end{aligned}
$$

By (5.14) and applying Theorem 5.5, we get (5.23).

In order to show (5.24), note that

$$
(A_1\sigma_1(0), \tau) + (r(0), \tau) = 0, \qquad \tau \in M_h,\ \operatorname{div}\tau = 0.
$$

From (5.15), $(A_1 e_{\sigma_1}(0), \tau) + (e_r(0), \tau) = 0$ for $\tau \in M_h$ such that $\operatorname{div}\tau = 0$. By (5.11) with $\operatorname{div}\tau = 0$ and the fundamental theorem of calculus, $(A_1 e_{\sigma_1}, \tau) + (e_r, \tau) = 0$ which is equivalent to

$$
(e^h_r, \tau) = -(A_1(e^h_{\sigma_1} + e^P_{\sigma_1}), \tau) - (e^P_r, \tau) = 0, \qquad \tau \in M_h,\ \operatorname{div}\tau = 0.
$$

By **(A2)**, there exists $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(q, \tau) = \|e^h_r\|^2$, and $\|\tau\| \le c\|e^h_r\|$. If we take such τ in this equation, and apply the Cauchy–Schwarz inequality, then (5.24) follows.

In the steps of the proof, it is not difficult to see that the constant c in the theorem is uniformly bounded as A_0' decays to zero. □

5.4 Full discretization

In this section, we use the Crank–Nicolson scheme for full discretization and prove a priori error estimates for the AFW elements of degree $k \ge 1$.

Let $(\Sigma^j_0, \Sigma^j_1, V^j, R^j)$ denote the numerical solution of $(\sigma_0, \sigma_1, v, r)$ at time t_j. In the Crank–Nicolson scheme, for given initial data $(\Sigma^0_0, \Sigma^0_1, V^0, R^0)$, the numerical solution $(\Sigma^j_0, \Sigma^j_1, V^j, R^j)$, $j \ge 1$ is defined inductively by

$$
\left(A_0\bar{\partial}_t\Sigma_0^{j+\frac{1}{2}}, \tau_0\right) + (A_0'\hat{\Sigma}_0^{j+\frac{1}{2}}, \tau_0) + (\hat{V}^{j+\frac{1}{2}}, \operatorname{div}\tau_0) + (\bar{\partial}_t R^{j+\frac{1}{2}}, \tau_0) = 0, \tag{5.28}
$$

$$
\left(A_1\bar{\partial}_t\Sigma_1^{j+\frac{1}{2}}, \tau_1\right) + (\hat{V}^{j+\frac{1}{2}}, \operatorname{div}\tau_1) + (\bar{\partial}_t R^{j+\frac{1}{2}}, \tau_1) = 0, \tag{5.29}
$$

$$
\left(\rho\bar{\partial}_t V^{j+\frac{1}{2}}, w\right) - (\operatorname{div}(\hat{\Sigma}_0^{j+\frac{1}{2}} + \hat{\Sigma}_1^{j+\frac{1}{2}}), w) = (\hat{f}^{j+\frac{1}{2}}, w), \tag{5.30}
$$

$$
(\bar{\partial}_t\Sigma_0^{j+\frac{1}{2}} + \bar{\partial}_t\Sigma_1^{j+\frac{1}{2}}, q) = 0, \tag{5.31}
$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$. The numerical solution $(\Sigma_0^j, \Sigma_1^j, V^j, R^j)$ in (5.28–5.31) is well-defined.

Theorem 5.7. *Let $M_h \times V_h \times K_h$ be the* AFW *elements of degree $k \geq 1$. Suppose*

$$\sigma_0, \sigma_1, v, r \in W^{1,1}([0, T_0]; H^m) \cap W^{3,1}([0, T_0]; L^2), \tag{5.32}$$

and given initial data $(\Sigma_0^0, \Sigma_1^0, V^0, R^0) \in M_h \times M_h \times V_h \times K_h$ satisfying (5.14–5.15). *Then the fully discrete solution $(\Sigma_0^j, \Sigma_1^j, V^j, R^j)$ with the initial data satisfies*

$$\begin{aligned} &\|\sigma_0^j - \Sigma_0^j, \sigma_1^j - \Sigma_1^j, v^j - V^j, r^j - R^j\| \\ &\qquad \leq c(\Delta t^2 + h^m)\|\sigma_0, \sigma_1, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}, \end{aligned} \tag{5.33}$$

for $1 \leq m \leq k$, where $c > 0$ depends on A_0, A_0', A_1, ρ_0, ρ_1 but is independent of h and Δt. Moreover, the constant c is uniformly bounded as A_0' decays.

The proof of this theorem will be given in the rest of this section.

5.4.1 Well-definedness

We first show the full discretization is well-defined. We need to check that $(\Sigma_0^{j+1}, \Sigma_1^{j+1}, V^{j+1}, R^{j+1})$ is well-defined as a solution of the linear system (5.28–5.31) when Σ_0^j, Σ_1^j, V^j, R^j, f^j, f^{j+1} are given. If we rewrite (5.28–5.31),

$$\begin{aligned} &\Big(A_0\Sigma_0^{j+1}, \tau_0\Big) + \frac{\Delta t}{2}(A_0'\Sigma_0^{j+1}, \tau_0) + \frac{\Delta t}{2}(V^{j+1}, \operatorname{div}\tau_0) + (R^{j+1}, \tau_0) \\ &\qquad = (A_0\Sigma_0^j, \tau_0) - \frac{\Delta t}{2}(A_0'\Sigma_0^j, \tau_0) - \frac{\Delta t}{2}(V^j, \operatorname{div}\tau_0) + (R^j, \tau_0), \\ &(A_1\Sigma_1^{j+1}, \tau_1) + \frac{\Delta t}{2}(V^{j+1}, \operatorname{div}\tau_1) + (R^{j+1}, \tau_1) \\ &\qquad = (A_1\Sigma_1^j, \tau_1) - \frac{\Delta t}{2}(V^j, \operatorname{div}\tau_1) + (R^j, \tau_1), \\ &(\rho V^{j+1}, w) - \frac{\Delta t}{2}(\operatorname{div}(\Sigma_0^{j+1} + \Sigma_1^{j+1}), w) \\ &\qquad = (\rho V^j, w) + \frac{\Delta t}{2}(\operatorname{div}(\Sigma_0^j + \Sigma_1^j), w) + \Delta t(\hat{f}^{j+\frac{1}{2}}, w), \\ &(\Sigma_0^{j+1} + \Sigma_1^{j+1}, q) = (\Sigma_0^j + \Sigma_1^j, q), \end{aligned}$$

for $(\tau_0, \tau_1, w, q) \in M_h \times M_h \times V_h \times K_h$. The above is a system of linear equations with same number of equations and unknowns, so in order to show

$(\Sigma_0^{j+1}, \Sigma_1^{j+1}, V^{j+1}, R^{j+1})$ is well-defined, it suffices to show that $\Sigma_0^{j+1} = \Sigma_1^{j+1} = V^{j+1} = R^{j+1} = 0$ if all right-hand sides are vanishing. Suppose that the right-hand sides vanish. By taking $\tau_0 = \Sigma_0^{j+1}$, $\tau_1 = \Sigma_1^{j+1}$, $w = V^{j+1}$, $q = -R^{j+1}$ and adding the four equations, we have

$$(A_0 \Sigma_0^{j+1}, \Sigma_0^{j+1}) + \frac{\Delta t}{2}(A_0' \Sigma_0^{j+1}, \Sigma_0^{j+1}) + (A_1 \Sigma_1^{j+1}, \Sigma_1^{j+1}) + (\rho V^{j+1}, V^{j+1}) = 0,$$

which yields $\Sigma_0^{j+1} = \Sigma_1^{j+1} = V^{j+1} = 0$. From the stability condition **(A2)** of mixed finite elements, there is $\tau \in M_h$ so that $\operatorname{div} \tau = 0$ and $(R^{j+1}, \tau) = (R^{j+1}, R^{j+1})$. If we use such τ in the first equation, we obtain $R^{j+1} = 0$ since $\Sigma_0^{j+1} = V^{j+1} = 0$. Hence the full discretization is well-defined.

5.4.2 Convergence

We now turn to the proof of a priori estimates in Theorem 5.7. Let us denote the error $(\sigma_0^j - \Sigma_0^j, \sigma_1^j - \Sigma_1^j, v^j - V^j, r^j - R^j)$ by

$$E_{\sigma_i}^j := \sigma_i^j - \Sigma_i^j = (\sigma_i^j - \sigma_{i,h}^{P,j}) + (\sigma_{i,h}^{P,j} - \Sigma_i^j) =: e_{\sigma_i}^{P,j} + \theta_{\sigma_i}^j, \quad i = 0, 1, \tag{5.34}$$

$$E_v^j := v^j - V^j = (v^j - v_h^{P,j}) + (v_h^{P,j} - V^j) =: e_v^{P,j} + \theta_v^j, \tag{5.35}$$

$$E_r^j := r^j - R^j = (r^j - r_h^{P,j}) + (r_h^{P,j} - R^j) =: e_r^{P,j} + \theta_r^j. \tag{5.36}$$

In Theorem 5.5, we already obtained error bounds of the projection errors $(e_{\sigma_0}^P, e_{\sigma_1}^P, e_v^P, e_r^P)$. Thus we only need to consider a priori estimates of the approximation errors, $(\theta_{\sigma_0}^j, \theta_{\sigma_1}^j, \theta_v^j, \theta_r^j)$, for Theorem 5.7. More specifically, we want to show the following.

Theorem 5.8. *Suppose the assumptions of Theorem 5.7 hold and $\theta_{\sigma_0}^i$, $\theta_{\sigma_1}^i$, θ_v^i, θ_r^i are defined as in (5.34–5.36). Then for $0 \le i \le N$ and $1 \le m \le k$,*

$$\|\theta_{\sigma_0}^i, \theta_{\sigma_1}^i, \theta_v^i, \theta_r^i\| \le c(\Delta t^2 + h^m)\|\sigma_0, \sigma_1, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}, \tag{5.37}$$

where the constant c depends on A_0, A_0', A_1, ρ_0, ρ_1 but is independent of h, Δt. Moreover, the constant is uniformly bounded as A_0' decays.

Proof. In order to show (5.37), consider the arithmetic mean of equations (5.4–

5.6) at $t = t_j$ and $t = t_{j+1}$, which are

$$(A_0 \hat{\dot{\sigma}}_0^{j+\frac{1}{2}}, \tau_0) + (A_0' \hat{\sigma}_0^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div} \tau_0, \hat{\dot{v}}^{j+\frac{1}{2}}) + (\hat{\dot{r}}^{j+\frac{1}{2}}, \tau_0) = 0,$$
$$(A_1 \hat{\dot{\sigma}}_1^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div} \tau_1, \hat{\dot{v}}^{j+\frac{1}{2}}) + (\hat{\dot{r}}^{j+\frac{1}{2}}, \tau_1) = 0,$$
$$(\rho \hat{\dot{v}}^{j+\frac{1}{2}}, w) - (\operatorname{div}(\hat{\sigma}_0^{j+\frac{1}{2}} + \hat{\sigma}_1^{j+\frac{1}{2}}), w) = (\hat{f}^{j+\frac{1}{2}}, w),$$

for $(\tau_0, \tau_1, w) \in M_h \times M_h \times V_h$. We subtract (5.28–5.30) from the above equations. If we rewrite the difference equations with $(E_{\sigma_0}, E_{\sigma_1}, E_v, E_r)$ defined in (5.34–5.36), then

$$\begin{aligned}(A_0 \bar{\partial}_t E_{\sigma_0}^{j+\frac{1}{2}}, \tau_0) + (A_0' \hat{E}_{\sigma_0}^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div} \tau_0, \hat{E}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t E_r^{j+\frac{1}{2}}, \tau_0) \\ = (A_0(\bar{\partial}_t \sigma_0^{j+\frac{1}{2}} - \hat{\dot{\sigma}}_0^{j+\frac{1}{2}}), \tau_0) + (\bar{\partial}_t r^{j+\frac{1}{2}} - \hat{\dot{r}}^{j+\frac{1}{2}}, \tau_0),\end{aligned} \tag{5.38}$$

$$\begin{aligned}(A_1 \bar{\partial}_t E_{\sigma_1}^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div} \tau_1, \hat{E}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t E_r^{j+\frac{1}{2}}, \tau_1) \\ = (A_1(\bar{\partial}_t \sigma_1^{j+\frac{1}{2}} - \hat{\dot{\sigma}}_1^{j+\frac{1}{2}}), \tau_1) + (\bar{\partial}_t r^{j+\frac{1}{2}} - \hat{\dot{r}}^{j+\frac{1}{2}}, \tau_1),\end{aligned} \tag{5.39}$$

$$(\rho \bar{\partial}_t E_v^{j+\frac{1}{2}}, w) - (\operatorname{div}(\hat{E}_{\sigma_0}^{j+\frac{1}{2}} + \hat{E}_{\sigma_1}^{j+\frac{1}{2}}), w) = (\rho \bar{\partial}_t v^{j+\frac{1}{2}} - \rho \hat{\dot{v}}^{j+\frac{1}{2}}, w). \tag{5.40}$$

Considering (5.34–5.36) with the reductions which are similar to the ones in (3.24), we have

$$\begin{aligned}(A_0 \bar{\partial}_t \theta_{\sigma_0}^{j+\frac{1}{2}}, \tau_0) + (A_0' \hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}, \tau_0) + (\operatorname{div} \tau_0, \hat{\theta}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t \theta_r^{j+\frac{1}{2}}, \tau_0) \\ = (A_0(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + A_0' \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}} + \omega_5^{j+\frac{1}{2}}, \tau_0),\end{aligned} \tag{5.41}$$

$$\begin{aligned}(A_1 \bar{\partial}_t \theta_{\sigma_1}^{j+\frac{1}{2}}, \tau_1) + (\operatorname{div} \tau_1, \hat{\theta}_v^{j+\frac{1}{2}}) + (\bar{\partial}_t \theta_r^{j+\frac{1}{2}}, \tau_1) \\ = (A_1(\omega_6^{j+\frac{1}{2}} + \omega_7^{j+\frac{1}{2}}) + \omega_4^{j+\frac{1}{2}} + \omega_5^{j+\frac{1}{2}}, \tau_1),\end{aligned} \tag{5.42}$$

$$(\rho \bar{\partial}_t \theta_v^{j+\frac{1}{2}}, w) - (\operatorname{div}(\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}} + \hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}), w) = (\omega_8^{j+\frac{1}{2}} + \omega_9^{j+\frac{1}{2}}, w), \tag{5.43}$$

where

$$\begin{gathered}\omega_1^{j+\frac{1}{2}} = \bar{\partial}_t \sigma_0^{j+\frac{1}{2}} - \hat{\dot{\sigma}}_0^{j+\frac{1}{2}}, \quad \omega_2^{j+\frac{1}{2}} = \bar{\partial}_t e_{\sigma_0}^{P,j+\frac{1}{2}}, \quad \omega_3^{j+\frac{1}{2}} = \hat{e}_{\sigma_0}^{P,j+\frac{1}{2}}, \\ \omega_4^{j+\frac{1}{2}} = \bar{\partial}_t r^{j+\frac{1}{2}} - \hat{\dot{r}}^{j+\frac{1}{2}}, \quad \omega_5^{j+\frac{1}{2}} = \bar{\partial}_t e_r^{P,j+\frac{1}{2}}, \quad \omega_6^{j+\frac{1}{2}} = \bar{\partial}_t \sigma_1^{j+\frac{1}{2}} - \hat{\dot{\sigma}}_1^{j+\frac{1}{2}}, \\ \omega_7^{j+\frac{1}{2}} = \bar{\partial}_t e_{\sigma_1}^{P,j+\frac{1}{2}}, \quad \omega_8^{j+\frac{1}{2}} = \rho(\bar{\partial}_t v^{j+\frac{1}{2}} - \hat{\dot{v}}^{j+\frac{1}{2}}), \quad \omega_9^{j+\frac{1}{2}} = \rho \bar{\partial}_t e_v^{P,j+\frac{1}{2}}.\end{gathered} \tag{5.44}$$

Letting $\tau_0 = \hat{\theta}_{\sigma_0}^{j+1/2}$, $\tau_1 = \hat{\theta}_{\sigma_1}^{j+1/2}$, $w = \hat{\theta}_v^{j+1/2}$ in (5.41–5.43), and adding those equations yield

$$
\begin{aligned}
&(\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2 + 2\Delta t\|\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}\|_{A_0'}^2 + \|\theta_{\sigma_1}^{j+1}\|_{A_1}^2 + \|\theta_v^{j+1}\|_\rho^2) \\
&\quad - (\|\theta_{\sigma_0}^{j}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j}\|_{A_1}^2 + \|\theta_v^{j}\|_\rho^2) \\
=&(A_0(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + A_0'\omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}} + \omega_5^{j+\frac{1}{2}}, \hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}) \\
&+ 2\Delta t(A_0(\omega_6^{j+\frac{1}{2}} + \omega_7^{j+\frac{1}{2}}) + \omega_4^{j+\frac{1}{2}} + \omega_5^{j+\frac{1}{2}}, \hat{\theta}_{\sigma_1}^{j+\frac{1}{2}}) + (\omega_8^{j+\frac{1}{2}} + \omega_9^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).
\end{aligned}
\tag{5.45}
$$

If we drop $2\Delta t\|\hat{\theta}_{\sigma_0}^{j+\frac{1}{2}}\|_{A_0'}^2$, divide both sides by

$$
(\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j+1}\|_{A_1}^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} + (\|\theta_{\sigma_0}^{j}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j}\|_{A_1}^2 + \|\theta_v^{j}\|_\rho^2)^{\frac{1}{2}},
$$

and apply a weighted Cauchy–Schwarz inequality, then

$$
\begin{aligned}
&(\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j+1}\|_{A_1}^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \\
&\qquad \le (\|\theta_{\sigma_0}^{j}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j}\|_{A_1}^2 + \|\theta_v^{j}\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{l=1}^{9} \|\omega_l^{j+\frac{1}{2}}\|,
\end{aligned}
\tag{5.46}
$$

for each $0 \le j \le N-1$, with c independent of h and Δt. Using (5.46) inductively,

$$
\begin{aligned}
&(\|\theta_{\sigma_0}^{i}\|_{A_0}^2 + \|\theta_{\sigma_1}^{i}\|_{A_1}^2 + \|\theta_v^{i}\|_\rho^2)^{\frac{1}{2}} \\
&\qquad \le (\|\theta_{\sigma_0}^{0}\|_{A_0}^2 + \|\theta_{\sigma_1}^{0}\|_{A_1}^2 + \|\theta_v^{0}\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{j=0}^{i-1}\sum_{l=1}^{9} \|\omega_l^{j+\frac{1}{2}}\|.
\end{aligned}
\tag{5.47}
$$

Since A_0, A_1 are coercive and $0 < \rho_0 \le \rho$ for a constant ρ_0, we can obtain $\|\theta_{\sigma_0}^i\| + \|\theta_{\sigma_1}^i\| + \|\theta_v^i\| \le c(\|\theta_{\sigma_0}^i\|_{A_0}^2 + \|\theta_{\sigma_1}^i\|_{A_1}^2 + \|\theta_v^i\|_\rho^2)^{1/2}$ where $c > 0$ depends only on A_0, A_1, and ρ_0. Note that $(\|\theta_{\sigma_0}^0\|_{A_0}^2 + \|\theta_{\sigma_1}^0\|_{A_1}^2 + \|\theta_v^0\|_\rho^2)^{1/2} \le ch^m$ for $1 \le m \le k$ by boundedness of A_0, A_1, ρ, the assumption (5.14), and the triangle inequality. Henceforth, if we show

$$
c\Delta t \sum_{j=0}^{i-1}\sum_{l=1}^{9} \|\omega_l^{j+\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \quad 1 \le m \le k,
\tag{5.48}
$$

then (5.37) for $\|\theta_{\sigma_0}^i\| + \|\theta_{\sigma_1}^i\| + \|\theta_v^i\|$ is proved. In order to show (5.48), we recall Taylor expansions (3.53) and (3.54). From (3.53) and the definitions of $\omega_l^{i+1/2}$'s in (5.44), we have

$$\Delta t\|\omega_1^{j+\frac{1}{2}}\| = \frac{1}{2}\|2\sigma_0^{j+1} - 2\sigma_0^j - \Delta t\dot{\sigma}_0^{j+1} - \Delta t\dot{\sigma}_0^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{\sigma}_0\|\,ds, \quad (5.49)$$

$$\Delta t\|\omega_6^{j+\frac{1}{2}}\| = \frac{1}{2}\|2\sigma_1^{j+1} - 2\sigma_1^j - \Delta t\dot{\sigma}_1^{j+1} - \Delta t\dot{\sigma}_1^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{\sigma}_1\|\,ds, \quad (5.50)$$

$$\Delta t\|\omega_4^{j+\frac{1}{2}}\| = \frac{1}{2}\|2r^{j+1} - 2r^j - \Delta t\dot{r}^{j+1} - \Delta t\dot{r}^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{r}\|\,ds, \quad (5.51)$$

$$\Delta t\|\omega_8^{j+\frac{1}{2}}\| = \frac{1}{2}\|\rho(2v^{j+1} - 2v^j - \Delta t\dot{v}^{j+1} - \Delta t\dot{v}^j)\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot{v}\|\,ds. \quad (5.52)$$

By (2.57) and the definitions of $\omega_l^{i+1/2}$'s in (5.44), for $1 \le m \le k$,

$$\Delta t\|\omega_2^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_{\sigma_0}^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot{e}_{\sigma_0}^P\,ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{\sigma}_0\|_m\,ds, \quad (5.53)$$

$$\Delta t\|\omega_3^{j+\frac{1}{2}}\| \le c\Delta t\|\hat{e}_{\sigma_0}^{P,j+\frac{1}{2}}\| \le ch^m\Delta t\|\sigma_0\|_{L^\infty H^m}, \quad (5.54)$$

$$\Delta t\|\omega_7^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_{\sigma_1}^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot{e}_{\sigma_1}^P\,ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{\sigma}_1\|_m\,ds, \quad (5.55)$$

$$\Delta t\|\omega_5^{j+\frac{1}{2}}\| = \Delta t\|\bar{\partial}_t e_r^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot{e}_r^P\,ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{r}\|_m\,ds, \quad (5.56)$$

$$\Delta t\|\omega_9^{j+\frac{1}{2}}\| = \Delta t\|\rho\bar{\partial}_t e_v^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \rho\dot{e}_v^P\,ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m\,ds. \quad (5.57)$$

Combining (5.49–5.57) with (5.47), Theorem 5.8 is proved for $\|\theta_{\sigma_0}^i, \theta_{\sigma_1}^i, \theta_v^i\|$.

In order to show (5.37) for $\|\theta_r^i\|$, we use the argument similar to the semidiscrete error analysis. From (5.15) $(A_1(\sigma_1(0) - \Sigma_1^0), \tau_1) + (r(0) - R^0, \tau_1) = 0$ for $\tau_1 \in M_h$ such that $\operatorname{div}\tau_1 = 0$. If we combine this with (5.29), then we get

$$(A_1E_{\sigma_1}^j, \tau) + (R^j, \tau) = 0, \qquad \tau \in M_h,\ \operatorname{div}\tau = 0,$$

which is equivalent to

$$(\theta_r^j, \tau) = -(A_1(e_{\sigma_1}^{P,j} + \theta_{\sigma_1}^j), \tau) - (e_r^{P,j}, \tau), \qquad \tau \in M_h,\ \operatorname{div}\tau = 0.$$

If we take $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(\tau, q) = \|\theta_r^j\|^2$, and $\|\tau\| \le c\|\theta_r^j\|$, then we have $\|\theta_r^j\| \le c\|e_{\sigma_1}^{P,j}, \theta_{\sigma_1}^j, e_r^{P,j}\|$, which yields (5.37) for $\|\theta_r^i\|$. The proof is completed. □

Corollary 5.9 (Numerical solution of displacement)**.** *Let $U^0 \in V_h$ be an approximation of initial displacement $u(0)$ with $\|u(0) - U^0\| \le ch^m$, $1 \le m \le k$. If we reconstruct displacement U^i from $\{V^j\}$, using the trapezoidal rule, as*

$$U^i = U^0 + \Delta t \sum_{j=1}^{i} \frac{V^j + V^{j-1}}{2} = U^0 + \Delta t \sum_{j=1}^{i} \hat{V}^{j+\frac{1}{2}},$$

then for $0 \le i \le N$, $1 \le m \le k$,

$$\|u^i - U^i\| \le c(h^m + \Delta t^2).$$

The proof is same as that of Corollary 3.9, so we omit it.

5.5 Error analysis for the GG elements

In this section we discuss the error analysis for the GG elements. As in previous chapters, we do not repeat all details of error analysis but focus on the steps that should be modified.

Theorem 5.10. *Let (M_h, V_h, K_h) be the* GG *elements of degree $k \ge 1$. Suppose that m is an integer of $1 \le m \le k+1$ and $m' = m - \delta_{m,k+1}$ where $\delta_{m,k+1}$ is the Kronecker delta. Suppose $\|\rho\|_{W_h^{1,\infty}} < \infty$ for $\|\rho\|_{W_h^{1,\infty}}$ defined in* (3.64) *and*

$$\begin{aligned} \sigma_0, \sigma_1, r \in W^{1,1}([0,T_0]; H^m) \cap W^{3,1}([0,T_0]; L^2), \\ v \in W^{1,1}([0,T_0]; H^{m'}) \cap W^{3,1}([0,T_0]; L^2). \end{aligned} \tag{5.58}$$

Assume the initial data are given to be $\Sigma_0^0 = \tilde{\Pi}_h \sigma_0(0)$, $\Sigma_1^0 = \tilde{\Pi}_h \sigma_1(0)$, $V^0 = P_h v(0)$, $R^0 = P_h' r(0)$. Then the fully discrete solution $(\Sigma_0^j, \Sigma_1^j, V^j, R^j)$ in (5.28–5.31) *is well-defined and for all $1 \le j \le N$,*

$$\begin{aligned} &\|\sigma_0^j - \Sigma_0^j, \sigma_1^j - \Sigma_1^j, P_h v^j - V^j, r^j - R^j\| && (5.59) \\ &\quad \le c(\Delta t^2 + h^m)(\|\sigma_0, \sigma_1, r\|_{W^{1,1}H^m \cap W^{3,1}L^2} + \|v\|_{W^{1,1}H^{m'} \cap W^{3,1}L^2}), \\ &\|v^j - V^j\| && (5.60) \\ &\quad \le c(\Delta t^2 + h^{m'})(\|\sigma_0, \sigma_1, r\|_{W^{1,1}H^m \cap W^{3,1}L^2} + \|v\|_{W^{1,1}H^{m'} \cap W^{3,1}L^2}), \end{aligned}$$

where c depends on A_0, A'_0, A_1, ρ_0, and $\|\rho\|_{W_h^{1,\infty}}$ but is independent of h and Δt. The constants c in (5.59–5.60) are uniformly bounded as A'_0 decays.

Well-definedness of fully discrete solutions is same as the proof of Theorem 5.7, so we only discuss a priori error estimates. We decompose the errors $(E^j_{\sigma_0}, E^j_{\sigma_1}, E^j_v, E^j_r)$ into the projection errors $(e^{P,j}_{\sigma_0}, e^{P,j}_{\sigma_1}, e^{P,j}_v, e^{P,j}_r)$ and the approximation errors $(\theta^j_{\sigma_0}, \theta^j_{\sigma_1}, \theta^j_v, \theta^j_r)$ as in (5.34–5.36).

Theorem 5.11. *There exists a constant $c > 0$ independent h so that the following inequalities hold.*

$$
\begin{aligned}
\|e^P_{\sigma_i}\| &\le ch^m \|\sigma_i\|_m, && 1 \le m \le k+1, \quad i = 0, 1,\\
\|e^P_v\| &\le ch^m \|v\|_m, && 0 \le m \le k,\\
\|e^P_r\| &\le ch^m \|r\|_m, && 0 \le m \le k+1.
\end{aligned}
$$

Furthermore, similar inequalities hold for the time derivatives of σ_0, σ_1, v, and r, respectively, as in Theorem 5.5.

We omit its proof because it is same as the proof of Theorem 5.5.

For Theorem 5.10, we only need to consider a priori estimates of the approximation errors $(\theta^j_{\sigma_0}, \theta^j_{\sigma_1}, \theta^j_v, \theta^j_r)$ by the triangle inequality.

Theorem 5.12. *Suppose the assumptions in Theorem 5.10 hold and $\theta^i_{\sigma_0}$, $\theta^j_{\sigma_1}$, θ^i_v, θ^i_r are defined as in (5.34–5.36). Then there exists a constant $c > 0$ depending on A_0, A'_0, A_1, ρ_0, $\|\rho\|_{W_h^{1,\infty}}$ but is independent of h and Δt so that*

$$
\begin{aligned}
&\|\theta^i_{\sigma_0}, \theta^i_{\sigma_1}, \theta^i_v, \theta^i_r\| \\
&\qquad \le c(\Delta t^2 + h^m)(\|\sigma_0, \sigma_1, r\|_{W^{1,1}H^m \cap W^{3,1}L^2} + \|v\|_{W^{1,1}H^{m'} \cap W^{3,1}L^2}), \quad (5.61)
\end{aligned}
$$

for $0 \le i \le N$, $1 \le m \le k+1$.

Proof. For a priori error estimates, we follow same argument in the proof of Theorem 5.7 and obtain (5.45).

To prove (5.61) for $\|\theta^i_{\sigma_0}, \theta^i_{\sigma_1}, \theta^i_v\|$, let ρ_c be the orthogonal L^2 projection of ρ into the space of piecewise constant functions associated to $\mathcal{T}_h$. Then we define $\tilde{\omega}_9^{j+1/2} = (\rho - \rho_c)\bar{\partial}_t e_v^{P,j+1/2}$. Note that $\rho_c \bar{\partial}_t e_v^{P,j+1/2} \perp V_h$ from the definition of $\bar{\partial}_t e_v^{P,j+1/2}$ and

$$
(\omega_9^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\rho \bar{\partial}_t v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = ((\rho - \rho_c)\bar{\partial}_t e_v^{P,j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}) = (\tilde{\omega}_9^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).
$$

Therefore we have an inequality analogous to (5.46) as follows.

$$
\begin{aligned}
&(\|\theta_{\sigma_0}^{j+1}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j+1}\|_{A_1}^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \\
&\qquad \le (\|\theta_{\sigma_0}^{j}\|_{A_0}^2 + \|\theta_{\sigma_1}^{j}\|_{A_1}^2 + \|\theta_v^{j}\|_\rho^2)^{\frac{1}{2}} + c\Delta t \left(\sum_{l=1}^{8} \|\omega_l^{j+\frac{1}{2}}\| + \|\tilde{\omega}_9^{j+\frac{1}{2}}\| \right).
\end{aligned}
$$

Repeating the steps in (5.46) and (5.47), for the proof of (5.61) for $\|\theta_{\sigma_0}^i, \theta_{\sigma_1}^i, \theta_v^i\|$, we only need to show

$$
c\Delta t \sum_{j=0}^{i-1} \left(\sum_{l=1}^{8} \|\omega_l^{j+\frac{1}{2}}\| + \|\tilde{\omega}_9^{j+\frac{1}{2}}\| \right) \le c(\Delta t^2 + h^m), \quad 1 \le m \le k+1.
$$

For the estimates of $\omega_l^{j+1/2}$, $l = 1, \cdots, 8$, we use (5.49–5.56) and note that the integer m ranges $1 \le m \le k+1$ for the GG elements. For $\tilde{\omega}_9^{j+1/2}$,

$$
\begin{aligned}
\Delta t \|\tilde{\omega}_9^{j+\frac{1}{2}}\| = \Delta t \|(\rho - \rho_c)\bar{\partial}_t e_v^{P,j}\| &= \left\| \int_{t_j}^{t_{j+1}} (\rho - \rho_c)(\dot{v}_h^P - \dot{v})\, ds \right\| \\
&\le ch^{m+1} \|\rho\|_{W_h^{1,\infty}} \int_{t_j}^{t_{j+1}} \|\dot{v}\|_m ds, \quad 1 \le m \le k.
\end{aligned}
$$

For the proof of (5.61) for $\|\theta_r^i\|$, we repeat the argument for $\|\theta_r^i\|$ in the proof of Theorem 5.8. Details are omitted. □

We also claim that a simple postprocessing can be used to find a better numerical solution of u as we have done in previous chapters. Its proof is same as the proof of Theorem 3.13, so we only state it below without a proof.

For V_h in the GG elements, let V_h^* be the space of (possibly discontinuous) piecewise polynomials of one degree higher than V_h, and $\tilde{V}_h$ be the orthogonal complement of V_h in V_h^*. The orthogonal L^2 projections onto V_h^* and $\tilde{V}_h$ are denoted by P_h^* and $\tilde{P}_h$.

Theorem 5.13. *Suppose that Theorem 5.10 holds with $m = k+1$ and also $\|v\|_{W^{1,1}H^{k+1}} < \infty$. Let U^i be a numerical solution of displacement in Corollary 5.9 with an assumption that $\|P_h u(0) - U^0\| \le ch^{k+1}$. We define $U^{*,i} \in V_h^*$ by*

$$
(\operatorname{grad}_h U^{*,i}, \operatorname{grad}_h w) = (A_1 \Sigma_1^i + R^i, \operatorname{grad}_h w), \qquad w \in \tilde{V}_h, \tag{5.62}
$$

$$
(U^{*,i}, w) = (U^i, w), \qquad w \in V_h, \tag{5.63}
$$

for each $0 \le i \le N$ *where* grad_h *is the piecewise gradient operator adapted to the triangulation* $\mathcal{T}_h$*. Then* $U^{*,i}$ *is well-defined and there exists* $c > 0$ *independent of* h *and* Δt *such that*

$$\|u^i - U^{*,i}\| \le c(\Delta t^2 + h^{k+1})\|\sigma_0, \sigma_1, v, r\|_{W^{1,1}H^{k+1}\cap W^{3,1}L^2}. \tag{5.64}$$

5.6 Numerical results

In this section, we present numerical results. We use $\Omega = [0,1] \times [0,1]$ and the AFW elements of degree $k = 2$ in all our numerical computations. We assume that the medium is homogeneous with density $\rho = 1$, and compliance tensors A_0, A_0', A_1 are given as in (2.7) with parameters μ_0, λ_0, μ_0', λ_0', μ_1, λ_1. For simplicity, we put $\mu_0 = \lambda_0 = 1$, $\mu_0' = \lambda_0' = 5$, and $\mu_1 = \lambda_1 = 10$ in all experiments.

For each spatial mesh size h, we take $\Delta t = h$ for time step Δt, so the expected order of convergence is 2 from our error bound $O(h^2+\Delta t^2)$. We present L^2 errors of σ_0, σ_1, v, and r at time $T_0 = 1$, with mesh sizes $h = 1/4, 1/8, 1/16, 1/32, 1/64$, and compute the order of convergences.

Finally, as in the previous chapters, all codes are implemented using the Dolfin Python module [1] of FEniCS project [2, 38].

Example 5.14. Let the displacement field be

$$u(t,x,y) = \begin{pmatrix} (1-x)x^2\sin(\pi y)\cos t \\ (1+t)\sin(\pi x)\sin(\pi y) \end{pmatrix}, \tag{5.65}$$

and $\sigma_0(0) = 0$. Then one can find v, σ_0, σ_1, and f using (5.1). For this exact solution, we compute a numerical solution with inhomogeneous displacement

Table 5.1: Order of convergence for the exact solution with the displacement as in (5.65) and $\sigma_0(0) = 0$ ($\mu_0 = \lambda_0 = 1$, $\mu_0' = \lambda_0' = 5$, $\mu_1 = \lambda_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	2.37e-01	–	1.08e+00	–	1.96e-02	–	5.77e-02	–
8	3.15e-02	2.91	1.83e-01	2.55	4.88e-03	2.01	1.46e-02	1.99
16	4.81e-03	2.71	3.82e-02	2.26	1.22e-03	2.00	3.64e-03	2.00
32	9.11e-04	2.40	8.93e-03	2.10	3.05e-04	2.00	9.09e-04	2.00
64	2.05e-04	2.15	2.18e-03	2.03	7.62e-05	2.00	2.27e-04	2.00

Table 5.2: Order of convergence for the exact solution with the displacement as in (5.66) and $\sigma_0(0) = 0$ ($\mu_0 = \lambda_0 = 1$, $\mu_0' = \lambda_0' = 5$, $\mu_1 = \lambda_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	1.79e-02	–	2.28e-01	–	8.41e-03	–	9.07e-03	–
8	4.44e-03	2.01	5.67e-02	2.01	2.09e-03	2.01	2.27e-03	2.00
16	1.11e-03	2.00	1.41e-02	2.00	5.17e-04	2.01	5.67e-04	2.00
32	2.76e-04	2.00	3.53e-03	2.00	1.29e-04	2.00	1.42e-04	2.00
64	6.90e-05	2.00	8.83e-04	2.00	3.22e-05	2.00	3.54e-05	2.00

boundary conditions using the weak formulation (5.6–5.9). The numerical result for (5.65) is in Table 5.1.

Example 5.15. As an example with inhomogeneous displacement boundary conditions, we let the displacement field be

$$u(t,x,y) = \begin{pmatrix} e^{-y}\cos t \sin x \\ e^{t+x} \end{pmatrix}, \tag{5.66}$$

and let $\sigma_0(0) = 0$. Then one can compute v, σ_0, σ_1, and f using (5.1). We compute a numerical solution with inhomogeneous displacement boundary conditions using the weak formulation (5.6–5.9). The numerical results for (5.66) are shown in Table 5.2.

Example 5.16. For a nonsmooth solution, let

$$u(t,x,y) = \begin{pmatrix} (1+t^2)x^{\frac{7}{3}}y \\ y^{\frac{7}{3}}\cos t \end{pmatrix}, \tag{5.67}$$

Table 5.3: Order of convergence for the exact solution with displacement as in (5.67) and $\sigma_0(0) = 0$ ($\mu_0 = \lambda_0 = 1$, $\mu_0' = \lambda_0' = 5$, $\mu_1 = \lambda_1 = 10$, $h = \Delta t$ and $T_0 = 1$).

$\frac{1}{h}$	$\|\sigma_0 - \sigma_{0,h}\|$		$\|\sigma_1 - \sigma_{1,h}\|$		$\|v - v_h\|$		$\|r - r_h\|$	
	error	order	error	order	error	order	error	order
4	3.61e-01	–	2.27e-01	–	1.07e-02	–	4.27e-03	–
8	1.06e-01	1.76	6.44e-02	1.82	2.63e-03	2.02	1.05e-03	2.03
16	3.06e-02	1.80	1.81e-02	1.83	6.52e-04	2.01	2.60e-04	2.01
32	8.69e-03	1.81	5.07e-03	1.84	1.62e-04	2.01	6.46e-05	2.01
64	2.45e-03	1.82	1.42e-03	1.84	4.04e-05	2.00	1.61e-05	2.00

and $\sigma_0(0) = 0$. The corresponding σ_0, σ_1 are

$$\begin{pmatrix} (70t-350)yx^{\frac{4}{3}} + \frac{35}{78}y^{\frac{4}{3}}(5\cos t - \sin t) & 10(-5+t)x^{\frac{7}{3}} \\ 10(-5+t)x^{\frac{7}{3}} & \frac{70}{3}(t-5)yx^{\frac{4}{3}} + \frac{35}{26}y^{\frac{4}{3}}(5\cos t - \sin t) \end{pmatrix},$$

$$10\begin{pmatrix} 7y((1+t^2)x^{\frac{4}{3}} + y^{\frac{1}{3}}\cos t) & (1+t^2)x^{\frac{7}{3}} \\ (1+t^2)x^{\frac{7}{3}} & \frac{7}{3}y((1+t^2)x^{\frac{4}{3}} + 3y^{\frac{1}{3}}\cos t) \end{pmatrix},$$

respectively. From the fractional order 4/3 of polynomial terms in σ_0 and σ_1, they belong to $H^{5/6-\delta}$ in space for any $\delta > 0$.

As in the previous example, we compute a numerical solution with inhomogeneous displacement boundary condition using (5.8–5.9). The numerical results for (5.67) are shown in Table 5.3. The orders of convergence of σ_0, σ_1 approach $11/6 \approx 1.833$ as we expected in our error analysis but the order of convergence of v is 2. It is consistent to the results we have seen for the examples of elastodynamics and the equations of the Kelvin–Voigt model in the previous chapters, so we again get a motivation to study a better estimate for the v error.

Chapter 6

Numerical simulations

In this chapter we present numerical simulations which are more closely involved in physical situations using the numerical schemes that we developed in previous chapters. In section 6.1, we show wave propagation in homogeneous isotropic, heterogeneous isotropic, and anisotropic elastic media. In section 6.2, we show the creep test of viscoelastic materials and one seismology model problem of wave propagation in viscoelastic media.

6.1 Elastodynamics

In this section, we present three numerical simulations. In the first simulation, we will see the difference of two different types of wave propagation, P and S waves, in homogeneous isotropic medium. In the second simulation, we consider two isotropic heterogeneous media and will see that different parameters of elastic media affect differently to P and S waves. In the last one, we consider wave propagation in two anisotropic materials and show that a radially symmetric initial data becomes asymmetric waves.

6.1.1 Wave propagation in homogeneous isotropic elastic media

In Figure 6.1, we see a series of screen-shots of wave propagation in an isotropic homogeneous linear elastic medium. For initial data, the initial displacement is vanishing (hence initial stress is vanishing) and the initial velocity is given as a bump function of horizontal direction. The magnitude of wave is described as height in Figure 6.1. The splitting of P and S waves is one of the features of

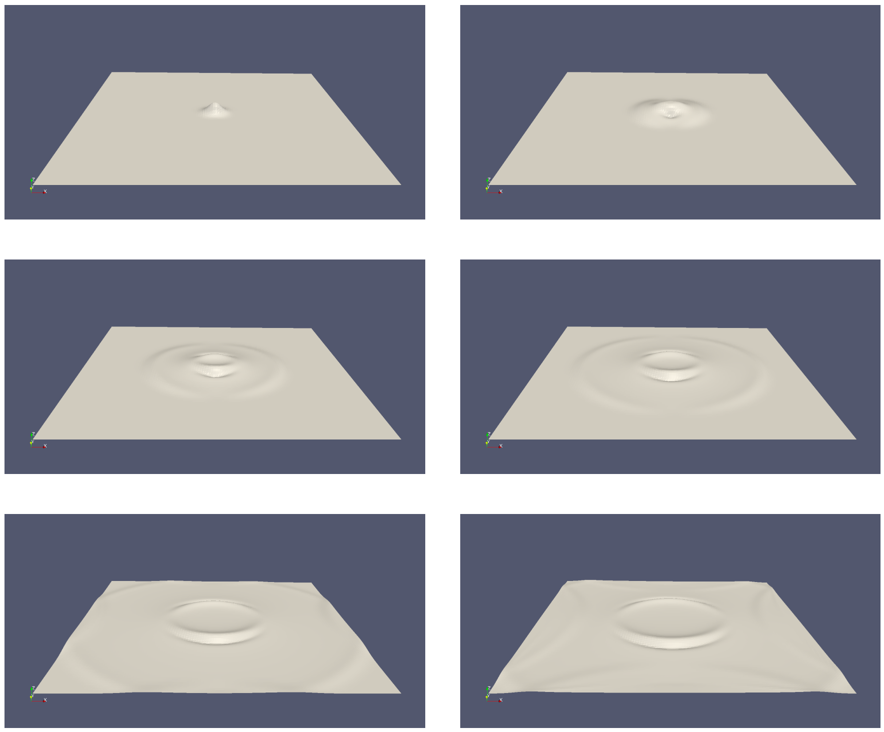

Figure 6.1: Magnitude of elastic waves in homogeneous isotropic elastic medium.

elastic wave propagation. It is known that P wave is faster than S wave, but S wave is more destructive. In the above figures, we observe these features of P and S waves. We can see that P wave mostly propagates horizontally with faster speed and small magnitude, but S wave propagates vertically with slower speed and big magnitude.

6.1.2 Wave propagation in isotropic heterogeneous elastic media

We compare wave propagation in two different heterogeneous media with same initial data. In these simulations, we can see different parameters of elastic medium influence differently to propagation of P and S waves.

The domain is $[-6, 6] \times [-6, 6]$ and the two media are heterogeneous with different parameters on the left and right of the vertical line $x = -1.2$. For convenience, we call the two subdomains, splited by $x = -1.2$, as left and right domains. In Figure 6.2, we see the wave propagation is not completely symmetric with respect to $x = -1.2$, but two crescent shape waves propagate symmetrically. It is because S wave propagation is not affected by the difference of material parameter, but P wave is affected and has increased propagation speed.

On the contrary, we can see both S and P waves propagates faster on right domain in Figure 6.3. Since there are reflections and refractions of waves, the wave propagation becomes more complicated in time and is not simply described.

6.1.3 Wave propagation in anisotropic media

Wave propagation in anisotropic materials is much more complicated than the one in isotropic materials. The elastic waves in anisotropic materials are not simply classified into P and S waves, so it is extremely difficult to classify waves and establish principles on wave propagation even in linear elastodynamics. This is a broad research area which still needs many works to be done in the future. Therefore we do not discuss theoretical parts of this topic but present wave propagation examples on the media which may somewhat represent features of components of the compliance tensor.

For initial data, we always give a radially symmetric initial velocity and vanishing stress, so vanishing displacement as well. In isotropic media, the wave propagation with this initial data is radially symmetric as we see in Figure 6.4. However, we see that wave propagation in anisotropic media may become strongly asymmetric up to the compliance tensor, which is a symmetric 3×3 matrix A satisfying

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{22} \end{pmatrix} = A \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}.$$

For the materials that we used in our numerical computations, compliance tensors are given by

$$\text{Orthotropic 1} : c \begin{pmatrix} 4 & 0 & -1/2 \\ 0 & 4 & 0 \\ -1/2 & 0 & 2 \end{pmatrix}, \quad \text{Orthotropic 2} : c \begin{pmatrix} 4 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix},$$

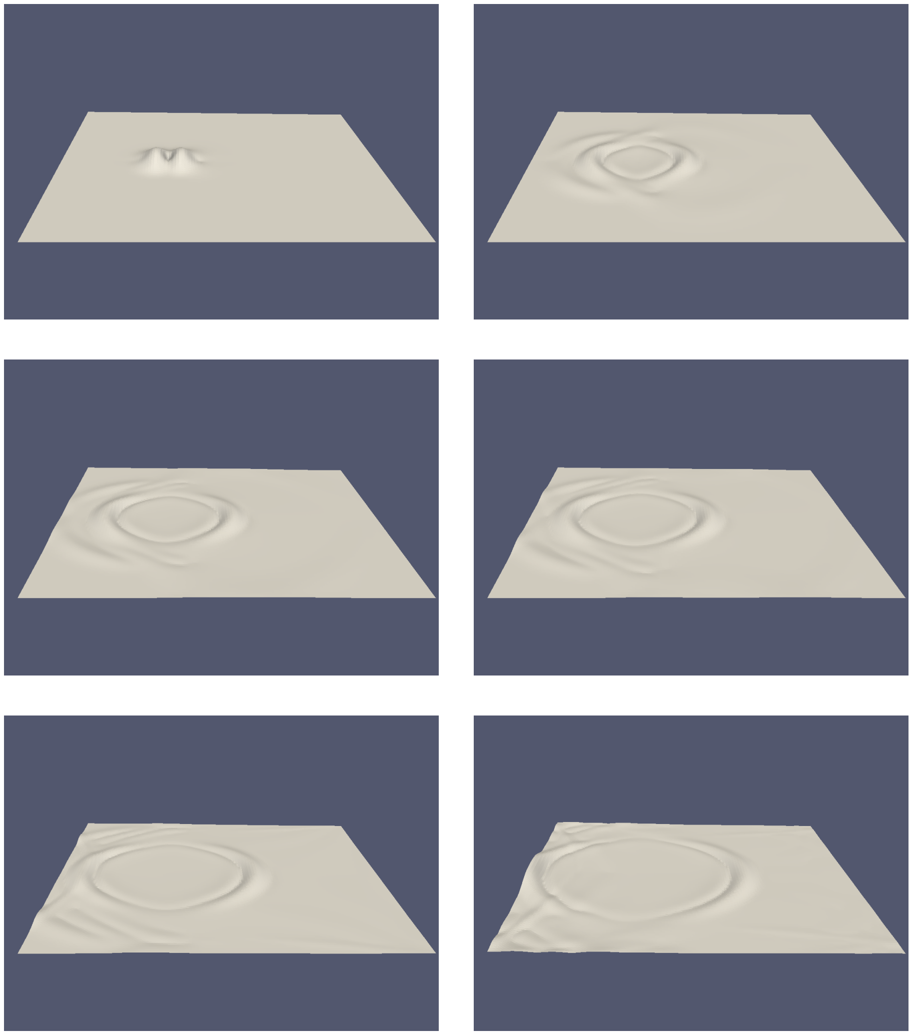

Figure 6.2: Magnitude of waves in heterogeneous medium 1. For the medium 1, the Lamé coefficient λ is 90 if $x \leq -1.2$ and is 10 if $x > -1.2$. Another Lamé coefficient μ is 10, and the mass density ρ is 5 on the whole domain.

where $c = 1.0\text{e} - 2$. These are examples of orthotropic materials which means that the material has two or three mutually orthogonal twofold axes of rotational symmetry. For more details on orthotropic materials, see [43]

In all examples, $\rho = 10$. For the domain of numerical computations, $[-5, 5] \times [-5, 5]$ is used with the triangulation that 100 meshes in horizontal and vertical directions. The AFW elements of degree 2 is used for spatial discretization

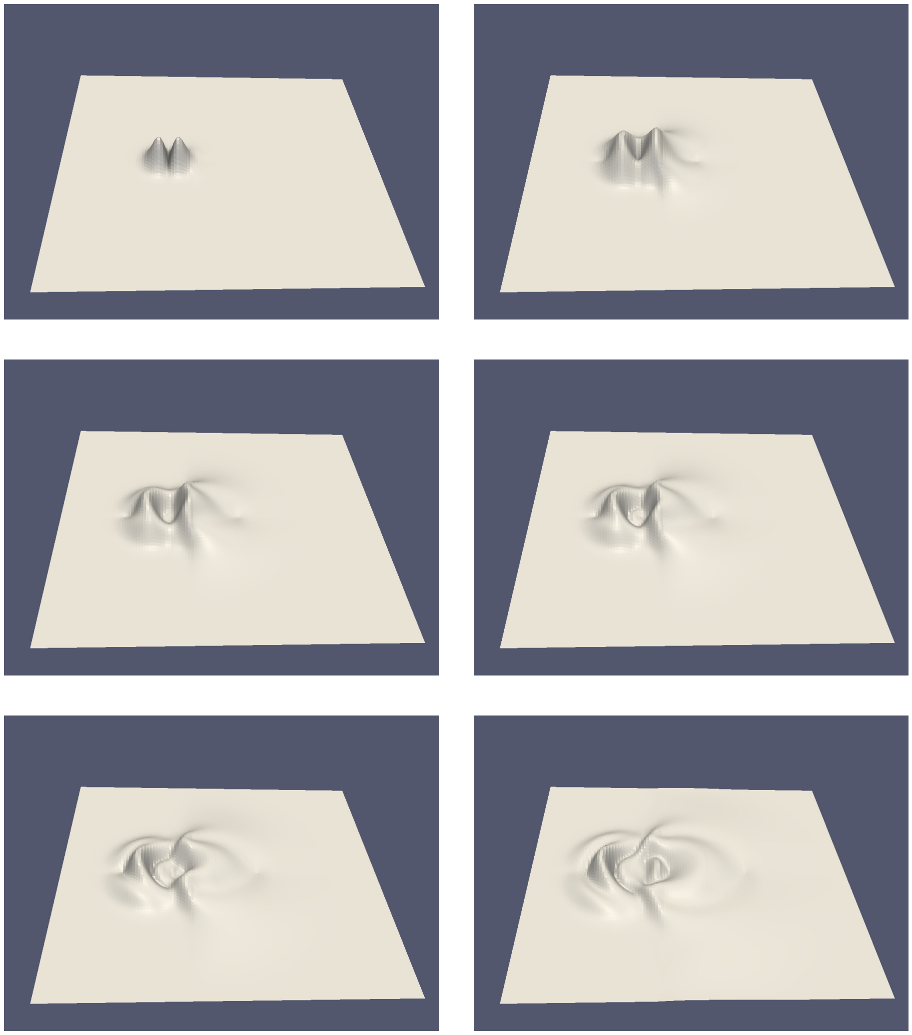

Figure 6.3: Magnitude of waves in heterogeneous medium 2. For the medium 2, the Lamé coefficients λ and μ are both 10 on the whole domain but the mass density is 5 if $x \leq -1.2$ and is 45 if $x > -1.2$.

and the Crank–Nicolson scheme is used for time discretization with time step $\Delta t = 0.01$. In all figures, the red–blue color range corresponds to the magnitude of displacement.

In Figures 6.4–6.6, we compare wave propagation in an isotropic medium, and two orthotropic media. The media Ortho. 1 and Ortho. 2 have compliance tensors which are given as above.

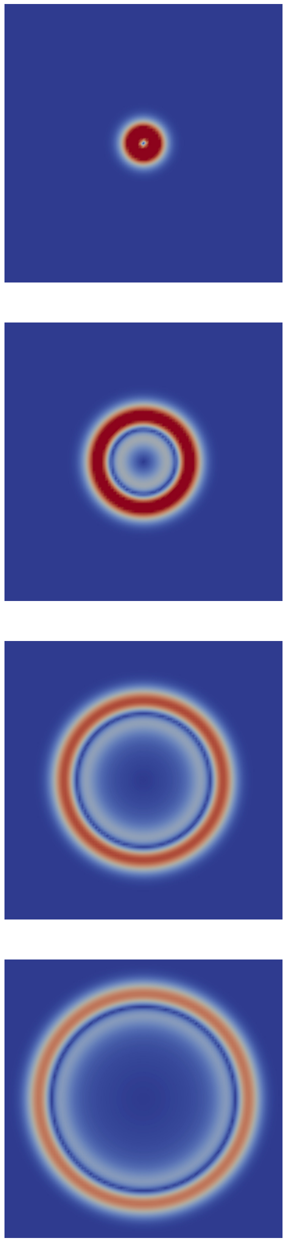
Figure 6.4: Isotropic

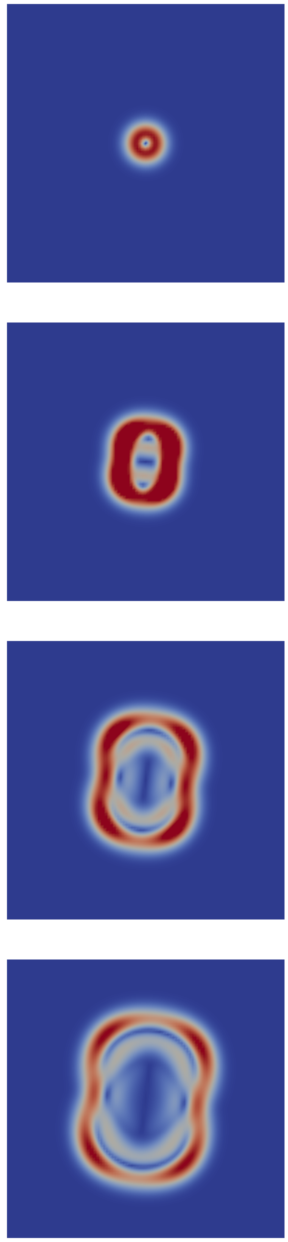
Figure 6.5: Ortho. 1

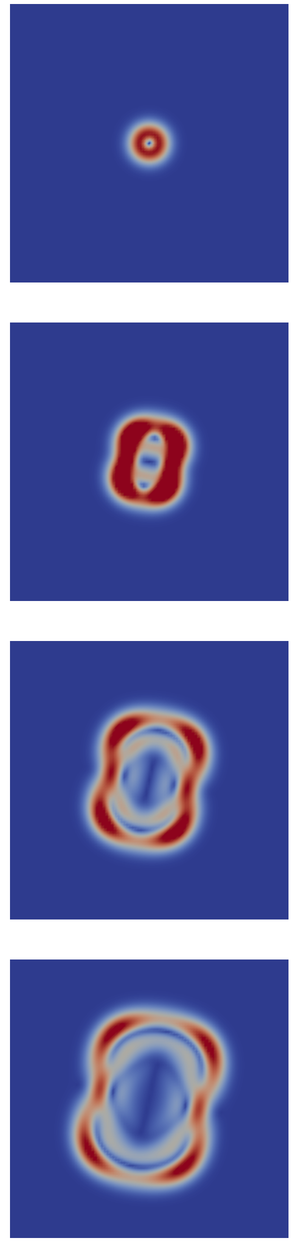
Figure 6.6: Ortho. 2

6.2 Viscoelasticity of materials

In this section, we show numerical experiments which are more involved in physical implications. In 6.2.1, we compare creep compliances of the viscoelastic models.

6.2.1 Creep compliance

We use our numerical schemes to test creep compliance of viscoelastic materials. As a model problem, we consider a rectangular object on $[-1, 1] \times [0, 1]$. We assume that the top is clamped, i.e., vanishing displacement boundary conditions are given. For the other sides, the left and right are traction-free and the bottom is exerted by a given normal force

$$\begin{pmatrix} 0 \\ 10e^{-10x^2} \end{pmatrix}$$

only from $t = 2$ to $t = 10$.

In our numerical computations, we used 16×32 mesh refinements, the AFW elements of degree 1, and $\Delta t = 0.1$ for time stepping. The mass density ρ is 1 in all simulations. The materials parameters are

$$\begin{aligned} \text{Zener} : \mu_0 = 500, \mu_1 = \mu_2 = 10, \lambda_0 = 500, \lambda_1 = \lambda_2 = 10, \\ \text{Kelvin–Voigt} : \mu_0 = \mu_1 = 10, \lambda_0 = \lambda_1 = 100, \\ \text{Maxwell} : \mu_0 = 1000, \mu_1 = 100, \lambda_0 = 1000, \lambda_1 = 100. \end{aligned}$$

Note that these parameters are not involved in specific physical motivations and are only based on purely numerical experiments for visualization.

In Figure 6.7 we compare vertical displacements at $(0, 0)$ in time of the three materials. In this figure, we can see that the Zener and Kelvin–Voigt models show similar behaviors. From the physical viewpoint, the shape begins deforming under the exerted force at $t = 2$ but the deformation process is not instantaneous. When the exerted force is removed at $t = 10$, the shape recovers to the original shape but this recovering is not instantaneous either. We can see that the Maxwell model shows a very different, fluid-like, behavior. The shape deformation is proportional to the exerted time and it does not recover after the exerted force is removed.

6.2.2 Attenuation of reflected waves

We consider a simple model problem for reflected waves. We compare reflected waves for two media. The first medium, say Medium I, consists of purely elastic materials and the second medium, Medium II, has an intermediate layer which is viscoelastic. In Figure 6.8, regions of elastic and viscoelastic materials are described. We call the three layers as Layer 1, 2, and 3.

For implementation, we use the Zener model with A_0' which is vanishing on regions that the medium is elastic. The domain is $[-15, 15] \times [-15, 0]$ with

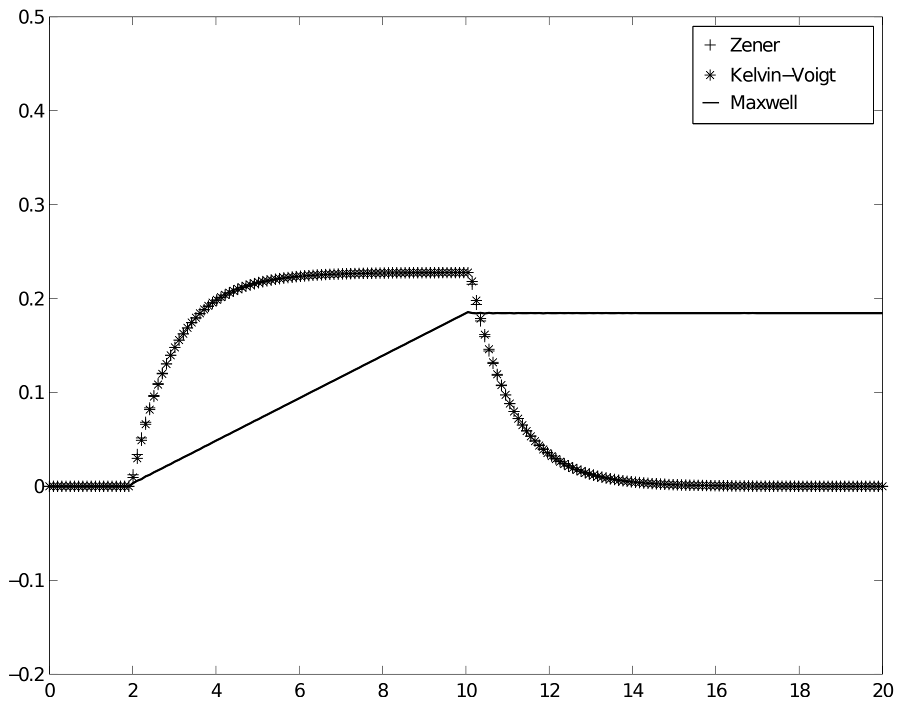

Figure 6.7: Comparison of creep compliances of the Zener, Kelvin–Voigt, and Maxwell models. ($\Delta t = 0.1$)

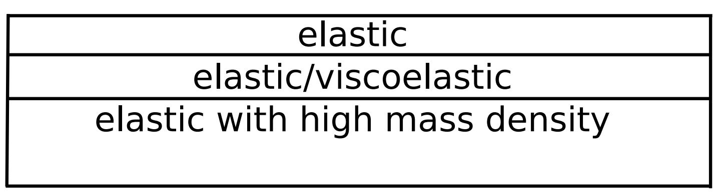

Figure 6.8: Regions of elastic and viscoelastic materials.

180×60 structured triangular meshes. The coordinates of two internal border lines of layers are $y = -1.5$, $y = -3.0$, respectively. We used the AFW elements of degree 1 and the time step interval is $\Delta t = 0.01$. The material parameters in the layers for Medium I and II are as follows.

Medium I

Layer 1 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \rho = 1$

Layer 2 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \rho = 1$

Layer 3 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \rho = 5.$

Medium II

Layer 1 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \rho = 1$

Layer 2 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \mu_1 = \lambda_1 = 500, \rho = 1$

Layer 3 : $\mu_0 = \mu_2 = \lambda_0 = \lambda_2 = 10, \rho = 5.$

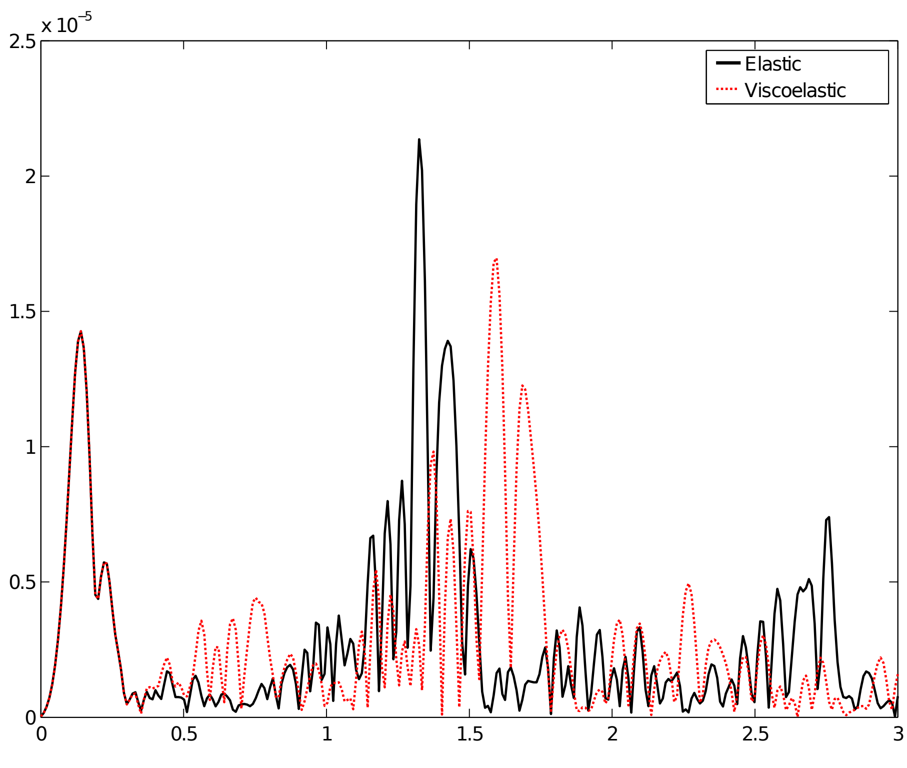

Figure 6.9: Magnitude of reflected waves for the media of elastic and viscoelastic intermediate layers.

In Figure 6.9, we recorded the magnitude of displacement at (0.0) for Medium I and II. The red dashed line, denoting the magnitude of the reflected waves for Medium II, has lower peak than the one for Medium I. Although this model is a schematic simplified one and the material parameters are not based on physical motivations, we expect that this attenuation phenomenon of reflected waves can be used for constructing more realistic models when it is combined with experimental data in practical situations. However, a further study of this modeling problem is beyond the scope of this dissertation.

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