Robin Boundary Conditions for the Hodge Laplacian and Maxwell's Equations

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Bo Yang

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

Advisor: Prof. Douglas N. Arnold

September, 2019

Acknowledgements

I am most grateful to my advisor Prof. Douglas N. Arnold. His enthusiasm toward mathematics, his superb lectures, his patience have inspired me greatly throughout my graduate study. Without his encouragement and support, it would be impossible for me to overcome many hard times and complete this work. I am also thankful to my family for their unconditional support for years.

Dedication

To my family.

Abstract

In this thesis we study two problems. First, we generalize the Robin boundary condition for the scalar Possoin equation to the vector case and derive two kinds of general Robin boundary value problems. We propose finite elements for these problems, and adapt the finite element exterior calculus (FEEC) framework to analyze the methods. Second, we study the time-harmonic Maxwell's equations with impedance boundary condition. We work with the function space $\mathscr{H}(\mathrm{curl})$ consisting of L^2 vector fields whose curl are square integrable in the domain and whose tangential traces are square integrable on the boundary. We will show convergence of our numerical solution using $\mathscr{H}(\mathrm{curl})$-conforming finite element methods.

Contents

List of Tables

Chapter 1

Introduction

In this thesis, we study finite element methods for two problems related to the Hodge-Laplace operator. First, the Hodge-Laplacian with two kinds of Robin boundary conditions, and second, the time-harmonic Maxwell's problem.

Partial differential equations, or PDEs, are widely used to describe laws in science and engineering. The Laplace operator occurs in many of the most common and fundamental ones. It is related to many phenomena in physics. For example, in the diffusion of heat flow, wave propagation, and Schrödinger's equation in quantum mechanics.

Often the PDEs in physical models are impossible to solve analytically. To overcome this and for practical purposes in science and engineering, people seek approximate numerical solutions that are close enough to the true solution. That is, instead of looking for a function u that solves the problem P, one uses a mesh in the domain, studies another finite dimensional problem P_h defined on the mesh and find its solution u_h, where the index h is a size parameter of the mesh. Such a discrete problem P_h is much easier to solve, as it is a matrix-vector equation. Then one measures some suitable error norm $\|u - u_h\|$. As refining the mesh and obtain a sequence of numerical solutions u_h, we hope u_h to converge to the real solution u. It is natural to consider these questions: Does the original problem P admits a unique solution? Does P_h do so as well? As the parameter $h \to 0$, does $\|u - u_h\| \to 0$? If so, how fast does it converge? The finite element method is one of the most useful numerical tools in study of PDEs. In this thesis, we will propose finite element methods for the Hodge-Laplace equation with two kinds of Robin boundary conditions, and the time-harmonic Maxwell equation with impedance boundary condition. We will analyze the continuous and the discrete problems, and establish the convergence of our methods.

1.1 Poisson problem and Hodge Laplacian

The scalar Poisson equation $-\Delta u = f$ is well studied with three kinds of boundary conditions on the boundary: the Dirichlet condition $u = 0$, the Neumann condition $\partial u/\partial n = 0$, and the Robin condition $u + \lambda\, \partial u/\partial n = 0$. When writing the Neumann problem into variational form, one has

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v dx = \int_\Omega f v dx,$$

with both trial and test functions $u, v \in H^1$. For the Dirichlet problem, one has the same variational form, but with $u, v \in \mathring{H}^1$. For the Robin problem, the variational form is

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v dx + \frac{1}{\lambda} \int_\Gamma u v ds = \int_\Omega f v dx.$$

Thus we see that the Neumann boundary condition is implicitly imposed in the variational form, while the Dirichlet boundary condition is explicitly imposed on the function space. For this reason, we say they are *natural* and *essential* boundary conditions, respectively[1].

In the vector case in three dimensions, the equation becomes

$$(\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f. \tag{1.1}$$

It is a known fact (cf. [3] for instance) that we have counterparts of the Neumann and Dirichlet boundary conditions for (1.1). One of our results in this thesis is establishment of two kinds of Robin boundary conditions for (1.1).

To treat scalar and vector Laplacian in three dimensions in a uniform way, we follow [3] and view scalar and vector functions as proxies of differential forms, given by Table 1.1. Our usual differential operators, grad, curl, and div, can be viewed as instances of the exterior differential d. We thus have the Hodge-Laplacian

$$(d\delta + \delta d)u = f$$

where δ is the formal adjoint of d. Moreover, we may impose the natural boundary condition

$$\operatorname{tr} \star du = 0, \quad \operatorname{tr} \star u = 0,$$

[1] The concept is relative to the variational form we adopt. If one tries mixed variational form instead, the Neumann boundary condition becomes essential, and the Dirichlet boundary condition becomes natural. See Chapter 2 for detail.

or the essential boundary condition

$$\operatorname{tr} u = 0, \quad \operatorname{tr} \delta u = 0.$$

The novelty in this thesis includes proposal and analysis of two types of Robin boundary conditions, namely the *semi-natural* Robin boundary condition

$$\operatorname{tr} \star du = 0, \quad \operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \delta u = 0,$$

and the *semi-essential* Robin boundary condition

$$\operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0, \quad \operatorname{tr} \delta u = 0.$$

We will explain the meaning of the trace operator tr and the Hodge star $\star$ in detail in Chapter 2.

degree k	differential form	function
0	u	u
1	$\sum_{i=1}^3 u_i\, dx_i$	(u_1, u_2, u_3)
2	$\sum_{i=1}^3 (-1)^{i+1} u_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$	$(u_1, -u_2, u_3)$
3	$u\, dx_1 \wedge dx_2 \wedge dx_n$	u

Table 1.1: Isomorphisms between k-forms ($k = 0, 1, 2, 3$) in three dimensions and functions.

Another discussion we will have in this thesis is the harmonic functions for these problems. As we know, depending on the topology of the domain, there may exist functions that satisfy the homogeneous Hodge-Laplace equation with homogeneous boundary conditions. Those are called harmonic functions. It is essential to include them in our formulation properly to derive well-posedness. One interesting result in our research is that the harmonic function spaces for our semi-natural and semi-essential Robin BVPs are identical to those for the natural and essential BVPs, respectively.

1.2 FEEC theory

As a large portion of our analysis heavily depends on the theory of *finite element exterior calculus* [4, 2], or *FEEC*, we briefly go over it in this section. Developed over the past decade by Arnold, Falk, and Winther, FEEC analyzes problems associated with general differential complexes, and provides a freamwork for analysis of concrete problems. One essential result is that if one has a complex (W, D) consisting of spaces W and differential operators D:

$$\cdots \xrightarrow{D^{k-2}} W^{k-1} \xrightarrow{D^{k-1}} W^k \xrightarrow{D^k} W^{k+1} \xrightarrow{D^{k+1}} \cdots, \tag{1.2}$$

and if D and their domains V possesses certain properties, which make (W, D) a *closed Hilbert complex*, then the Hodge-Laplacian is well-posed.

As an example, the natural BVP corresponds to the L^2 de Rham complex, with domains being spaces $H\Lambda^k$, where $H\Lambda^k$ represents the spaces of differential forms that are square integrable and whose exterior differentials, $(k+1)$-forms, are also square integrable. In three dimensions, these domain spaces correspond to the function spaces H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$, and L^2. For the essential BVPs, we have similar results, except that we impose vanishing boundary traces on all these spaces. In this thesis, we will introduce the Hilbert spaces $\mathscr{H}\Lambda^k$, which consists functions in $H\Lambda^k$ whose trace is square integrable on the boundary. In three dimensions, $\mathscr{H}\Lambda^k$ corresponds to H^1, $\mathscr{H}(\mathrm{curl})$, $\mathscr{H}(\mathrm{div})$, and L^2, where $\mathscr{H}(\mathrm{curl})$ (resp. $\mathscr{H}(\mathrm{div})$) is the subspace of $H(\mathrm{curl})$ (resp. $H(\mathrm{div})$) consist of functions with square integrable tangential (resp. normal) traces. We will provide Hilbert complexes associated with Robin BVPs.

In order to apply the theory, one non-trivial part is to verify a general Poincaré inequality. In general, we need to prove there is $C > 0$ such that for all $u \in W^k$ orthogonal to the null space of D^k, it holds that $\|u\| \leq C\|Du\|$. For both natural and essential BVPs, Poincaré inequalities are established in [3]. In Chapter 3 of this thesis, we will validate various Poincaré inequalities for our Robin BVPs.

Once we have a closed Hilbert complex, FEEC also studies a discrete complex (V_h, d) where $V_h \subset V$ are finite dimensional. Relating (V, D) and (V_h, D) is a cochain projection Π with commuting property, as illustrated below,

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{D^{k-2}} & V^{k-1} & \xrightarrow{D^{k-1}} & V^k & \xrightarrow{D^k} & V^{k+1} & \xrightarrow{D^{k+1}} & \cdots \\
 & & \big\downarrow{\scriptstyle \Pi^{k-1}} & & \big\downarrow{\scriptstyle \Pi^k} & & \big\downarrow{\scriptstyle \Pi^{k+1}} & & \\
\cdots & \xrightarrow{D^{k-2}} & V_h^{k-1} & \xrightarrow{D^{k-1}} & V_h^k & \xrightarrow{D^k} & V_h^{k+1} & \xrightarrow{D^{k+1}} & \cdots
\end{array} \tag{1.3}$$

The discrete complex (V_h, D) gives us a mixed finite element method. If these Π are uniformly bounded, then we have stability of the mixed method, and convergence of the discrete solution to the true solution. Efforts have been put into finding such uniformly bounded cochain projections. For L^2 de Rham complexes, Arnold, Falk, and Winther [3] first provided such projection, based on earlier works of Chistiansen [9] and Schöberl [30]. Later, Christiansen and Winther [11] found a uniformly bounded cochain projection for de Rham complexes with vanishing traces. Unfortunately, neither projection fits our need for the Robin BVPs, because our problems involve nonvanishing traces on the boundary. Nevertheless, thanks to [4, Theorem 3.7], we can overcome this issue by verifying two things. First, we need to prove some discrete Poincaré inequality. Second, we verify the *gap* between continuous and discrete harmonic function spaces is not too large, cf. (3.8). As we will see in Chapter 3, establishment of discrete Poincaré inequalities requires the most effort. For the harmonic function space gap, we can apply some known results.

1.3 Finite element spaces

For our application, two families of finite elements are useful, the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ spaces. These notations are from the exterior calculus point of view, and unifies some well-known finite elements, such as the Lagrange element, the Nédélec elements [27, 28], and the Discontinuous Galerkin element. Studies of these spaces can be found in many works, for instance [3, 10, 19]. In particular, for 0 and 3 forms, we have $\mathcal{P}_r\Lambda^0 = \mathcal{P}_r^-\Lambda^0$ and $\mathcal{P}_r\Lambda^3 = \mathcal{P}_{r+1}^-\Lambda^3$. Thus we only have the Lagrange element (for $k = 0$) and the Discontinuous Galerkin element (for $k = 3$).

In our analysis, it is common that we need to use some projection from our continuous function space $\mathscr{H}\Lambda^k$ to the finite element. A difficulty is to find a suitable bounded projection. As we mentioned before, the smoothed cochain projection cannot control the boundary traces. Thus we will use the canonical projection. However, it is not bounded for all $H\Lambda^k$, as for functions in those spaces, we do not necessarily have bounded traces on faces, edges, or vertex values. Fortunately, Amrouche *et al.* [1] proved that with extra regularity, the canonical projection is defined and bounded. Therefore, we will need to refer and prove a number of lemmas to establish such regularity.

1.4 Maxwell's equations

Maxwell's equations, first completed by James C. Maxwell in 1873, consists of four fundamental physical laws in electromagnetism, Gauss' laws for electricity and for magnetism, Faraday's law, and Ampere's law. This set of differential equations has numerous uses and applications. Almost everything in modern technology involving electricity or magnets, for instance, magnetic tape, electricity generation, computers, and MRI scanners, is based on understanding of electromagnetism and hence governed by these equations.

Owing to Maxwell's equations' high importance, there is a huge amount of related researches. Among them, numerical analysis of the equations is a useful and active branch. Various finite difference and finite element methods have been proposed and studied [21, 29, 12, 13, 22, 14, 23]. In this thesis, we will approach the problem with the Nédélec edge element, and prove its convergence.

In order to uniquely determine a solution to the system, we need to impose a suitable condition on the boundary. Different boundary conditions for Maxwell's equations reflects the surface characters of different materials. For the simplest example, if we have a perfect conductor, then on the boundary, we will have normal electric field and tangential magnetic field, that is, $E \times n = 0$ and $B \cdot n = 0$. However, if the surface is coated with some material that allows the electric fields to penetrate a small distance, a more appropriate boundary condition that models the electromagnetic behavior of the coating is

$$H \times n = \lambda n \times E \times n. \tag{1.4}$$

This is the impedance boundary condition, which we will analyze. As we have tangential traces on the boundary, our function space will be $\mathscr{H}(\mathrm{curl})$.

In general, Maxwell's equations are a time-dependant system, and it is possible to recognized it as a Hodge wave equation, and apply finite element methods on the system (cf. [2] for instance). Alternatively, if we assume all source fields are sinusoids with a fixed frequency ω, then so are the net fields. In this *time-harmonic* case, which we study in the thesis, every fields can be viewed as the real part of the product of a purely-spacial function and a time-dependent factor $e^{-i\omega t}$. We will derive and analyze a method for the time-harmonic Maxwell's equations with impedance boundary condition (1.4) in Chapter 4. The major difference between our result and Monk's [26] is the boundary condition we use in our analysis.

1.5 Outline

We divide the remainder of the thesis into three parts.

The first part consists of Chapters 2 and 3. We will study Hodge-Laplacian in three dimensions with two types of Robin boundary conditions.

The first half of Chapter 2 is mostly dedicated to review of natural and essential boundary value problems and related concepts in exterior calculus. In the second half of this chapter, we will introduce the two kinds of Robin BVPs. In particular, we will see that these two sets of boundary conditions can be viewed as natural and essential boundary conditions with perturbation terms. In addition, when the domain has nontrivial geometry, we will encounter harmonic functions. We will see that our problems of interest have the same harmonic function space as the natural and essential BVPs. Thus we will give them the names *semi-natural* and *semi-essential* Robin BVPs.

Chapter 3 first covers some preliminary results from the FEEC theory. Then a lot of effort is put into constructing suitable closed Hilbert complexes that will fits our two kinds of Robin BVPs. This will leads to well-posedness of our problems. Further, we propose finite elements for our problem, and give proof of stability of the discrete problems and convergence of the discrete solutions.

The second part is Chapter 4 alone. We will give a brief introduction to Maxwell's equations and boundary conditions Then we will focus on the case of time-harmonic fields with impedance boundary condition and derive the special formulation for our problem. We follow [26] to give proof of well-posedness of the continuous Problem 4.10. Next we propose to use the $H(\text{curl})$-conforming Nédélec elements as our numerical method. We will analyze the stability and convergence of our method. The result generalizes that of [26].

In the last part, Chapter 5, we provide several numerical examples that verify our theory.

Chapter 2

Hodge-Laplace equation with Robin boundary conditions

2.1 Introduction

In this Chapter, we will introduce the Hodge-Laplace equation with Robin boundary conditions and related finite elements. In a domain $\Omega \in \mathbb{R}^3$, the Poisson equation is a simple second order differential equation that has the form

$$\Delta u = \sum_i \frac{\partial^2}{\partial x_i^2} u = f, \quad \text{in } \Omega. \tag{2.1}$$

Depending on whether u and f are scalar or vector functions, (2.1) may stand for two different equations. As discussed in [3], we can establish correspondence between differential forms and functions. Thus, all these equations can be included in a more general problem of solving

$$(\delta d + d\delta)u = f, \tag{2.2}$$

where u and f are differential forms, and in the operator $(\delta d + d\delta)$, called the Hodge-Laplacian, d is the exterior differential, and δ is its formal adjoint. Besides its uniformity, (2.2) also suggest a way of rewriting the original problem in a variational formulation in which there are a pair of unknowns $(\sigma, u) := (\delta u, u)$. Details will be given in following sections.

For the scalar Poisson equation, three boundary conditions are typically considered, Dirichlet condition, Neumann condition, and Robin condition, as shown below:

$$u = 0, \quad \text{or} \quad \frac{\partial u}{\partial n} = 0, \quad \text{or} \quad u + \lambda \frac{\partial u}{\partial n} = 0.$$

As we come to vector version of Poisson equation, or Hodge-Laplace equation, we will see counterparts of these boundary conditions. In the light of [3], we will call the first two boundary conditions and their counterparts the natural and the essential conditions. The third kind of boundary condition for Hodge-Laplace equation is not as well-known, and is a major topic of this thesis. We will show it further splits into two kinds for vector equations. We will called them the first and the second kinds of Robin boundary condition and study finite element methods regarding such boundary conditions.

We will start by reviewing existing theories in exterior calculus in Section 2.2. Besides basic concepts, we will see correspondence between usual functions and differential forms. Differential operators, such as grad, curl, div will be interpreted as exterior differentials and their formal adjoint operators. With this knowledge, we can define some useful function spaces in the language of exterior calculus. Next, we will review some results for the Hodge Laplace problems with natural or essential boundary conditions in Section 2.3. In Section 2.4, two sets of finite element spaces are introduced due to [3]. Consisting of known finite element spaces, these sets provide a new perspective that connect those spaces. We will also recall some results regarding projections from function spaces to finite elements. Finally, we generalize the scalar Poisson equation with the well-known Robin boundary condition in two ways, which gives us two kinds of Robin boundary value problems in Section 2.5.

2.2 Definitions and notations from exterior calculus

Throughout this thesis, we denote by Ω a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, and by Γ the boundary of Ω. For any such domain Ω, we can consider differential k-forms, each element u of which is an alternating k-linear map at every $x \in \Omega$: $u_x : T_x\Omega \times \dots T_x\Omega \to \mathbb{R}$. Here $T_x\Omega$ is the tangent space at x, which is $\mathbb{R}^n$ under our assumption on Ω. We can define the wedge product of a k_1-from u_1 and a k_2-form u_2, denoted by $u_1 \wedge u_2$, to be the following (k_1+k_2)-form: For any (k_1+k_2) vectors $v_1, \dots, v_{k_1+k_2} \in T_x\Omega$,

$$
\begin{aligned}
&(u_1 \wedge u_2)_x(v_1, \dots, v_{k_1+k_2}) \\
&\qquad = \sum_\sigma (\operatorname{sgn}\sigma)\, u_{1,x}(v_{\sigma(1)}, \dots, \sigma(k_1))\, u_{2,x}(\sigma(k_1+1), \dots, \sigma(k_1+k_2)),
\end{aligned}
$$

where σ denotes the permutations of $(1, \dots, k_1+k_2)$.

We denote $e_1, \dots, e_n$ an orthonormal basis of $T_x\Omega = \mathbb{R}^n$. Naturally, at each $x \in \Omega$, a differential k-forms has the following basis:

$$
\{dx_{i_1} \wedge \dots \wedge dx_{i_k} \mid 1 \le i_1 < \dots < i_k \le n\},
$$

where $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ maps the k-tuple $(e_{j_1}, \ldots, e_{j_k})$ to 0 unless $\{j_1, \ldots, j_k\}$ is a permutation of $\{i_1, \ldots, i_k\}$, in which case the image is the sign of that permutation. Thus each differential k-form can be written as

$$u = \sum_{|I|=k} u_I \, dx_I,$$

for functions u_I defined in Ω. Here we used the multi-index $I = (i_1, \ldots, i_k)$, whose components are in $\{1, \ldots, n\}$, and dx_I denotes $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

In n dimensions, there are interesting and useful isomorphisms between 0-, 1-, $(n-1)$-, and n-forms and usual functions. In particular, 0- and n-forms can be viewed as scalar functions, while 1- and $(n-1)$-forms correspond to n-dimensional vector functions. We give the isomorphisms in Table 2.1.

degree k	differential form	function
0	u	u
1	$\sum_{i=1}^{n} u_i \, dx_i$	$(u_1, \ldots, u_n)$
$n-1$	$\sum_{i=1}^{n} u_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$	$(u_1, -u_2, \ldots, (-1)^{n+1} u_n)$
n	$u \, dx_1 \wedge \ldots \wedge dx_n$	u

Table 2.1: Isomorphisms between k-forms ($k = 0, 1, n-1, n$) in n dimensions and functions.

We can define integrability and regularity for differential forms. A k-form u is said to be in $L^p(\Omega)$, denoted by $u \in L^p\Lambda^k(\Omega)$, if and only if for any k smooth vector fields $v_1, \ldots, v_k$, the function $x \mapsto u(v_1, \ldots, v_k)$ is in $L^p(\Omega)$. Because our domain Ω is a compact sub-manifold of $\mathbb{R}^3$, thus given any basis of $\mathbb{R}^3$, a form $u = \sum_I u_I dx_I$ is in $L^p(\Omega)$ if and only each u_I, called a *coefficient*, is in $L^p(\Omega)$. Similarly, we can define $C^\infty\Lambda^k(\Omega)$, $H^s\Lambda^k(\Omega)$, etc. All these properties can be characterized by the regularity of the coefficients u_I[1]. As convention, we denote by d^k the exterior derivative from k-forms to $(k+1)$-forms, i.e., assuming that $u = \sum_I u_I dx_I$, then $d^k u$ is defined to be the following $(k+1)$-form:

$$d^k u = \sum_I \sum_{j=1}^{n} \frac{\partial u_I}{\partial x_j} dx_j \wedge dx_I.$$

We may abuse notation to write du when the degree of u is clear.

With the isomorphism introduced just before, we can identify some usual differential operators as the exterior derivative. If u is a 0-form, which is also a scalar function, du is isomorphic to $\operatorname{grad} u$. If u is an $(n-1)$-form, du is $\operatorname{div} u$. For a 1 form u, du can be used to define the curl operator in n dimensions. In particular, it is identical to the

[1]The regularity of such u_I is independent of the choice of basis.

usual rotation operator in 3 dimensions:

$$\operatorname{curl}(u_1, u_2, u_3) = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).$$

We have a natural connection between k-forms and $(n-k)$-forms via the Hodge star, $\star$. For any dx_I, $\star dx_I$ is defined to be $\pm dx_J$, where I and J partitions $N = \{1, \ldots, n\}$, and the sign of the image is the same as the sign of the permutation (I, J) of $(1, \ldots, n)$. Then by linearity, we can define $\star$ for all k-forms.

With the definition of the Hodge star operator, we have the following for any k-form u_1 and any $(n-k)$-form u_2 at each $x \in \Omega$:

$$\langle \star u_1, u_2 \rangle \, dx_1 \wedge \ldots \wedge dx_n = u_1 \wedge u_2.$$

Thus we can consider the inner product of two k-forms u and v over the whole domain Ω:

$$\langle u, v \rangle_\Omega = \int_\Omega u \wedge \star v.$$

Using the Hodge star operator, we can also introduce the coderivative operator, denoted by δ, which is an essential ingredient for the Hodge Laplace problem. For any k-form u, δu is defined to be the $(k-1)$-form that satisfies

$$\star \delta u = (-1)^k d \star u.$$

Next, in order to implement proper boundary conditions in our problems, we need to consider the trace operator of differential forms. For any domain Ω, we have the natural embedding of Γ into Ω: $i : \Gamma \hookrightarrow \Omega$. Thus for any k-form u defined on Ω, we can define the trace of a k-form u to be the pullback of u under i: for any $v_1, \ldots, v_k \in T_x\Gamma$, we have $v_1, \ldots, v_k \in T_x\Omega$, and

$$\operatorname{tr} u_x(v_1, \ldots, v_k) = u_x(v_1, \ldots, v_k).$$

Thus, at each point x, $\operatorname{tr} u$ acts as an alternating k-linear map on the tangent space $T_x\Gamma$. In other words, $\operatorname{tr} u$ is a differential k-form on Γ.

In the view of exterior calculus, one can write various differential equations in a unified form. In order to formulate those problems in one framework, we need the *domain* of d:

$$H\Lambda^k(\Omega) = \left\{ u \in L^2\Lambda^k(\Omega) \mid du \in L^2\Lambda^{k+1}(\Omega) \right\}.$$

That is, $H\Lambda^k(\Omega)$ is the space of k-forms whose exterior differentials are in L^2. Similarly, we define the domain of δ, denoted by $H^*\Lambda^k$, to be the space of k-forms whose coderivatives are also in L^2:

$$H^*\Lambda^k(\Omega) = \left\{ u \in L^2\Lambda^k(\Omega) \mid \delta u \in L^2\Lambda^{k-1}(\Omega) \right\}.$$

These spaces are useful when one deals with natural boundary value problem, as later we will see examples of.

We remark that the trace operator defined above maps everything in $H\Lambda^k(\Omega)$ into $H^{-1/2}\Lambda^k(\Gamma)$. For any $u \in C^\infty\Lambda^k$, and any $\rho \in H^{1/2}\Lambda^k(\Gamma)$, we have $\star_\Gamma\rho \in H^{1/2}\Lambda^{n-1-k}(\Gamma)$. Hence we can find $v \in H^1\Lambda^{n-1-k}(\Omega)$ such that $\operatorname{tr} v = \star_\Gamma\rho$. Then, we have

$$\begin{aligned}\langle \operatorname{tr} u, \rho\rangle_\Gamma &= \int_\Gamma \operatorname{tr} u \wedge \star_\Gamma\rho = \int_\Gamma \operatorname{tr} u \wedge \operatorname{tr} v = \int_\Omega d(u \wedge v) \\ &= \int_\Omega \Big[du \wedge v + (-1)^k u \wedge dv \Big] \le 2\|u\|_{H\Lambda}\|v\|_{H^1} \le C\|u\|_{H\Lambda}\|\rho\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Next, we can extend the definition of tr to $H\Lambda^k$ by density of $C^\infty\Lambda^k$ in that space. The above inequality shows the extended trace operator gives us an element in $H^{-1/2}\Lambda^k(\Gamma)$.

It is also crucial to consider essential boundary value problems. In this case, we need to introduce another space, $\mathring{H}\Lambda^k(\Omega)$. It is a subspace of $H\Lambda^k(\Omega)$, whose elements have vanishing traces:

$$\mathring{H}\Lambda^k(\Omega) = \left\{ u \in H\Lambda^k(\Omega) \mid \operatorname{tr} u = 0 \right\}.$$

We also have a counterpart for $H^*\Lambda^k(\Omega)$:

$$\mathring{H}^*\Lambda^k(\Omega) = \left\{ u \in H^*\Lambda^k(\Omega) \mid \operatorname{tr} \star u = 0 \right\}.$$

In order to formulate the *Robin* boundary value problem, we define a space that lies between $H\Lambda^k$ and $\mathring{H}\Lambda^k$:

$$\mathscr{H}\Lambda^k(\Omega) = \left\{ u \in H\Lambda^k(\Omega) \mid \operatorname{tr} u \in L^2\Lambda^k(\Gamma) \right\}. \tag{2.3}$$

Similarly, we define the following subspace of $H^*\Lambda^k$:

$$\mathscr{H}^*\Lambda^k(\Omega) = \left\{ u \in H^*\Lambda^k(\Omega) \mid \operatorname{tr} \star u \in L^2\Lambda^{n-k}(\Gamma) \right\}.$$

When analyzing the well-posedness of the problems we consider, two spaces of harmonic forms will be needed. They are

$$\mathfrak{H}^k(\Omega) = \left\{ u \in H\Lambda^k(\Omega) \mid du = 0, \langle u, dv \rangle = 0, \forall v \in H\Lambda^{k-1}(\Omega) \right\}, \tag{2.4}$$

and

$$\mathring{\mathfrak{H}}^k(\Omega) = \left\{ u \in \mathring{H}\Lambda^k(\Omega) \mid du = 0, \langle u, dv \rangle = 0, \forall v \in \mathring{H}\Lambda^{k-1}(\Omega) \right\}. \tag{2.5}$$

We shall prove that a k-form is harmonic if and only if it vanishes under both d and δ. An important utility for the proof is integration by parts. It will be used in many other places in this thesis. For this reason, we state it here as a lemma.

Lemma 2.2.1. *For all $u \in H\Lambda^k$ and $v \in H^*\Lambda^{k+1}$. If one of them has H^1-regularity, there holds*

$$\langle du, v \rangle = \langle u, \delta v \rangle + \int_\Gamma \operatorname{tr} u \wedge \operatorname{tr} \star v,$$

If one of them has vanishing trace on Γ, there holds

$$\langle du, v \rangle = \langle u, \delta v \rangle.$$

Proof. We note that the first equation is valid if $u \in H\Lambda^k$, $v \in H^*\Lambda^{k+1}$, and one of them is smooth (in C^∞). Then, recalling the well-known result that C^∞ is dense in H^1, we can validate the first part. The proof to the second case is similar. Starting with $u \in H\Lambda^k$ and $v \in H^*\Lambda^{k+1}$ such that one of them is in C_0^∞, applying the density argument, we have the proof. □

Lemma 2.2.2. *The harmonic spaces with natural or essential boundary conditions have the following equivalent definitions:*

$$\mathfrak{H}^k(\Omega) = \left\{ u \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \mid du = 0, \delta u = 0 \right\},$$
$$\mathring{\mathfrak{H}}^k(\Omega) = \left\{ u \in \mathring{H}\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid du = 0, \delta u = 0 \right\}.$$

Proof. We only prove the first equation, since the second is similar. For any $u \in \mathfrak{H}^k$, and any $v \in C^\infty\Lambda^{k-1}(\Omega)$, we have, by Lemma 2.2.1, that

$$0 = \langle dv, u \rangle = \langle v, \delta u \rangle + \int_\Gamma \operatorname{tr} v \wedge \operatorname{tr} \star u.$$

This implies both $\delta u = 0$, and $\operatorname{tr} \star u = 0$. Thus "$\subset$" is proven in the first equation. To prove the other direction, one just reverse the arguments, and use the density of $C^\infty\Lambda^{k-1}$ in $H\Lambda^{k-1}$. □

2.2.1 Interpretation in 3 dimensions

We have already seen that 0-, 1-, $(n-1)$-, and n-forms are isomorphic to scalar- or vector-valued functions in general. In particular, if $\Omega \in \mathbb{R}^3$, we know how all forms are isomorphic to functions, or *proxy (scalar or vector) fields*. We also have interpretation of d and δ for $k = 0, 1, 2$, as shown in Table 2.2.

k	0	1	2	3
d	grad	curl	div	0
δ	0	$-$ div	curl	$-$ grad

Table 2.2: Correspondence between d, δ and usual differential operator in 3 dimensions.

The trace operator can be interpreted by appropriate operations applied on those proxy fields, as well. The trace of a 0-form ω, identical to a function w, is the same as the restriction on the boundary, $w|_\Gamma$. Just as a 1-form on a domain in $\mathbb{R}^2$ can be viewed as a 2-vector field, on the 2-dimensional manifold Γ, a 1-form may be viewed as a tangential vector field (since it acts linearly on the tangent space at each point). If ω is a differential form on Ω, and w is the associated vector field, then the vector field associated to $\operatorname{tr}\omega$ is the tangent vector field $n \times (w \times n)|_\Gamma$[2]. Similarly, if ω is a 2-form and w is the associated vector field, then $\operatorname{tr}\omega$ is associated to the scalar projection on the unit outer normal, i.e., $w \cdot n$. Finally, all 3-forms have vanishing traces. This is summarized in Table 2.3.

k	0	1	2	3
$\operatorname{tr}\omega$	$w\|_\Gamma$	$n \times (w \times n)\|_\Gamma$	$w \cdot n$	0

Table 2.3: Interpretation of the trace operators in 3 dimensions. Assuming the differential form ω has a proxy field w, we present the relationship between $\operatorname{tr}\omega$ and w.

Then, we explain the meanings of the Hodge star $\star$. If w is a scalar function, i.e., a 0-form, then $\star w = w\, dx_1 \wedge dx_2 \wedge dx_3$ is the 3-form associated to the same function. In other words, if we identify 3-forms with scalar functions, then the Hodge star operator from 0-forms to 3-forms is just the identity map. Similarly the Hodge star applied on a 1- or 2-form ω can be viewed as an identity operator on vector fields, because the proxy fields of ω and $\star\omega$ are identical vector fields. We also need to interpret the Hodge star $\star_\Gamma$ that applies on forms on Γ. If u is a 0- or 2-form, which can be viewed as a scalar field on the boundary, $\star_\Gamma u$ is a 2- or 0-form on Γ that represents an identical scalar field as u does. For a 1-form u on Γ, which represents a vector field, $\star_\Gamma u$ is another vector field, which is generated by rotating the original one for 90 degrees with respect to the

[2] Although the tangential projection corresponds to the trace, we do not call it the tangential trace. In literature (cf. [6][8] for instance), the term tangential trace is used for the tangent vector field $n \times w$, which differs from the tangential projection by a 90-degree rotation.

normal vector $n = e_1 \times e_2$ of Γ following right-hand rule, i.e., the vector field that $\star_\Gamma$ stands for is $n \times w$. When it is clear that we take the Hodge star of a form defined on the boundary, we may omit the subscript and just write $\star$ for $\star_\Gamma$.

2.3 Hodge Laplacian with natural or essential boundary conditions

The Hodge Laplace equation

$$Lu = (d\delta + \delta d)u = f \tag{2.6}$$

is a generalization of the usual Poisson equation. The equation by itself is not a well-posed problem, unless suitable boundary conditions and other restrictions are given. In general, given such boundary conditions, we are interested in the problem: Given $f \in L^2\Lambda^k$, we will seek u in the domain of the Hodge Laplacian

$$D(L) = \left\{ u \in H\Lambda^k \cap H^*\Lambda^k \mid \delta u \in H\Lambda^{k-1}, du \in H^*\Lambda^{k+1} \right\}, \tag{2.7}$$

such that the equation (2.6), as well as the boundary conditions, is satisfied.

The goal of this chapter is to introduce two kinds of Hodge Laplacian with *Robin* boundary conditions, a term that we will make clearer later. Before turning to these, we first recall the simpler case of natural boundary value problem (natural BVP) and the case of essential boundary value problem (essential BVP) in this section.

2.3.1 Natural BVP

Given any $f \in L^2\Lambda^k$, we will seek $u \in D(L)$ that satisfies the Hodge Laplacian (2.6) with the boundary conditions

$$\operatorname{tr} \star u = 0, \quad \operatorname{tr} \star du = 0 \text{ on } \Gamma. \tag{2.8}$$

This problem may not be well-posed. One can consider the homogeneous problem: Find $u \in D(L)$ such that $Lu = 0$ and (2.8) are satisfied. We have the solution space is $\mathfrak{H}^k$, the space of harmonic k-forms, as defined before.

Lemma 2.3.1. *Any $u \in D(L)$ satisfies $Lu = 0$ and (2.8), if and only if $u \in \mathfrak{H}^k$.*

Proof. The “if” part is straightforward from Lemma 2.2.2. We now prove the “only if” part. If $u \in D(L)$ satisfies $Lu = 0$, we have

$$\langle d\delta u, u\rangle + \langle \delta du, u\rangle = 0.$$

We integrate by parts (cf. Lemma 2.2.1), and write

$$\langle d\delta u, v\rangle = \langle \delta u, \delta v\rangle + \int_\Gamma \operatorname{tr}\delta u \wedge \operatorname{tr}\star v, \quad \forall v \in H^1\Lambda^k,$$
$$\langle w, \delta du\rangle = \langle dw, du\rangle - \int_\Gamma \operatorname{tr} w \wedge \operatorname{tr}\star du = \langle du, dw\rangle, \quad \forall w \in H^1\Lambda^k,$$

where we applied (2.8) in the last equation. In the first equation, we can take $v = v_n \to u$ in $H^*\Lambda^k$ by density of $H^1\Lambda^k$ in $H^*\Lambda^k$, and hence obtain that $\langle d\delta u, u\rangle = \|\delta u\|^2$. In the second equation, we similarly let $w = w_n \to u$ in $H\Lambda^k$ by density of $H^1\Lambda^k$ in $H\Lambda^k$, and thus have $\langle \delta u, \delta u\rangle = \|du\|^2$. Then, we have $du = 0$ and $\delta u = 0$, and proven that $u \in \mathfrak{H}^k$. □

Similar to the scalar Neumann problem $-\Delta u = f$, $\partial u/\partial n = 0$, in which we imposed compatibility conditions to guarantee well-posedness, we take into account of the harmonic space $\mathfrak{H}^k$, and have the well-posed natural BVP.

Problem 2.3.2 (Natural BVP)**.** *Given $f \in L^2\Lambda^k$, find $u \in D(L)$ such that $u \perp \mathfrak{H}^k$, and that*

$$(d\delta + \delta d)u = f - P_{\mathfrak{H}^k} f \text{ in } \Omega, \quad \operatorname{tr}\star u = 0, \quad \operatorname{tr}\star du = 0 \text{ on } \Gamma. \tag{2.9}$$

We point out that this problem unifies a class of (scalar or vector) Laplace problems by "translating" it to the language of partial differential equations using the correspondences that we determined before. For simplicity, we assume $f \perp \mathfrak{H}^k$, and omit the auxiliary condition $u \perp \mathfrak{H}^k$. If $k = 0$, it is the Laplace equation $-\Delta u = f$ with Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. If $k = 1$, it becomes the vector Laplace equation $(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})u = f$ with boundary conditions $u \cdot n = 0$ and $\operatorname{curl} u \times n = 0$. If $k = 2$, the equation is again the vector Laplacian, and the boundary conditions are $u \times n = 0$ and $\operatorname{div} u = 0$. If $k = 3$, it reduces to the scalar Laplacian with Dirichlet boundary condition $u = 0$. These interpretation are summarized in Table 2.4.

k	equation	BC1	BC2
0	$-\Delta u = f$	-	$\frac{\partial u}{\partial n} = 0$
1	$(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})u = f$	$u \cdot n = 0$	$\operatorname{curl} u \times n = 0$
2	$(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})u = f$	$u \times n = 0$	$\operatorname{div} u = 0$
3	$-\Delta u = f$	$u = 0$	-

Table 2.4: Interpretation of the natural BVP 2.3.2 in 3 dimensions.

In practice, it is not the best to directly analyze these two problems with those spaces, equations, and boundary condition. Much more preferable is to analyze their *weak formulations*.

Problem 2.3.3 (Weak form of the natural BVP)**.** *Find* $(\sigma, u, p) \in H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ *that satisfies*

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, d\tau \rangle &= 0, \quad \forall \tau \in H\Lambda^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in H\Lambda^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.10}$$

The formulation explains the origin of the name "natural" BVP. As we see from above, one does not impose any boundary condition in the space $H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$, while we will see in the next section that one needs to impose boundary conditions in the space for the essential BVP. Given the names *strong* and *weak* forms, one natural question is the relationship between them. As a matter of fact, these formulations are equivalent, as the following theorems states more precisely.

Theorem 2.3.4. *If* u *solves Problem 2.3.3, then* $(u, \sigma = \delta u, p = P_{\mathfrak{H}} f)$ *solves Problem 2.3.2. If* (u, σ, p) *solves Problem 2.3.2, then* u *solves Problem 2.3.3, and* $\sigma = \delta u$, $p = P_{\mathfrak{H}} f$.

We will not present the proof. However, we will have similar results (Theorems 2.5.4 and 2.5.7) for the Robin problems, which is the main course of this chapter and will be analyzed later. Since all these theorems have similar proof, we will only prove Theorems 2.5.4 and 2.5.7 later.

2.3.2 Essential BVP

In this section, we briefly review a second type of BVP for the Hodge Laplacian, the essential BVP. Everything discussed in this section is in strong analogy to the natural BVP, and hence we will omit details.

For any given $f \in L^2\Lambda^k$, we seek $u \in D(L)$ that satisfies the Hodge Laplace equation (2.6), and the boundary conditions

$$\operatorname{tr} \delta u = 0, \quad \operatorname{tr} u = 0. \tag{2.11}$$

Just as in the natural boundary conditions, there may be nontrivial solutions to the homogeneous problem. The space $\mathring{\mathfrak{H}}^k$ defined by (2.5) consists of all solutions of the homogeneous problem.

Lemma 2.3.5. *Any* $u \in D(L)$ *satisfies* $Lu = 0$ *and (2.11), if and only if* $u \in \mathring{\mathfrak{H}}^k$*.*

Proof. The "if" statement is obvious. Thus we just prove the "only if" part, as follows. Any $u \in D(L)$ such that $Lu = 0$ must satisfy

$$\langle d\delta u, u \rangle + \langle \delta du, u \rangle = 0.$$

We integrate by parts, and write

$$\langle d\delta u, v\rangle = \langle \delta u, \delta v\rangle + \int_\Gamma \operatorname{tr}\delta u \wedge \operatorname{tr}\star v = \langle \delta u, \delta v\rangle, \quad \forall v \in \mathring{H}^1\Lambda^k,$$
$$\langle w, \delta du\rangle = \langle dw, du\rangle - \int_\Gamma \operatorname{tr} w \wedge \operatorname{tr}\star du = \langle dw, du\rangle, \quad \forall w \in \mathring{H}^1\Lambda^k,$$

where (2.11) is used in the first equation. Then by density of $\mathring{H}^1\Lambda^k$ in $H\Lambda^k$ and in $H^*\Lambda^k$, we obtain that $\|du\|^2 + \|\delta u\|^2 = 0$, as before. Hence the lemma is proven. □

Now we can state the strong form of the essential BVP.

Problem 2.3.6 (Essential BVP). *Given $f \in L^2\Lambda^k$, find $u \in D(L)$ such that $u \perp \mathring{\mathfrak{H}}^k$, and*

$$(d\delta + \delta d)u = f - P_{\mathring{\mathfrak{H}}^k} f \text{ in } \Omega, \quad \operatorname{tr}\delta u = 0, \operatorname{tr} u = 0 \text{ on } \Gamma. \tag{2.12}$$

We note that $\operatorname{tr}\delta u$ and $\operatorname{tr} u$ are $(k-1)$- and k-forms on Γ, respectively. Hence the total dimension of the boundary conditions is given by

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$

where the last coincides with the dimension of k-forms in Ω. This means we have the right number of (scalar) boundary conditions in (2.12).

We now interpret the problem in 3 dimensions. For simplicity, we assume $f \perp \mathring{\mathfrak{H}}^k$, and omit that $u \perp \mathring{\mathfrak{H}}^k$. The meanings of Problem 2.3.6 are summarized in Table 2.5.

k	equation	BC 1	BC 2
0	$-\Delta u = f$	-	$u = 0$
1	$(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})u = f$	$\operatorname{div} u = 0$	$u \times n = 0$
2	$(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})u = f$	$\operatorname{curl} u \times n = 0$	$u \cdot n = 0$
3	$-\Delta u = f$	$\frac{\partial u}{\partial n} = 0$	-

Table 2.5: Interpretation of the essential BVP 2.3.6 in 3 dimensions.

From the tables (2.4) and (2.5), we see that the Hodge Laplacian with natural or essential boundary conditions covers several important boundary value problems: The (scalar) Laplace equation with Neumann and Dirichlet boundary condition, the vector Laplace equation with magnetic and electric boundary conditions. Nevertheless, some other related types of boundary conditions are not covered by these formulations. For instance, the scalar Laplace problem may come with Robin boundary condition. Another example comes from the impedance boundary condition from electromagnetics. These problems will motivate us to the *generalized Robin problem*, which we will discuss in detail later.

We will end this section with the weak formulation of the essential BVP and a theorem of equivalence of the two formulations.

Problem 2.3.7 (Weak form of the essential BVP)**.** *Find* $(\sigma, u, p) \in \mathring{H}\Lambda^{k-1} \times \mathring{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$ *that satisfies*

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, d\tau \rangle &= 0, \quad \forall \tau \in \mathring{H}\Lambda^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in \mathring{H}\Lambda^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathring{\mathfrak{H}}^k. \end{aligned} \tag{2.13}$$

Here we see the boundary conditions (2.11) are explicitly imposed in the space $\mathring{H}\Lambda^{k-1} \times \mathring{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$, which explains the name "essential". We also have the equivalence between the strong and weak formulations for the essential BVP, which is stated in the next theorem.

Theorem 2.3.8. *If u solves Problem 2.3.7, then $(u, \sigma = \delta u, p = P_{\mathring{\mathfrak{H}}} f)$ solves Problem 2.3.6. If (u, σ, p) solves Problem 2.3.6, then u solves Problem 2.3.7, and $\sigma = \delta u$, $p = P_{\mathring{\mathfrak{H}}} f$.*

2.4 Finite element spaces

We will consider two families of finite element spaces which will be used in our analysis and computation. First we need to introduce some function spaces.

Just as we defined $C^\infty\Lambda^k$, $H^s\Lambda^k$, etc., we define $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$(resp. $\mathcal{H}_r\Lambda^k(\mathbb{R}^n)$) to be the space of k-forms whose coefficients are polynomials of degree at most r (resp. homogeneous polynomials of degree r). As before, we may omit $\mathbb{R}^n$ in these notations when the context is clear.

Besides $\mathcal{P}_r\Lambda^k$ and $\mathcal{H}_r\Lambda^k$, we also need to define $\mathcal{P}_r^-\Lambda^k$, a space that lies between $\mathcal{P}_r\Lambda^{k-1}$ and $\mathcal{P}_r\Lambda^k$. Let us start by defining an operator, denoted by κ, that maps any k-form ω to a $(k-1)$-form $\kappa\omega$. For any $x \in \Omega$, and any $(k-1)$ vectors $v_1, \ldots, v_{k-1}$ in the tangent space $T_x\Omega$, we define

$$(\kappa\omega)_x(v_1, \ldots, v_{k-1}) = \omega_x(-X(x), v_1, \ldots, v_{k-1}),$$

where $X(x)$ is the vector that starts at x and terminates at the origin. The vector $X(x)$ in the definition is in $T_x\Omega$, as $\Omega \subset \mathbb{R}^n$. Thus the right-hand side of the last equation is meaningful.

An important property of κ is that it maps $\mathcal{H}_r\Lambda^k$ to $\mathcal{H}_{r+1}\Lambda^{k-1}$. In fact for each k-form $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, one can verify that $\kappa\omega = f \sum_{j \in I} (-1)^{j+1} x_j dx_{i_1} \wedge \ldots \wedge$

$\widehat{dx_j} \wedge \ldots \wedge dx_{i_k}$. The property hence follows. Consequently, for each k, we can define

$$\mathcal{P}_r^- \Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa \mathcal{P}_{r-1}\Lambda^{k+1} = \mathcal{P}_{r-1}\Lambda^k + \kappa \mathcal{H}_{r-1}\Lambda^{k+1}.$$

It is obvious that this space satisfies the desired property that $\mathcal{P}_{r-1}\Lambda^k \subset \mathcal{P}_r^-\Lambda^k \subset \mathcal{P}_r\Lambda^k$. In most cases, these inclusions are proper. However, we have special cases for $k = 0$ and n:

$$\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0, \quad \mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n. \tag{2.14}$$

The second equation is straightforward, as all $(n+1)$-forms vanish. To validate the first equation, it suffices to show that $\kappa\mathcal{H}_{r-1}\Lambda^1 = \mathcal{H}_r\Lambda^0$. In fact, for any 1-form $\sum_i f_i \, dx_i$, the image of κ is $\sum_i (-1)^{i+1} f_i x_i$. The result follows from the fact that

$$\left\{ \sum_i (-1)^{i+1} f_i x_i \mid f_i \in \mathcal{H}_{r-1}, \forall i \right\} = \mathcal{H}_r.$$

Thus we have verified the equations.

After the brief introduction to the spaces $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$, we next use them as our shape function space, and define appropriate degrees of freedom to turn them into useful finite element spaces. When context is clear, we will refer these spaces as the $\mathcal{P}^-$ and $\mathcal{P}$ spaces, without degrees of polynomials and differential forms. In the next subsections, we will take functions in $\mathcal{P}^-$ spaces to give degrees of freedom of $\mathcal{P}$ spaces, and vice versa.

2.4.1 The $\mathcal{P}_r\Lambda^k$ finite element space

On an n-dimensional simplex T, for each subsimplex $f \in \Delta(T)$, we consider the following functional ϕ_v, where $v \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}$, on $\mathcal{P}_r\Lambda^k$:

$$u \mapsto \int_f \operatorname{tr}_f u \wedge v. \tag{2.15}$$

This $\mathcal{P}^-$ space vanishes if $r + k - \dim f \leq 1$ or $\dim f - k < 0$ holds. In addition, f being a subsimplex, we must have $\dim f < n$. Thus only the functionals with $k \leq \dim f \leq \min(r + k - 1, n)$ are of interest. It is proved [3, Theorem 4.8] that any $u \in \mathcal{P}_r\Lambda^k$ with $\phi_v(u) = 0$ for all such v must vanish. Besides, [3, Theorem 4.9] verifies that

$$\sum_{f \in \Delta(T)} \dim \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f) = \dim \mathcal{P}_r\Lambda^k(T).$$

Thus we have obtained a set of degrees of freedom on T: $\{\phi_v\}$ with v being basis functions of $\mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}$.

2.4.2 The $\mathcal{P}^-_r\Lambda^k$ finite element space

The $\mathcal{P}^-$ element is defined in a similar way to the $\mathcal{P}$ element. On a simplex $T \subset \mathbb{R}^n$, and any subsimplex $f \in \Delta(T)$, for each $v \in \mathcal{P}_{r+k+\dim -1}\Lambda^{\dim f-k}$, the following ϕ_v defines a functional on $\mathcal{P}^-_r\Lambda^k$:

$$u \mapsto \int_f \operatorname{tr}_f u \wedge v. \tag{2.16}$$

As before, only those functionals with $k \leq \dim f \leq \min(r+k-1, n)$ are of interest, because others are associated with vanishing $\mathcal{P}$ spaces. A unisolvance property is given by [3, Theorem 4.12]: if $u \in \mathcal{P}^-_r\Lambda^k$ satisfies $\phi_v(v) = 0$ for all such v above, then u must vanish. We also have the dimension equation

$$\sum_{f\in\Delta(T)} \dim \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f) = \dim \mathcal{P}^-_r\Lambda^k(T).$$

Hence if we let v be all basis functions of those $\mathcal{P}$ spaces associated with $\Delta(T)$, such $\{\phi_v\}$ is a set of degrees of freedom of $\mathcal{P}^-_r\Lambda^k$.

2.4.3 The $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}^-_r\Lambda^k$ spaces in three dimensions

As this thesis focuses on problems in three dimensions, we want to associated these $\mathcal{P}$ and $\mathcal{P}^-$ spaces with some well-known finite element spaces. For $k = 0$, we have proved that $\mathcal{P}_r\Lambda^0 = \mathcal{P}^-_r\Lambda^0$ in (2.14). Both the $\mathcal{P}$ and $\mathcal{P}^-$ spaces are the Lagrange element. They are elements for the H^1 function space. For $k = 1$ and $k = 2$, the $\mathcal{P}^-$ spaces are the two kinds of Nédélec edge elements, and the $\mathcal{P}$ spaces are the two kinds of Nédélec face elements. Moreover, the 1-form spaces $\mathcal{P}^-\Lambda^1$ and $\mathcal{P}\Lambda^1$ are elements for $H(\mathrm{curl})$, and the 2-form spaces $\mathcal{P}^-\Lambda^2$ and $\mathcal{P}\Lambda^2$ are elements for $H(\mathrm{div})$. For $k = 3$, thanks to (2.14), $\mathcal{P}_r\Lambda^3 = \mathcal{P}^-_{r+1}\Lambda^3$, and both spaces are the discontinuous Lagrange element. They do not assume any continutiy accorse elements. These instances of finite elements are summarized in the following table.

2.4.4 Projections to finite element spaces

We recall two kinds of projections from the continuous function spaces (e.g., $H\Lambda^k$, $\mathscr{H}\Lambda^k$, $\mathring{H}\Lambda^k$) to the finite element spaces (e.g., $P_r\Lambda^k$, $P^-_r\Lambda^k$). The first kind is the canonical projection, denoted by π^k, or simply π when there is no confusion. It is the interpolation via degrees of freedom. Unfortunately, such projections may not be continuous. For

	k	Finite element space name	degree	regularity
$P_r^-\Lambda^k$	0	Lagrange element	r	H^1
	1	Nédélec edge element of the 1st kind	r	$H(\text{curl})$
	2	Nédélec face element of the 1st kind	r	$H(\text{div})$
	3	Discontinuous Lagrange element	$r-1$	L^2
$P_r\Lambda^k$	0	Lagrange element	r	H^1
	1	Nédélec edge element of the 2nd kind	r	$H(\text{curl})$
	2	Nédélec face element of the 2nd kind	r	$H(\text{div})$
	3	Discontinuous Lagrange element	r	L^2

Table 2.6: Correspondence of finite element spaces in three dimensions

instance, in order to project an arbitrary function $u \in H(\text{curl})$ to the first-kind Nédélec edge element $\mathcal{P}^-\Lambda^1$, one needs to consider the moment $\int_e u \cdot tq\, ds$, where t is a unit vector along an edge e of some triangulation of the domain. Such moment requires u to have L^2 tangential traces along the edge e, which property u may not possess. This illustrate that in general, extra regularity of $u \in H\Lambda^k$ is needed when we consider πu. Some well-known results can be found in [1]. We do not list all results in the thesis, but will refer to such ones in future when needed, e.g. Lemma 3.3.10 in this thesis.

The canonical projection has two important additional properties. It commutes with the exterior derivative:

$$d^k \circ \pi^k u = \pi^{k+1} \circ d^k u, \quad \text{for all } u \in H\Lambda^k \text{ smooth enough.} \tag{2.17}$$

We sometimes just write $d \circ \pi = \pi \circ d$ for short. The other important property of the canonical projection is that it preserves vanishing traces:

$$\operatorname{tr} u = 0 \implies \operatorname{tr} \pi u = 0, \quad \text{for all } u \text{ smooth enough.}$$

These properties will be used later in our analysis.

Another kind of projection, called *smoothed projection*, is rather new. Unlike the canonical projection, the smoothed projection is bounded on all of $H\Lambda^k$. In fact, as given in [3, Section 5] such projections are even bounded from $L^2\Lambda^k$ to the finite element spaces. These operators, denoted by $\tilde{\Pi}$ in [3] or simply Π in this thesis, also commute with exterior derivatives:

$$d \circ \Pi u = \Pi \circ du, \quad \text{for all } u \in L^2\Lambda^k.$$

However, unlike the canonical projection, the smoothed projection does not preserve vanishing traces. Even if we have $u \in C_0^\infty\Lambda^k$, it is not guaranteed that $\operatorname{tr} \Pi u = 0$. Thus, this class of projection is ideal only when we are interested in $H\Lambda^k$ and their finite dimensional subspaces. When it comes to essential boundary conditions, thanks to Christiansen and Winther (cf. [11, Section 6]), a similar smoothed projection will help. We abuse notation and still denoted the other class of projection by Π in this thesis. It preserves vanishing trace, and commutes with d:

$$d \circ \Pi u = \Pi \circ du, \quad \operatorname{tr} \Pi u = 0, \quad \text{for all } u \in \mathring{H}\Lambda^k.$$

2.5 Two types of Robin problem

In this section we will start with the scalar Poisson equation with Robin boundary condition. Then we consider two types of generalization of this problem.

2.5.1 Semi-essential Robin problem

For the scalar Poisson equation,

$$-\Delta u = f,$$

the most commonly studied boundary conditions are the Dirichlet condition $u = 0$, and the Neumann condition $\partial u/\partial n = 0$. Viewing the differential equation as the Hodge Laplacian for 0-forms, these are exactly the cases of essential and natural boundary conditions, respectively, as we have seen in the preceding section (cf. Tables 2.4 and 2.5). Another important boundary condition for the Poisson equation is Robin boundary condition, as appears in the next problem.

Problem 2.5.1. *Given $f \in L^2(\Omega)$ and a constant $\lambda > 0$, find $u \in H^1(\Omega)$ that satisfies:*

$$-\Delta u = f \ \text{ in } \Omega, \quad \lambda\, u + \frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma.$$

All three kinds of BVPs arise in physics. For example, in the case of steady-state heat flow, which is one application of the Poisson equation, natural boundary conditions describe an insulated boundary, essential boundary conditions describe a boundary held at a fixed temperature, and Robin boundary conditions describe a Newton's law of cooling, that the heat flow is proportional to the temperature drop.

In this section we study the following set of boundary conditions for the Hodge Laplacian which generalize these Robin boundary conditions:

$$\operatorname{tr} \delta u = 0 \quad \text{and} \quad \operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0, \tag{2.18}$$

where $\lambda > 0$ is a constant.

We first verify that (2.18) is indeed a generalization of the boundary condition in the model Problem 2.5.1. In the case of 0-forms, the first condition in (2.18) is null and the second becomes the usual Robin boundary condition. We also remark that another interesting case is for 1-forms in 3 dimensions, where these boundary conditions are

$$\operatorname{div} u = 0 \quad \text{and} \quad \lambda n \times (u \times n) + \operatorname{curl} u \times n = 0. \tag{2.19}$$

We will see in section Chapter 4 that these are closely related to impedance boundary conditions for Maxwell's equations.

We next take a closer look at the conditions. We have two different Hodge stars in the second condition, the usual $\star$ that maps the $(k+1)$-form du to a $(n-k-1)$-form in Ω, and the boundary $\star_\Gamma$ that maps $\operatorname{tr} \star du$, an $(n-k-1)$-form defined on the boundary Γ, to $\star_\Gamma \operatorname{tr} \star du$, a k-form on Γ. Thus we see the addition in the second condition makes sense, since both terms are k-forms defined on Γ. The conditions (2.18) are closely related to the essential boundary conditions. As a matter of fact, while keeping one piece of the essential boundary conditions, one can obtain (2.18) by adding a perturbation $\frac{1}{\lambda} \star_\Gamma \operatorname{tr} \star du$ the other piece of essential boundary condition. Thus, the constant λ can be viewed as a permutation parameter that indicates how far (2.18) is from the essential conditions. As $\lambda \to \infty$ in (2.18), we get the essential boundary condition. Because of these observations, we call this type of Robin problem semi-essential BVP.

Just as for the natural and essential BVPs, these problems may be ill-posed, unless we take into account the solution space of the homogeneous problem. We have seen that the solution space is $\mathring{\mathfrak{H}}$ (cf. (2.5)). Thus the strong form of our first kind of Robin problem states as follows.

Problem 2.5.2. *: Find $u \in D(L) \cap \mathscr{H}\Lambda^k$ such that $u \perp \mathring{\mathfrak{H}}^k$, and*

$$(d\delta + \delta d)u = f - P_{\mathring{\mathfrak{H}}^k} f \ \text{ in } \Omega, \quad \operatorname{tr} \delta u = 0, \operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0 \ \text{ on } \Gamma. \tag{2.20}$$

The problem has the following weak formulation.

Problem 2.5.3. *Find $(\sigma, u, p) \in \mathring{H}\Lambda^{k-1} \times \mathscr{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$ that satisfies*

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, d\tau \rangle &= 0, \quad \forall \tau \in \mathring{H}\Lambda^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \lambda \langle \operatorname{tr} u, \operatorname{tr} v \rangle_\Gamma + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in \mathscr{H}\Lambda^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathring{\mathfrak{H}}^k. \end{aligned} \tag{2.21}$$

Here $\lambda > 0$.

We have pointed out for the natural and essential BVPs that the strong and weak formulations are equivalent. We have a similar result for this Robin problem.

Theorem 2.5.4. *If u solves Problem 2.5.2, then $(u, \sigma = \delta u, p = P_{\mathring{\mathfrak{H}}} f)$ solves Problem 2.5.3. If (u, σ, p) solves Problem 2.5.3, then u solves Problem 2.5.2, and $\sigma = \delta u$, $p = P_{\mathring{\mathfrak{H}}} f$.*

Proof. We prove the first statement first. Assuming such a solution u, we set $\sigma = \delta u \in H\Lambda^{k-1}$ and $p = P_{\mathring{\mathfrak{H}}} f \in \mathring{\mathfrak{H}}$. Then we have immediately that $\sigma \in \mathring{H}\Lambda^{k-1}$ from the boundary condition. Next, from the two equations

$$\sigma - \delta u = 0, \quad d\sigma + \delta du = f - p,$$

taking appropriate test functions, integrating by parts (cf. Lemma 2.2.1), we obtain

$$\langle \sigma, \tau \rangle - \langle \delta u, \tau \rangle = \langle \sigma, \tau \rangle - \langle u, d\tau \rangle = 0, \quad \forall \tau \in \mathring{H}\Lambda^{k-1}, \tag{2.22}$$

$$\langle d\sigma, v \rangle + \langle \delta du, v \rangle = \langle d\sigma, v \rangle + \langle du, dv \rangle - \int_\Gamma \operatorname{tr} v \wedge \operatorname{tr} \star du = \langle f, v \rangle - \langle p, v \rangle, \quad \forall v \in C^\infty \Lambda^k. \tag{2.23}$$

Here we have used the fact that $\langle \operatorname{tr} u, \operatorname{tr} v \rangle = \int_\Gamma \operatorname{tr} v \wedge \star \operatorname{tr} u$. The first part of (2.21) follows from (2.22) . Applying boundary condition (2.20) to (2.23), we obtain

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \lambda \int_\Gamma \operatorname{tr} v \wedge \star \operatorname{tr} u + \langle p, v \rangle = \langle f, v \rangle, \quad \forall v \in C^\infty \Lambda^k.$$

Since $d\sigma, du, p, f \in L^2\Lambda^k(\Omega)$ and $\operatorname{tr} u \in L^2_{\tan}\Lambda^k(\Gamma)$, by density of $C^\infty\Lambda^k$ in $\mathscr{H}\Lambda^k$, we can validate the second equation in (2.5.4) from the last identity. The last equation in (2.5.4) is part of the theorem hypothesis.

The other statement can be proven in a similar way. Given (2.21), one has (2.22), hence $\sigma = \delta u$, which also implies the boundary condition $\operatorname{tr} \delta u = 0$. Besides, the second equation of (2.21) indicates that

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \lambda \int_\Gamma \operatorname{tr} v \wedge \star \operatorname{tr} u = \langle f, v \rangle - \langle p, v \rangle, \quad \forall v \in C^\infty \Lambda^k.$$

Integrating the term $\langle du, dv \rangle$ by parts, one obtains

$$\langle d\sigma, v \rangle + \langle \delta du, v \rangle + \int_\Gamma \operatorname{tr} v \wedge \operatorname{tr} \star du + \lambda \int_\Gamma \operatorname{tr} v \wedge \star \operatorname{tr} u = \langle f, v \rangle - \langle p, v \rangle, \quad \forall v \in C^\infty \Lambda^k.$$

Thus we must have

$$\begin{aligned} &\langle d\sigma, v \rangle + \langle \delta du, v \rangle = \langle f, v \rangle - \langle p, v \rangle, \quad \forall v \in C^\infty \Lambda^k, \\ &\operatorname{tr} \star du + \lambda \star \operatorname{tr} u = 0, \quad \text{on } \Gamma, \end{aligned}$$

where the first equation leads to the second part of (2.21), and the other is the desired boundary condition. Finally, taking test functions $v \in P_{\mathring{\mathfrak{H}}^k}$ in (2.21), ones sees $f - p \perp \mathring{\mathscr{H}}^k$, and must have $p = P_{\mathring{\mathfrak{H}}^k} f$. □

From the proof we know that we can analyze the weak formulation to obtain well-posedness, which is a much easier task in practice. We will analyze the semi-essential Robin problem and its discretization in Section 3.3.

2.5.2 Interpretation of the semi-essential Robin BVPs in 3 dimensions

Before we do any analysis, we interpret Problem 2.5.2 in 3 dimensions to get more concrete examples. We have considered the cases $k = 0$ and 1. If $k = 0$, then $\mathring{\mathfrak{H}}^k$ is vacuous, and the equation is just the scalar Laplacian with Robin boundary condition

$$-\Delta u = f - P_{\mathring{\mathfrak{H}}^0} f = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \lambda u = 0, \quad \text{on } \Gamma.$$

If $k = 1$, the equation is the vector Laplacian

$$(-\operatorname{grad}\operatorname{div} + \operatorname{curl}\operatorname{curl})u = f - P_{\mathring{\mathfrak{H}}^1} f. \tag{2.24}$$

The boundary conditions become

$$\operatorname{div} u = 0, \quad \operatorname{curl} u \times n + \lambda\, n \times (u \times n) = 0, \quad \text{on } \Gamma. \tag{2.25}$$

This is the same as the conditions in (2.19). In fact, using the unit normal n to cross the second equation above, we can validate the equivalence.

If $k = 2$, the equation is again the vector Laplacian, but with a different boundary condition

$$\operatorname{curl} u \times n = 0, \quad \operatorname{div} u + \lambda u \cdot n = 0, \quad \text{on } \Gamma. \tag{2.26}$$

If $k = 3$, we have the scalar Laplacian again. Now the second boundary condition in (2.20) is vacuous, since the left hand side is a 3-form on a 2-dimension boundary, so the only boundary condition that applies is $\operatorname{Tr} \delta u = 0$, or

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \Gamma.$$

Thus we have the Neumann problem, or the essential BVP. The space $\mathring{\mathfrak{H}}^3$ of harmonic forms is simply a set of piecewise constant functions: each element of $\mathring{\mathfrak{H}}^3$ takes a constant value on each connected components of Ω.

Another thing we can consider is to split the problem into subproblems, depending on the space that f lies in. They are called the $\mathfrak{B}$ and $\mathfrak{B}^*$ problems, whose names will be explained after we introduced the concept of Hilber complex in the next Chapter. For the $\mathfrak{B}$-problem, we assume $f \in d\mathring{H}\Lambda^k$. For any $v \in \mathscr{H}\Lambda^k$, we take $v_\perp$, the L^2-projection of v into the L^2 orthogonal compliment of $d\mathring{H}\Lambda^{k-1} \oplus \mathring{\mathfrak{H}}^k$, to be the test function in the second equation of (2.21). By the definition of $v_\perp$ and the fact that $\sigma \in \mathring{H}\Lambda^{k-1}$, we have $\langle d\sigma, v_\perp \rangle = 0$, Thus the second equation in (2.21) now becomes

$$\langle du, dv_\perp \rangle + \lambda \langle \operatorname{tr} u, \operatorname{tr} v_\perp \rangle = 0. \tag{2.27}$$

Besides, since $v - v_\perp \in d\mathring{H}\Lambda^{k-1} \oplus \mathring{\mathfrak{H}}^k$, we have

$$d(v - v_\perp) = 0,\ \operatorname{tr}(v - v_\perp) = 0 \quad \Longrightarrow \quad dv = dv_\perp,\ \operatorname{tr} v = \operatorname{tr} v_\perp.$$

So (2.27) becomes

$$\langle du, dv \rangle + \lambda \langle \operatorname{tr} u, \operatorname{tr} v \rangle = 0.$$

Choosing $v = u$, we see that $du = 0$ and $\operatorname{tr} u = 0$. Therefore, the strong form of this problem is

$$du = 0,\ d\delta u = f \text{ in } \Omega, \quad \operatorname{tr} u = 0,\ \operatorname{tr} \delta u = 0 \text{ on } \Gamma.$$

Now let us write down the problem in 3 dimensions. If $k = 0$, we do not have σ, and we have

$$\operatorname{grad} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

This leads to the trivial solution $u = 0$.

If $k = 1$, we have

$$\operatorname{curl} u = 0,\ -\operatorname{grad} \operatorname{div} u = f \text{ in } \Omega, \quad u \times n = 0,\ \operatorname{div} u = 0 \text{ on } \Gamma.$$

If $k = 2$, we have

$$\operatorname{div} u = 0,\ \operatorname{curl} \operatorname{curl} u = f \text{ in } \Omega, \quad u \cdot n = 0,\ \operatorname{curl} u \times n = 0 \text{ on } \Gamma.$$

If $k = 3$, we have

$$-\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma.$$

This is the Neumann BVP of the scalar Laplacian.

For the $\mathfrak{B}^*$-problem, we have $f \perp d\mathring{H}\Lambda^k \oplus \mathring{\mathfrak{H}}^k$. Let $v = d\sigma$ in (2.21), we have $d\sigma = 0$.

Then we set $\tau = \sigma$, and see that $\sigma = 0$ from the first equation. Thus we have

$$\langle u, d\tau \rangle = 0 \quad \text{and} \quad \langle du, dv \rangle + \lambda \langle \operatorname{tr} u, \operatorname{tr} v \rangle_\Gamma = \langle f, v \rangle.$$

Consequently, we have the strong form of this problem:

$$\delta u = 0, \quad \delta du = f \text{ in } \Omega, \quad \operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0.$$

We also interpret this problem in 3 dimensions. If $k = 0$, we have

$$-\Delta u = f \text{ in } \Omega, \frac{\partial u}{\partial n} + \lambda u = 0 \text{ on } \Gamma.$$

This is the usual scalar Laplace equation with Robin boundary condition.

If $k = 1$, we have

$$\operatorname{div} u = 0, \operatorname{curl} \operatorname{curl} u = f \text{ in } \Omega, \operatorname{curl} u \times n + \lambda \quad n \times (u \times n) = 0 \text{ on } \Gamma.$$

If $k = 2$, we have

$$\operatorname{curl} u = 0, -\operatorname{grad} \operatorname{div} u = f \text{ in } \Omega, \quad \operatorname{div} u + \lambda\, u \cdot n = 0 \text{ on } \Gamma.$$

If $k = 3$, we have $\operatorname{grad} u = 0$ with no boundary conditions. However, the auxiliary condition $u \perp \mathring{\mathfrak{H}}^3$ implies that $u = \operatorname{div} \phi$ for some $\phi \in \mathring{H}\Lambda^2$. We have $\langle \operatorname{grad} \operatorname{div} \phi, w \rangle = 0$ for all $w \in H\Lambda^2$. In particular, taking $w = \phi$, integrating by parts, we obtain that $u = \operatorname{div} \phi = 0$. Thus we have a trivial problem in this case.

Thus, we have introduced our Robin BVP of the first kind, in both strong and weak (mixed) formulation. We interpreted the abstract problem as concrete BVPs in three dimensions. We also considered breaking up the problem to two types of subproblems. We will postpone our analysis of this BVP in the next Chapter. Prior to that, we will introduce another Robin BVP in the next section, and make a few comparisons among these BVPs.

2.5.3 Semi-natural Robin problem

In this section we will introduce another problem that generalizes the usual scalar Robin BVP.

Problem 2.5.5. *Find $u \in D(L)$ such that $\delta u \in \mathscr{H}\Lambda^{k-1}$, $u \perp \mathfrak{H}^k$, and*

$$(d\delta + \delta d)u = f - P_{\mathfrak{H}^k} f \text{ in } \Omega, \quad \operatorname{tr} \star du = 0, \operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \delta u = 0 \text{ on } \Gamma. \tag{2.28}$$

As we will see in the next subsection, in the case of 3-form, this problem becomes the usual Robin problem.

Similar to the relationship between the semi-essential and essential boundary conditions, the boundary condition in (2.28) can be viewed as a perturbation of the natural boundary condition. As $\lambda \to 0$, this boundary condition formally approaches the natural boundary condition. Therefore, we call this type of Robin problem semi-natural BVP.

We recall that the space $\mathscr{H}\Lambda^{k-1}$ is the subspace of $H\Lambda^{k-1}$ such that the traces of its elements on the boundary are all in L^2 (cf. (2.3)). The weak formulation associated with this problem is

Problem 2.5.6. *Find $(\sigma, u, p) \in \mathscr{H}\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ that satisfies*

$$\begin{aligned} -\langle \sigma, \tau\rangle - \lambda\langle \operatorname{tr}\sigma, \operatorname{tr}\tau\rangle_\Gamma + \langle u, d\tau\rangle &= 0, \quad \forall \tau \in \mathscr{H}\Lambda^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, \quad \forall v \in H\Lambda^k, \\ \langle u, q\rangle &= 0, \quad \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.29}$$

Here $\lambda > 0$.

Same as in the preceding section, we have equivalence of these two formulations.

Theorem 2.5.7. *If u solves Problem 2.5.6, then $(u, \sigma = \delta u, p = P_{\mathfrak{H}^k} f)$ solves Problem 2.5.5. If (u, σ, p) solves Problem 2.5.5, then u solves Problem 2.5.6, and $\sigma = \delta u$, $p = P_{\mathfrak{H}^k} f$.*

Proof. The proof is similar to that of Theorem 2.5.4. For any u solving Problem 2.5.6, we let $\sigma = \delta u \in \mathscr{H}\Lambda^{k-1}$ and $p = P_{\mathfrak{H}^k} f$. Taking appropriate test functions, by Lemma 2.2.1, we have

$$\begin{aligned} -\langle \sigma, \tau\rangle + \langle \delta u, \tau\rangle &= -\langle \sigma, \tau\rangle + \langle d\tau, u\rangle - \int_\Gamma \operatorname{tr}\tau \wedge \operatorname{tr}\star u = 0, \quad \forall \tau \in C^\infty\Lambda^{k-1}, \\ \langle d\sigma, v\rangle + \langle \delta du, v\rangle &= \langle d\sigma, v\rangle + \langle du, dv\rangle - \int_\Gamma \operatorname{tr} dv \wedge \operatorname{tr}\star du = \langle f - p, v\rangle, \quad \forall v \in C^\infty\Lambda^k. \end{aligned}$$

By the boundary conditions in (2.28) and the density of $C^\infty\Lambda^{k-1}$ in $\mathscr{H}\Lambda^{k-1}$ and $C^\infty\Lambda^k$ in $H\Lambda^k$, one can derive the first two equations in (2.29). The last equation in (2.29) is obvious from the definition of p.

On the other hand, if we have a solution $(\sigma, u, p) \in \mathscr{H}\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ satisfying (2.29), we immediately have $u \perp \mathfrak{H}^k$ from the last equation. Letting $v \in \mathfrak{H}^k \subset H\Lambda^k$, we can conclude that $f - p \perp \mathfrak{H}^k$, which implies $p = P_{\mathfrak{H}^k} f$. Thus by Lemma 2.2.1, for all $v \in C^\infty\Lambda^k$, we have

$$\langle d\sigma, v\rangle + \langle \delta du, v\rangle + \int_\Gamma \operatorname{tr} dv \wedge \operatorname{tr}\star du = \langle f - P_{\mathfrak{H}^k} f, v\rangle.$$

This shows that

$$d\sigma + \delta du = f - P_{\mathfrak{H}^k} f \tag{2.30}$$

and the boundary condition $\operatorname{tr} \star du = 0$. Finally, taking $\tau \in C^\infty \Lambda^{k-1}$ in (2.29), Lemma 2.2.1 yields

$$-\langle \sigma, \tau \rangle - \lambda \int_\Gamma \operatorname{tr} \tau \wedge \star_\Gamma \operatorname{tr} \sigma + \langle \delta u, \tau \rangle + \int_\Gamma \operatorname{tr} \tau \wedge \operatorname{tr} \star u = 0. \tag{2.31}$$

This implies $\sigma = \delta u$, and the boundary condition $\operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \sigma = \operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \delta u = 0$. Substituting $\sigma = \delta u$ in (2.30), we have $(d\delta + \delta d)u = f - P_{\mathfrak{H}^k} f$. Thus we verified that $u \in H\Lambda^k$ solves Problem 2.5.5. □

The analysis of semi-natural Robin problem will also be done in Section 3.4.

2.5.4 Interpretation of semi-natural Robin BVP in 3 dimensions

Similar to Problem 2.5.2, we can interpret Problem 2.5.5 in 3 dimensions, as follows. If $k = 0$, the equation is the scalar Laplacian

$$-\Delta u = f - P_{\mathfrak{H}^0} f.$$

The second piece of the boundary conditions in (2.28) drops out, and our boundary condition hence becomes

$$\frac{\partial u}{\partial n} = 0.$$

Thus for $k = 0$, semi-natural BVP reduces to just the natural BVP (Neumann problem).

If $k = 1$, we have the vector Laplace equation

$$(-\operatorname{grad} \operatorname{div} + \operatorname{curl} \operatorname{curl})u = f - P_{\mathfrak{H}^1} f.$$

The boundary condition become

$$\operatorname{curl} u \times n = 0, \quad u \cdot n + \lambda \operatorname{div} u = 0.$$

Notice that the boundary condition for $k = 1$ is the same as the boundary condition of the semi-essential problem for $k = 2$, except that the perturbation factor is λ here, whereas $\frac{1}{\lambda}$ there.

If $k = 2$, we have the vector Laplace equation still, but different boundary conditions:

$$\operatorname{div} u = 0, \quad n \times (u \times n) + \lambda \operatorname{curl} u \times n = 0.$$

This boundary condition is the same as the boundary condition of the semi-essential problem for $k = 1$, except that we have the reciprocal $\frac{1}{\lambda}$ there.

If $k = 3$, we have the scalar Laplacian again. The boundary condition is

$$u + \lambda \frac{\partial u}{\partial n} = 0,$$

which is the usual Robin boundary value problem.

Next, we also consider splitting the problem into the $\mathfrak{B}$ and $\mathfrak{B}^*$ problem. For the $\mathcal{B}$ problem of Problem 2.5.6, we consider the case when f is in the range of d. Then for any v, we may take $v_\perp$ to be the L^2-projection of v into the L^2-compliment of $d\mathscr{H}\Lambda^k \oplus \mathfrak{H}^k$. Let $v_\perp$ be the test function in the second equation of (2.29), we then have $\langle du, dv_\perp \rangle = 0$, which is equivalent to $\langle du, dv \rangle = 0$. Choosing $v = u$, we can see that $du = 0$. Thus the strong form of this problem is

$$du = 0, d\delta u = f \text{ in } \Omega, \quad \operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \delta u = 0.$$

We interpret this sub-problem in 3 dimensions. If $k = 0$, we have $\operatorname{grad} u = 0$, which indicates u is a constant on each connected component of Ω. However, the auxiliary condition $u \perp \mathfrak{H}^0$ guarantees that $u = 0$. So $k = 1$ is a trivial case.

If $k = 1$, we have

$$\operatorname{curl} u = 0, -\operatorname{grad} \operatorname{div} u = f \text{ in } \Omega, \quad u \cdot n + \lambda \operatorname{div} u = 0 \text{ on } \Gamma.$$

If $k = 2$, we have

$$\operatorname{div} u = 0, \operatorname{curl} \operatorname{curl} u = f \text{ in } \Omega, \quad n \times (u \times n) + \lambda \operatorname{curl} u \times n = 0 \text{ on } \Gamma.$$

If $k = 3$, we have

$$-\Delta u = 0 \text{ in } \Omega, \quad u + \lambda \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma.$$

For the $\mathfrak{B}^*$-problem, setting $v = d\sigma$ in the second equation of (2.29), we have $d\sigma = 0$. Then we take $\tau = d\sigma$ in the first equation, and obtain $\sigma = 0$, or $\delta u = 0$ equivalently. Thus our problem becomes

$$\delta u = 0, \quad \delta du = 0 \text{ in } \Omega, \quad \operatorname{tr} \star du = 0, \quad \operatorname{tr} \star u = 0 \text{ on } \Gamma.$$

If $k = 0$, the first equation and the second boundary condition vanish. We thus have the scalar Neumann BVP.

$$-\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma.$$

If $k = 1$, we have

$$\operatorname{div} u = 0, \operatorname{curl}\operatorname{curl} u = f \text{ in } \Omega, \quad u \cdot n = 0, \operatorname{curl} u \times n = 0 \text{ on } \Gamma.$$

If $k = 2$, we have

$$\operatorname{curl} u = 0, -\operatorname{grad}\operatorname{div} u = f \text{ in } \Omega, \quad u \times n = 0, \operatorname{div} u = 0 \text{ on } \Gamma.$$

If $k = 3$, the second equation and the first boundary condition vanish. The remaining problem is to find u such that $\operatorname{grad} u = 0$ in Ω and u vanishes on the boundary, which is obviously a trivial problem.

Thus, we have finished introducing the semi-natural Robin BVP, including the strong and weak formulation, interpretation in three dimensions, and have splitting the original problem into subproblems. So far we have four kinds of Hodge Laplace problem, the natural BVP 2.3.2, the essential BVP 2.3.6, and two kinds of Robin BVPs (cf. Problems 2.5.2 and 2.5.5). We will give a few remarks in the next section on their connections, and leave the analysis of the Robin BVPs in the next Chapter.

2.5.5 Comparison of the two Robin BVPs

We will close this Chapter by comparing the problems 2.5.2 and 2.5.5 discussed just before, as well as the natural and essential BVPs (cf. Problems 2.3.2 and 2.3.6). Looking at the boundary conditions of Problem 2.5.2, we see the boundary condition

$$\operatorname{tr} \delta u = 0$$

is exactly the same as one from the essential BVP (cf. (2.12)). The other boundary condition in (2.20),

$$\operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0,$$

can be viewed as adding a perturbation term to the boundary condition

$$\operatorname{tr} u = 0.$$

The last equation is the other boundary condition in the essential BVP. Besides, the perturbation is related to one boundary condition from the natural BVP, cf. Problem 2.5.5 and Equation (2.28). Thus we see that the boundary conditions associated with the semi-essential Robin BVP is in fact the boundary conditions from the essential BVP, with a perturbation term from the natural BVP.

Another observation we make is that the functions spaces in the mixed formulations of these problems are related. In the first kind of Robin BVP, we choose the space

$$\mathring{H}\Lambda^{k-1} \times \mathscr{H}\Lambda^k \times \mathring{\mathfrak{H}}\Lambda^k.$$

This is very like the space we used for the essential BVP, except that the middle one $\mathscr{H}\Lambda^k$ is a super set of $\mathring{H}\Lambda^k$, which we used for the essential BVP. Still, the space $\mathscr{H}\Lambda^k$ is a subset of $H\Lambda^k$, which is used for the natural BVP. Thus, we see that, in terms of function spaces, the first kind of Robin BVP is a perturbed version of the essential BVP.

Based on the above observations, we can say the first Robin BVP is semi-essential. From similar comparison, we can view the second kind of Robin BVPs as a perturbed version of the natural BVP, and hence we say it is semi-natural. We list all boundary conditions and function spaces in these problems in the following table.

BVP	boundary conditions	function spaces
Natural	$\operatorname{tr} \star du = 0,\ \operatorname{tr} \star u = 0$	$H\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$
Second Robin (Semi-natural)	$\operatorname{tr} \star du = 0,\ \operatorname{tr} \star u - \lambda \star_\Gamma \operatorname{tr} \delta u = 0$	$\mathscr{H}\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$
First Robin (Semi-essential)	$\operatorname{tr} \star du + \lambda \star_\Gamma \operatorname{tr} u = 0,\ \operatorname{tr} \delta u = 0$	$\mathring{H}\Lambda^{k-1} \times \mathscr{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$
Essential	$\operatorname{tr} u = 0,\ \operatorname{tr} \delta u = 0$	$\mathring{H}\Lambda^{k-1} \times \mathring{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$

Table 2.7: Boundary conditions in strong formulation and function spaces in weak formulation of four BVPs in three dimensions.

Just as we saw that the k-form natural BVP and the $(n-k)$-form essential BVP generates the same BVP (cf. Tables 2.4 and 2.5), we also have such relation between the semi-natural BVP and semi-essential BVP. This is not surprising. One can consider the Hodge star $\star$ as a mapping between k-forms and $(n-k)$-forms. If u satisfies the boundary conditions in (2.20), then $w = \star u$ satisfies the boundary conditions in (2.28).

Chapter 3

Hilbert Complex Approach

3.1 Introduction

In this chapter, we analyze our two Robin BVPs introduced in Section 2.5 following a approach based on the *finite element exterior calculus* (FEEC) framework established in [4]. The key concept in the FEEC theory is Hilbert complex, which generalizes de Rham complexes and can represents more function spaces and operators.

Our main goal in the chapter is to fit the semi-natural and -essential BVPs into the general FEEC framework established in [4]. Once we see that there are Hilbert complexes with appropriate operators behind the concrete BVPs, we can directly apply the general theory to our new Hilbert complexes, and immediately have wellposedness of the original problems 2.5.5 and 2.5.2, stability of the corresponding discrete problems, and approximation results.

3.2 Preliminaries

In this section we review some general results in FEEC from [4]. We consider a sequence of Hilbert spaces W^k with (maybe unbounded) linear operators $D^k : W^k \to W^{k+1}$,

$$W^{k-1} \xrightarrow{D^{k-1}} W^k \xrightarrow{D^k} W^{k+1}. \tag{3.1}$$

If the operators satisfy that $D^{k+1} \circ D^k = 0$, the sequence is then called a complex. We denote such a complex by (W, D). For each D^k, we consider a suitable subspace $V^k \subset \left\{u \in W^k \mid D^k u \in W^{k+1}\right\}$. This V^k is called the domain of D^k. For the domain space V^k, we consider the inner product $\langle \cdot, \cdot \rangle$:

$$\langle u, v\rangle_{V^k} = \langle u, v\rangle_{W^k} + \langle D^k u, D^k v\rangle_{W^{k+1}},$$

and the resulting norm, called the V-norm or graph norm. Thus we have the domain complex (V, D). It is straightforward that $D^k|_{V^k}$ maps V^k into V^{k+1}. We also consider the (still maybe unbounded) adjoint operators $D_k^*: W^k \to W^{k-1}$, and have the adjoint complex

$$W^{k-1} \xleftarrow{D_k^*} W^k \xleftarrow{D_{k+1}^*} W^{k+1}, \tag{3.2}$$

For each D_k^*, we have its domain

$$V_k^* = \left\{ u \in W^k \mid D_k^* u \in W^{k-1} \right\}.$$

Definition 3.2.1. *A complex (W, D) is a Hilbert complex, if the following properties hold:*

1. *Each D^k is a closed operator. This means the resulting graph $\left\{(u, D^k u) \mid u \in V^k\right\}$ is a closed set;*
2. *Each D^k is densely-defined in W^k, i.e., the corresponding domain V^k is a dense subset of W^k.*

In addition, if the operators D^k have closed ranges, we say that (W, D) is a closed Hilbert complex.

Remark 3.2.1. *It can be checked that for a Hilbert complex (W, D), the resulting domain complex (V, D), equipped with the graph norm, is a bounded Hilbert complex. A Hilbert complex (W, D) is closed if and only if so is (V, D).*

The closedness of a Hilbert complex, or the closed range property, plays an essential role in our analysis. However, usually we do not verify this property directly. Instead, we can check the *Poincaré inequality.* To introduce the general form of the inequality, we shall define a few more spaces. For each D^k continuously defined on V^k, we denote $\mathfrak{B}^{k+1}$ the range $D^k(V^k)$, and denote $\mathfrak{Z}^k$ the null space of D^k. One can check that both $\mathfrak{B}^k$ and $\mathfrak{Z}^k$ are subspaces of V^k. Moreover, each $\mathfrak{Z}^k$ is always closed (under W-norm), and every $\mathfrak{B}^k$ is closed if and only if the Hilbert complex is closed. For a Hilbert complex (W, D), each domain V^k can be decomposed as $V^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k,\perp}$, where $\perp$ denotes the W-orthogonality. Now we can introduce the General Poincaré inequality.

Definition 3.2.2 (General Poincaré inequality)**.** *Given a Hilbert complex (W, D) and domain spaces V, We say that Poincaré inequality holds for k, if there exists a constant $C > 0$, such that every $u \in \mathfrak{Z}^{k,\perp}$ satisfies*

$$\|u\| \leq C \|D^k u\|. \tag{3.3}$$

It has been pointed out in [4][1] that closedness of a Hilbert complex is equivalent to Poincaré inequality (for all k). However, for the sake of conciseness, only a proof of sufficiency was provided in that paper. We state the property as a lemma below and give a complete proof.

Lemma 3.2.2. *A Hilbert complex (W, D) with domains V is closed if and only if the Poincaré inequality (3.3) holds.*

Proof. We will prove the "if" part first for D^k. Suppose $Du_i \to w \in W^{k+1}$ for $u_i \in V^k$, our goal is to show that there exists a $u \in V^k$ such that $Du = w$. For each u_i, we decompose

$$u_i = u_{i,0} + u_{i,\perp} = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k,\perp}.$$

Applying Poincaré inequality (3.3) on $u_{i,\perp} - u_{j,\perp}$, we have $\|u_{i,\perp} - u_{j,\perp}\| \leq C\|Du_{i,\perp} - Du_{j,\perp}\|$, where $C > 0$ is some constant that does not depends on $\{u_i\}$. Note that $Du_{i,\perp} = Du_i$, which means $\{u_{i,\perp}\}$ is a Cauchy sequence. Since W^k is a Hilbert space, there exists a $u \in W^k$ such that $u_i \to u$. Next, because D is a closed operator, $u_i \to u$ in W^k implies $Du_i \to Du$ in W^{k+1}. The last convergence shows that $u \in V^k$, and $Du = w$ from our hypothesis $Du_i \to w$.

Next, we will prove the "only if" part. Obviously, D^k maps $\mathfrak{Z}^{k,\perp}$ to $\mathfrak{B}^{k+1}$ bijectively. It is clear that $\mathfrak{Z}^{k,\perp}$ is a closed subspace of W^k. Moreover, because (W, D) is closed, the range $\mathfrak{B}^{k+1}$ of D^k is a closed subspace of W^{k+1}. Hence, both $\mathfrak{Z}^{k,\perp}$ and $\mathfrak{B}^k$ are Banach spaces. Thus, by the well-known bounded inverse theorem, we have a constant $C > 0$, depending on those two spaces and the operator D^k only, such that $\|u\| \leq C\|D^k u\|$. $\square$

We will see (cf. Lemmas 3.3.4 and 3.4.5) that validating an appropriate Poincaré inequality makes up a nontrivial part of the analysis of both Robin BVPs.

Next we introduce the general Hodge Laplacian in the Hilbert complex context. This will be useful, as our plan is to fit the two kinds of Robin BVPs in this framework, and take advantages of analysis that has been done in [4]. We call the operator $L = D^*_{k+1} \circ D^k + D^{k-1} \circ D^*_k$ the abstract Hodge Laplacian. The domain of L is

$$D(L) = \left\{ u \in V^k \cap V^*_k \mid Du \in V^*_{k+1}, D^*u \in V^{k-1} \right\}.$$

The abstract Hodge Laplace problem is to find $u \in D(L)$ satisfying $Lu = f$ for given $f \in W^k$. In general, the problem may not be well-posed. To obtain a well-posed Hodge

[1] See the remark after equation (16) in this paper.

Laplacian, we need to take into account the space of abstract harmonic functions

$$\mathfrak{H}^k = \left\{ u \in V^k \cap V_k^* \mid Du = 0, D^*u = 0 \right\} \tag{3.4}$$
$$= \left\{ u \in V^k \mid D^u = 0, \langle u, D\sigma \rangle = 0, \forall \sigma \in V^{k-1} \right\}. \tag{3.5}$$

Remark 3.2.3. *Despite the similarity of notations, the space $\mathfrak{H}^k$ defined above is not necessarily the same space given by (2.4). However, as we will see later in this chapter, our (more abstract) harmonic function space $\mathfrak{H}^k$ above may equal the spaces $\mathfrak{H}^k$ given by (2.4) or $\mathring{\mathfrak{H}}^k$given by (2.5), given appropriate complex (W, D) and domains V.*

Remark 3.2.4. *We see that $\mathfrak{H}^k$ is the W-complement of $\mathfrak{B}^k$. If (W, D) is a closed Hilbert complex, $\mathfrak{B}^k$ is a closed subspace of $\mathfrak{Z}^k$, we hence have $\mathfrak{Z}^k = \mathfrak{B}^k \oplus \mathfrak{H}^k$, and consequently the Hodge decomposition*

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k,\perp}.$$

Now we can consider the well-posed Hodge Laplacian in strong and mixed formulation, assuming closedness of the associated Hilbert complex. All brackets $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$ stand for the W-inner products and W-norm, unless otherwise mentioned.

Problem 3.2.5 (Strong formulation)**.** *Assume that (W, D) is a closed Hilbert complex. Given $f \in W^k$, find $u \in D(L)$ such that $u \perp \mathfrak{H}^k$, and that*

$$Lu = f \mod \mathfrak{H}^k.$$

Problem 3.2.6 (Mixed formulation)**.** *Assume that (W, D) is a closed Hilbert complex. Given $f \in W^k$, find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ that satisfies*

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, D\tau \rangle &= 0, \quad \forall \tau \in V^{k-1}, \\ \langle D\sigma, v \rangle + \langle Du, Dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in V^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathfrak{H}^k. \end{aligned}$$

We have the following well-posedness from [4, Theorem 3.1].

Theorem 3.2.7. *For any closed Hilbert complex (W, D) and its domains V, there is a constant $C > 0$, such that for all $f \in W^k$, Problem 3.2.6 has a unique solution $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, and moreover,*

$$\|\sigma\|_{V^{k-1}} + \|u\|_{V^k} + \|p\| \le C\|f\|.$$

To apply the finite element methods to this problem, we need to discretize the

Hilbert complex (V, D) with a discrete complex (V_h, D), where each $V_h^k \subset V^k$ is a finite-dimensional subspace. Corresponding to this discretization is the abstract discrete harmonic space

$$\mathfrak{H}_h^k = \left\{ u \in V_h^k \mid Du = 0, \langle u, D\sigma \rangle = 0, \forall \sigma \in V_h^{k-1} \right\}. \tag{3.6}$$

The discrete mixed problem is defined as below.

Problem 3.2.8. *Given $f \in W^k$, find $(\sigma, u, p) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ that satisfies*

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, D\tau \rangle &= 0, \quad \forall \tau \in V_h^{k-1}, \\ \langle D\sigma, v \rangle + \langle Du, Dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in V_h^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathfrak{H}_h^k. \end{aligned}$$

We need a series of maps that connect the complex (V, D) and its discretization (V_h, D). A series of projections $\Pi^k : V^k \to V_h^k$, denoted by Π, is called a cochain projection from (V, D) to (V_h, D), if it commutes with the D operators:

$$\Pi^{k+1} \circ D^k u = D^{k+1} \circ \Pi^k u, \quad \forall u \in V^k, \forall k.$$

This property is illustrated by the following commuting diagram.

$$\begin{tikzcd} V^{k-1} \arrow[r, "D^{k-1}"] \arrow[d, "\Pi^{k-1}"] & V^k \arrow[r, "D^k"] \arrow[d, "\Pi^k"] & V^{k-1} \arrow[d, "\Pi^{k+1}"] \\ V_h^{k-1} \arrow[r, "D^{k-1}"] & V_h^k \arrow[r, "D^k"] & V_h^{k-1} \end{tikzcd}$$

Moreover, a cochain projection Π is said to be bounded, if $\|\Pi^k\|_{\mathscr{L}(V^k, V_h^k)} \leq C$ for some $C > 0$ independent of k.

If we have a family of subcomplexes (V_h, D) of (V, D), and there exist Π_h uniformly bounded projections from (V, D) to (V_h, D), we then have convergence of finite element methods. The next convergence theorem is [4, Theorem 3.9].

Theorem 3.2.9. *Let (V, D) be a closed Hilbert complex, and (V_h, D) be a family of Hilbert subcomplexes indexed by h. Assume that there exist a uniformly bounded cochain projection $\Pi : (V, D) \to (V_h, D)$. There exist constant $C > 0$, such that for any $f \in W^k$, $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ the unique solution to Problem 3.2.6, and $(\sigma_h, u_h, p_h) \in$*

$V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ the unique solution to Problem 3.2.8, we have

$$\begin{aligned}&\|\sigma-\sigma_h\|_{V^{k-1}}+\|u-u_h\|_{V^k}+\|p-p_h\|\\ &\le C\left(\inf_{\tau\in V_h^{k-1}}\|\sigma-\tau\|+\inf_{v\in V_h^k}\|u-v\|+\inf_{q\in V_h^k}\|p-q\|+\mu\inf_{v\in V_h^k}\|\mathrm{Proj}_{D(V^{k-1})}u-v\|\right),\end{aligned}$$

where $\mu = \sup_{r\in\mathfrak{H}^k,\|r\|=1}\|(I-\Pi)r\|$.

The existence of such uniform bounded cochain projection is usually not trivial. Thanks to [4, Theorem 3.7], as stated below, we need a discrete Poincaré inequality, and a lower bound of the projection from $\mathfrak{H}_h^k$ to $\mathfrak{H}^k$.

Theorem 3.2.10. *Let (V, D) be a closed Hilbert complex, and (V_h, D) be a family of Hilbert subcomplexes indexed by h. Assume that we have the discrete Poincaré inequality*

$$\|v\|_V \le C_1\|dv\|_V, \forall v \in \mathfrak{Z}_h^k, \tag{3.7}$$

and for the projection $P_{\mathfrak{H}} : \mathfrak{H}_h^k \to \mathfrak{H}^k$:

$$\|q\|_V \le C_2\|P_{\mathfrak{H}}q\|, \forall q \in \mathfrak{H}_h^k, \tag{3.8}$$

Then there exists a cochain projection $\pi_h : V \to V_h$, whose norm is bounded in terms of C_1 and C_2. The reverse of this theorem is also true.

Remark 3.2.11. *We remark that the harmonic function spaces $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ in the preceding theorem is the abstract spaces, as defined by (3.4) and (3.6). In the next subsection, we will give a proof for (3.8) for a few combination of concrete harmonic function spaces, based on analysis of the natural and essential BVPs in literature . The validation of the discrete Poincaré inequality (3.7) is more non-trivial. We will prove it on a case-by-case basis in the next two subsections.*

3.2.1 Some harmonic space gaps

Before we move on to the analysis of the two Robin BVPs and their discretization, we want to prove (3.8) for the combinations of $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ from the natural and essential BVPs. Later we will see that these concrete harmonic spaces are exactly the same from semi-natural and semi-essential Robin BVPs. Thus we can just apply the gap estimates there.

To start with, we recall that for the natural BVP, we have the de Rham complex (W, d), which in three dimensions is

$$L^2\Lambda^0 \xrightarrow{\text{grad}} L^2\Lambda^1 \xrightarrow{\text{curl}} L^2\Lambda^2 \xrightarrow{\text{div}} L^2\Lambda^3. \tag{3.9}$$

The corresponding domain complex is (V, d) where $V^k = H\Lambda^k$. The harmonic function space is given by (2.4). With a triangulation $\mathcal{T}$ of Ω, and one of the following sets of polynomial spaces $V_h^{k-1} \times V_h^k$:

$$\mathcal{P}_r^-\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k, \quad \mathcal{P}_r\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k,$$
$$\mathcal{P}_{r+1}^-\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k, \quad \mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k,$$

where the $\mathcal{P}$ and $\mathcal{P}^-$ spaces are defined in Section 2.4, we have uniformly bounded cochain projection from (V, d) to (V_h, d). By theorem 3.2.10, this implies (3.8) in the following special form:

$$\|q\|_V \leq C\|P_{\mathfrak{H}^k} q\|, \forall q \in \mathfrak{H}_h^k$$

where either

$$\mathfrak{H}_h^k = \{u \in \mathcal{P}_r \mid du = 0, \langle u, dv \rangle = 0, \forall v \in \mathcal{P}_{r-1}\}, \tag{3.10}$$

or

$$\mathfrak{H}_h^k = \left\{u \in \mathcal{P}_r^- \mid du = 0, \langle u, dv \rangle = 0, \forall v \in \mathcal{P}_{r-1}^-\right\}, \tag{3.11}$$

The story for the essential BVP is similar. With the same de Rham complex (3.9), and a different domain complex (V, d), where $V^k = \mathring{H}\Lambda^k$, we have the essential BVP. The corresponding harmonic function space is given by (2.5). Choosing one of the following sets of $V_h^{k-1} \times V_h^k$:

$$\mathring{\mathcal{P}}_r^-\Lambda^{k-1} \times \mathring{\mathcal{P}}_r^-\Lambda^k, \quad \mathring{\mathcal{P}}_r\Lambda^{k-1} \times \mathring{\mathcal{P}}_r^-\Lambda^k,$$
$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1} \times \mathring{\mathcal{P}}_r\Lambda^k, \quad \mathring{\mathcal{P}}_{r+1}\Lambda^{k-1} \times \mathring{\mathcal{P}}_r\Lambda^k,$$

we have a uniform bounded cochain projection from (W, d) to (V, d). Now (3.8) in Theorem 3.2.10 becomes the following special form:

$$\|q\|_V \leq C\|P_{\mathring{\mathfrak{H}}^k} q\|, \forall q \in \mathring{\mathfrak{H}}_h^k,$$

where either

$$\mathring{\mathfrak{H}}_h^k = \left\{u \in \mathring{\mathcal{P}}_r \mid du = 0, \langle u, dv \rangle = 0, \forall v \in \mathring{\mathcal{P}}_{r-1}\right\}, \tag{3.12}$$

or

$$\mathring{\mathfrak{H}}_h^k = \left\{ u \in \mathring{\mathcal{P}}_r^- \mid du = 0, \langle u, dv \rangle = 0, \forall v \in \mathring{\mathcal{P}}_{r-1}^- \right\}, \tag{3.13}$$

Summarizing the preceding argument, the known results for the natural and essential BVP give us (3.8) for the following special cases:

Lemma 3.2.12. *Assume Ω a Lipschitz domain with triangulation $\mathcal{T}$. For the harmonic spaces defined in (2.4), (2.5), and the discrete harmonic spaces given by (3.10), (3.11)), (3.12)), and (3.13)), we have*

$$\|q_1\| \leq C\|P_{\mathfrak{H}^k} q_1\|, \forall q_1 \in \mathfrak{H}_h^k,$$
$$\|q_2\| \leq C\|P_{\mathring{\mathfrak{H}}^k} q_2\|, \forall q_2 \in \mathring{\mathfrak{H}}_h^k,$$

3.3 Analysis of the semi-essential Robin BVP

In this section, we will show that Problem 2.5.2 is the abstract Hodge Laplacian for a certain closed Hilbert complex. Consider the complex (W, D) (as in (3.1)) with Hilbert spaces

$$W^{k-1} = L^2\Lambda^{k-1}, \quad W^k = L^2\Lambda^k, \quad W^{k+1} = L^2\Lambda^{k+1} \times L^2\Lambda^k(\Gamma),$$

and operators

$$D^{k-1} = d^{k-1} \quad \text{and} \quad D^k = (d^k, \mathrm{tr}).$$

Here W^{k+1} is equipped with the norm $\|(u, \sigma)\|^2 = \|u\|^2 + \lambda\|\sigma\|_\Gamma^2$, the lower-case d represents the usual exterior differentials, and tr denotes the trace operator of k-forms. In order to have boundedness of D, we choose the following domains:

$$V^{k-1} = \mathring{H}\Lambda^{k-1}, \quad \text{and} \quad V^k = \mathscr{H}\Lambda^k.$$

With these spaces and operators, we have a Hilbert complex, as stated and proved below.

Proposition 3.3.1. *The sequence (W, D) with domains V is a Hilbert complex.*

Proof. It is obvious that $D^k \circ D^{k-1} = 0$. Thus, (W, D) is a Hilbert complex if we can verify the following two properties:

1. The operators D^{k-1} and D^k are closed, i.e., the resulting graph $\{(u, Du)\}$ is a closed set;

2. The domains V^{k-1} and V^k are dense in W^{k-1} and W^k, respectively.

For the closedness, we need to check for D^k only, because the closedness of $D^{k-1} = d^{k-1}$ is known (cf. [4, Section 6.2] for instance). Suppose we have a sequence $(u_i, du_i, \operatorname{tr} u_i)$ with $u_i \in \mathscr{H}\Lambda^k$ converging to $(u, w, \rho) \in W^k \times W^{k+1} = L^2\Lambda^k \times L^2\Lambda^{k+1} \times L^2\Lambda^k(\Gamma)$. In order to prove closedness, we need to show that $u \in \mathscr{H}\Lambda^k$, $du = w$, and $\operatorname{tr} u = \rho$. In fact, we have

$$u_i \to u \text{ in } L^2\Lambda^k \implies du_i \to du \text{ in } H^{-1}\Lambda^{k+1},$$

and comparing the latter with our hypothesis $du_i \to w$ in $L^2\Lambda^{k+1}$, we have $w = du$. Thus we know $u \in H\Lambda^k$, and $u_i \to u$ in $H\Lambda^{k+1}$. Consequently, $\operatorname{tr} u_i \to \operatorname{tr} u$ in $H^{-1/2}\Lambda^k(\Gamma)$. However, we assumed $\operatorname{tr} u_i \to \rho$ in $L^2\Lambda^k(\Gamma)$, so we know $\operatorname{tr} u = \rho$. The first property is checked.

Next, we prove the density property. We know that $C_0^\infty\Lambda^{k-1}$ and $C_0^\infty\Lambda^k$ are dense in $L^2\Lambda^{k-1}$ and $L^2\Lambda^k$, respectively. In addition, we obviously have $C_0^\infty\Lambda^{k-1} \subset V^{k-1}$ and $C_0^\infty\Lambda^k \subset V^k$. The property hence follows. Thus, we showed (W, D) is a Hilbert complex. $\square$

We need to prove furthermore that (W, D) is a closed Hilbert complex, in order to apply Theorem 3.2.7. As pointed out by Lemma 3.2.2, we shall first show a Poincaré inequality (cf. Lemma 3.3.4). To serve that purpose, we need a compactness property (cf. Lemma 3.3.3). We start with the following theorem proven by Costabel [15].

Theorem 3.3.2. *For any $\Omega \subset \mathbb{R}^3$ Lipschitz domain, there exists C, dependent only on Ω, such that*

$$\|u\|_{H^{1/2}} \le C\left(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|u \times n\|_\Gamma\right), \tag{3.14}$$

$$\|u\|_{H^{1/2}} \le C\left(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|u \cdot n\|_\Gamma\right) \tag{3.15}$$

for all vector field $u \in H(\mathrm{curl}) \cap H(\mathrm{div})$).

Recalling the correspondence Tables 2.2 and 2.3, we can restate (3.14) and (3.15) as

$$\|u\|_{H^{1/2}} \le C\left(\|u\| + \|du\| + \|\delta u\| + \|\operatorname{tr} u\|_\Gamma\right). \tag{3.16}$$

for $u \in \mathscr{H}\Lambda^k \cap H^*\Lambda^k$ in three dimensions, where $k = 1, 2$. We note that (3.16) still holds for the cases $k = 0, 3$. In fact, u is in $H^1\Lambda^k$ if $k = 0$ or 3, and the right-hand side of (3.16) contains the $\|u\|_{H^1}$. Therefore, (3.16) holds for all k in three dimensions. In other words, $\mathscr{H}\Lambda^k \cap H^*\Lambda^k$ is continuously embedded in $H^{1/2}\Lambda^k$. By the Rellich theorem, $H^{1/2}\Lambda^k$ is compact in $L^2\Lambda^k$. Thus we have proved the following lemma:

Lemma 3.3.3. *For any Ω Lipschitz domain in $\mathbb{R}^3$, the space $\mathscr{H}\Lambda^k \cap H^*\Lambda^k$ is compactly embedded in $L^2\Lambda^k$.*

We note that the null spaces for D^{k-1} and D^k in (V, D) are $\left\{u \in \mathring{\mathscr{H}}\Lambda^{k-1} | du = 0\right\} = \mathring{\mathfrak{Z}}^{k-1}$ and $\left\{u \in H\Lambda^k | du = 0, \operatorname{tr} u = 0\right\} = \mathring{\mathfrak{Z}}^k$, respectively. So we shall prove the following Poincaré inequality.

Lemma 3.3.4. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain.*

- *(Poincaré inequality for k.) For all $u \in \mathring{\mathfrak{Z}}^{k,\perp_{\mathscr{H}\Lambda}}$, we have*

$$\|u\| \leq C\|D\sigma\| = C(\|du\| + \|\operatorname{tr} u\|_\Gamma), \tag{3.17}$$

  where the constant C depends on Ω and k only.

- *(Poincaré inequality for $k-1$.) For all $\sigma \in \mathring{\mathfrak{Z}}^{k-1,\perp_{\mathring{H}\Lambda}}$, we have*

$$\|\sigma\| \leq C\|D\sigma\| \leq C\|d\sigma\|,$$

  where the constant C depends on Ω and k only.

Proof. The second part is a standard result. Thus we only need to prove 3.17.

For $k = 0$, we see that u is a H^1 function such that it is orthoganal to constants: $\int_\Omega u = 0$. The desired result 3.17 is just a standard Poincaré inequality.

Now we prove for $k > 1$. If the statement is not true, we can find a sequence $\{u_n\} \subset \mathring{\mathfrak{Z}}^{\perp_{\mathscr{H}\Lambda^k}}$ satisfying

$$\|du_n\| + \|\operatorname{tr} u_n\|_\Gamma \to 0 \quad \text{as } n \to \infty, \quad \|u_n\| = 1. \tag{3.18}$$

Since $\mathring{\mathfrak{Z}}^{\perp_{\mathscr{H}\Lambda^k}}$ is (L^2-)orthogonal to $d\mathscr{H}\Lambda^{k-1}$, we know in particular that $\mathring{\mathfrak{Z}}^{\perp_{\mathscr{H}\Lambda^k}}$ is orthogonal to $dC_0^\infty\Lambda^{k-1}$. For any $u \in \mathring{\mathfrak{Z}}^{\perp_{\mathscr{H}\Lambda^k}}$ and any test function $v \in C_0^\infty\Lambda^{k-1}$, we obtain from Lemma 2.2.1 that

$$\langle \delta u, v\rangle = -\langle u, dv\rangle + \int_\Gamma \operatorname{tr} v \wedge \operatorname{tr} \star u = 0.$$

Because $C_0^\infty\Lambda^{k-1}$ is dense in $L^2\Lambda^{k-1}$, δu is orthogonal to $L^2\Lambda^{k-1}$, which implies $\delta u = 0$. Now since $u_n \in \mathscr{H}\Lambda^k \cap H^*\Lambda^k$ are bounded from (3.18), by Lemma 3.3.3, we know that u_n has a subsequence, which we still denote as u_n, that converges to some $u_0 \in L^2\Lambda^k$ (in L^2-norm).

Now from $u_n \to u_0$ in $L^2\Lambda^k$, we have $du_n \to du_0$ in $H^{-1}\Lambda^k$. Besides, we know $du_n \to 0$ from (3.18). These facts imply that $u_0 \in H\Lambda^k$, $du_0 = 0$, and $u_n \to u_0$ in $H\Lambda$-norm. Then from the boundedness of $\operatorname{tr} : H\Lambda^k(\Omega) \to H^{-1/2}\Lambda^k(\Gamma)$, we obtain

$\operatorname{tr} u_n \to \operatorname{tr} u_0$ in $H^{-1/2}\Lambda^k$. We also have $\operatorname{tr} u_n \to 0$ in $L^2(\Gamma)$ from (3.18), thus we know $\operatorname{tr} u_0 = 0$, which shows $u_0 \in \mathring{H}\Lambda^k$.

Next, $u_n \perp d\mathring{H}\Lambda^{k-1}$ implies $u_0 \perp d\mathring{H}\Lambda^{k-1}$. Thus we have $u_0 \in \mathring{\mathfrak{H}}^k$. However, since $u_n \perp \mathring{\mathfrak{H}}^k$, we have $u_0 \perp \mathring{\mathfrak{H}}^k$. Thus $u_0 = 0$, which contradicts with $1 = \|u_n\| \to \|u_0\|$ from (3.18). Therefore, we proved the lemma. □

By Lemma 3.2.2, we hence have proved the following.

Proposition 3.3.5. *The sequence (W, D) with domains V is a closed Hilbert complex.*

3.3.1 Adjoint complex of the semi-essential Robin BVP

As we have seen in Section 3.2, the abstract Hodge-Laplacian is made up with several operators and their adjoints. In this section we determine the operators and domains of the adjoint complex (W, D^*). With these, we shall see that the complex introduced earlier indeed corresponds to the semi-essential Robin BVP.

First, sincei $D^{k-1} = d^{k-1} : \mathring{H}\Lambda^{k-1} \to L^2\Lambda^k$, we know that its adjoint operator is $D_k^* = d_k^* = \delta^k$ with domain $V_k^* = H^*\Lambda^k$. Second, for $D^k = (d^k, \operatorname{tr})$, by definition, the domain of its adjoint is

$$V_{k+1}^* = \left\{(u,\sigma) \in W^{k+1} \mid \exists w \in W^k, \langle (u,\sigma), D^k v\rangle = \langle w, v\rangle \quad \forall v \in V^k\right\},$$

i.e.,

$$V_{k+1}^* = \left\{(u,\sigma) \in W^{k+1} \mid \exists w \in W^k, \langle u, dv\rangle + \lambda\langle \sigma, \operatorname{tr} v\rangle_\Gamma = \langle w, v\rangle \quad \forall v \in \mathscr{H}\Lambda^k\right\}. \tag{3.19}$$

A particular case is that for any $(u,\sigma) \in V_{k+1}^*$, there exists $w \in L^2\Lambda^k$ satisfying

$$\langle u, dv\rangle = \langle w, v\rangle, \quad \forall v \in \mathring{H}\Lambda^k.$$

The last equation shows that $u \in H^*\Lambda^{k+1}$, and $\delta u = w$. From (3.19) and Lemma 2.2.1, we obtain

$$\langle \delta u, v\rangle + \int_\Gamma \operatorname{tr} v \wedge \operatorname{tr} \star u + \lambda \int_\Gamma \operatorname{tr} v \wedge \star\sigma = \langle \delta u, v\rangle, \quad \forall v \in H^1\Lambda^k.$$

Hence we have

$$\operatorname{tr} \star u + \lambda \star \sigma = 0.$$

Since we assumed $\sigma \in L^2\Lambda^k(\Gamma)$, we have proved $u \in \mathscr{H}^*\Lambda^{k+1}$, and therefore

$$V_{k+1}^* \subset \left\{(u,\sigma) \in \mathscr{H}^*\Lambda^{k+1} \times L^2\Lambda^k(\Gamma) \mid \operatorname{tr}\star u + \lambda \star \sigma = 0\right\}.$$

On the other hand, for any pair $(u,\sigma) \in \mathscr{H}^*\Lambda^{k+1} \times L^2\Lambda^k(\Gamma)$ with the compatibility condition $\operatorname{tr}\star u + \lambda \star \sigma = 0$, we let $w = \delta u \in L^2\Lambda^k$. Then, Lemma 2.2.1 yields

$$\langle u, dv\rangle + \lambda\langle \sigma, \operatorname{tr} v\rangle_\Gamma = \langle w, v\rangle, \quad \forall v \in H^1\Lambda^k.$$

Next, by compactness of $H^1\Lambda^k$ in $\mathscr{H}\Lambda^k$, the preceding equation holds for all $v \in \mathscr{H}\Lambda^k$. Thus we have

$$V_{k+1}^* \supset \left\{(u,\sigma) \in \mathscr{H}^*\Lambda^{k+1} \times L^2\Lambda^k(\Gamma) \mid \operatorname{tr}\star u + \lambda \star \sigma = 0\right\},$$

and conclude that

$$V_{k+1}^* = \left\{(u,\sigma) \in \mathscr{H}^*\Lambda^{k+1} \times L^2\Lambda^k(\Gamma) \mid \operatorname{tr}\star u + \lambda \star \sigma = 0\right\}.$$

From the above argument, we also see that

$$D_{k+1}^*(u,\sigma) = \delta^{k+1}u, \quad \forall (u,\sigma) \in V_{k+1}^*.$$

With these ingredients, we can check that the abstract harmonic form now is in fact $\mathring{\mathfrak{H}}$, as defined in (2.5). We immediately have the following well-posedness from Theorem 3.2.7.

Theorem 3.3.6 (Well-posedness of the semi-essential Robin BVP)**.** *Assume that $\Omega \in \mathbb{R}^3$ is a Lipschitz domain, and $\lambda > 0$. There exist $C > 0$, depending on Ω and λ only, such that for any $f \in L^2\Lambda^k$, there exists a solution $(\sigma, u, p) \in \mathring{H}\Lambda^{k-1} \times \mathscr{H}\Lambda^k \times \mathring{\mathfrak{H}}^k$ for Problem 2.5.3. Moreover,*

$$\|\sigma\|_{H\Lambda^{k-1}} + \|u\|_{\mathscr{H}\Lambda^k} + \|p\| \le C\|f\|.$$

Remark 3.3.7. *We introduced the $\mathcal{B}$ problem and $\mathcal{B}^*$ problem for the semi-essential Robin BVP in Chapter 2. They have such names because the function f is assumed to be in the range of the D and D^* operators. This is also the reason for the semi-natural BVP, whose associated Hilbert complex will be discussed later in this chapter.*

3.3.2 Discretization

As we have proved the well-posedness of the semi-essential Robin BVP, we move on to introduce the finite element method we are going to use to solve it. We consider the finite element space $\mathring{\Lambda}_h^{k-1} \times \Lambda_h^k \times \mathring{\mathfrak{H}}_h^k$, where $\mathring{\Lambda}^{k-1} \times \Lambda^k$ are one of the following:

$$\mathring{\mathcal{P}}_r^- \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k, \quad \mathring{\mathcal{P}}_r \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k,$$
$$\mathring{\mathcal{P}}_{r+1}^- \Lambda^{k-1} \times \mathcal{P}_r \Lambda^k, \quad \mathring{\mathcal{P}}_{r+1} \Lambda^{k-1} \times \mathcal{P}_r \Lambda^k,$$

and

$$\mathring{\mathfrak{H}}_h^k = \left\{ u \in \Lambda_h^k \mid du = 0, \quad \langle u, d\sigma \rangle = 0 \quad \forall \sigma \in \mathring{\Lambda}_h^{k-1} \right\}.$$

Here the spaces $P_r \Lambda^k$ and $P_r^- \Lambda^k$ are the finite element spaces introduced in Section 2.4. A ring above a space, e.g., $\mathring{\mathcal{P}}_r \Lambda^k$, indicates that we impose boundary conditions by setting the degrees of freedom associated with vertices, edges, and faces on the boundary Γ of the domain to be zero.

Our first discrete Robin problem is the following.

Problem 3.3.8. *Find $(\sigma, u, p) \in \mathring{\Lambda}_h^{k-1} \times \Lambda_h^k \times \mathring{\mathfrak{H}}_h^k$ that satisfies*

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, d\tau \rangle &= 0, \quad \forall \tau \in \mathring{\Lambda}_h^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \lambda \langle \operatorname{tr} u, \operatorname{tr} v \rangle_\Gamma + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in \Lambda_h^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathring{\mathfrak{H}}_h^k. \end{aligned} \tag{3.20}$$

Here $\lambda > 0$.

In order to apply Theorems 3.2.10 and 3.2.9 to derive our discrete semi-natural BVP's stability and convergence , we need a discrete Poincaré inequality, which will be given and proven in the next section.

3.3.3 Discrete Poincaré inequalities

In this section we prove the following result.

Theorem 3.3.9 (Discrete Poincaré inequality for k)**.** *Given $\Omega \in \mathbb{R}^3$ a Lipschitz polyhedral domain and $\mathcal{T}$ a quasi-uniform mesh on it. For each $k = 0$, 1, 2, or 3, there exists a $C > 0$, such that for all $u_h \in \Lambda_h^k$ that satisfies $u_h \perp d\mathring{\Lambda}_h^{k-1}$ and $u_h \perp \mathring{\mathfrak{H}}_h^k$, we have*

$$\|u_h\| \leq C \left(\|du_h\| + \|\operatorname{tr} u_h\| \right). \tag{3.21}$$

We first note that the cases $k = 0$ or 3 are straightforward. If $k = 0$, the statement is identical to the continuous case, cf. Lemma 3.3.4. If $k = 3$, such u_h must vanish, and hence the result. Thus we only need to prove for $k = 1$ and 2. We need a few lemmas.

In general, to prove a discrete Poincaré inequality, it is a standard approach to use some projection between the continuous and discrete function spaces, and take advantage of the continuous Poincaré inequality. We will use this approach. However, as we know from Section 2.4, the canonical projection is not a bounded operator on $H(\mathrm{curl})$ or even $\mathscr{H}(\mathrm{curl})$. However, on a subset of $H(\mathrm{curl})$ with suitable extra assumptions, the projection is defined. Precisely, we have the following lemma (cf. [26, Lemma 5.38]).

Lemma 3.3.10. *Assume $\mathcal{T} = \{T\}$ is a triangulation of the domain Ω. The canonical projection πu of $u \in H(\mathrm{curl})$ into Λ_h^1 is defined and bounded, if u satisfies $u \in (H^\delta(T))^3$ and $\mathrm{curl}\, u \in L^p(T)^3$ for each tetrahedron T in the mesh for some $\delta > 1/2$ and $p > 2$.*

In order to apply the previous lemma, we need to show extra regularity for the function in the finite element space. The next two lemmas provide us with desired regularity. The first one, due to Bramble *et. al.*[5, Section 10], is an inverse estimate in planer domains.

Lemma 3.3.11. *Let $F \in \mathbb{R}^2$ be a Lipschitz polygon with a quasi-uniform triangulation $\mathcal{T}$ of size h. For $0 \leq t < 1/2$, there exists $C > 0$, independent of h, such that for all u piecewise polynomials, it holds that*

$$\|u\|_{H^t(F)} \leq Ch^{-t}\|u\|_F.$$

The next lemma generalize the result [1, Proposition 3.7] that $\mathring{H}(\mathrm{curl}) \bigcap H(\mathrm{div})$ is continuously embedded in $H^{1/2+\delta}$ for some $\delta > 0$ that depends on the domain Ω only. It is first given in [20, Equation 4.9]. However, for a more detailed and correct discussion, one should also refer to [25, Corollary 5.5.2].

Lemma 3.3.12. *Assume $\Omega \in \mathbb{R}^3$ is a polyhedral domain. There exist $C > 0$ and $0 < s < 1/2$, depending on Ω only, such that for all $u \in H(\mathrm{curl}) \cap H(\mathrm{div})$ satisfying $u \times n \in H^t(\Gamma)$, where $0 < t < s$, one has $u \in H^{t+1/2}(\Omega)$, and*

$$\|u\|_{H^{t+1/2}} \leq C\left(\|u\| + \|\mathrm{curl}\, u\| + \|\mathrm{div}\, u\| + \|u \times n\|_{H^t(\Gamma)}\right).$$

We note that in Lemma 3.3.11, regularity is given only on each face F, not the whole boundary Γ. However, in Lemma 3.3.12, regularity on Γ is needed in the hypothesis. We remark that these two are equivalent for $0 < t < 1/2$.

When proving Theorem 3.3.9, we will need to project $u_h \in \mathring{\mathfrak{Z}}_h^{1,\perp_{\Lambda_h^1}}$ into the space $\mathring{\mathfrak{Z}}^{1,\perp}$, and use several properties of the projection. Such projection and its properties will also be used when we analyze Maxwell's equations, cf. Theorem 4.4.5. Therefore, we state these properties as the following lemma.

Lemma 3.3.13. *Assume that Ω is a Lipschitz polyhedral domain with a quasi-uniform* *mesh, and* $u_h \in \Lambda_h^1$ *is orthogonal to* $\mathring{\mathfrak{Z}}^1$. *Let* $u = u_h - P_{\mathring{\mathfrak{Z}}^1} u_h$. *Then* u *satisfies*

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = \operatorname{curl} u_h, \quad u \times n = u_h \times n.$$

Moreover, πu *is defined, and*

$$\|u - u_h\| \le C\|u - \pi u\| \tag{3.22}$$

Proof. We first establish (3.27). By definition, u is orthogonal to the null space $\mathring{\mathfrak{Z}}^1$, thus it is in the range $\operatorname{curl}(H(\operatorname{curl}))$, and consequently, it is div-free. Again by definition of u, we know $u - u_h \in \mathring{\mathfrak{Z}}^1$, which gives $\operatorname{curl}(u - u_h) = 0$ and $(u - u_h) \times n = 0$. The other two equations in (3.27) hence follows.

Next, we will show πu is defined. The above argument has already implied that $u \in H(\operatorname{curl}) \bigcap H(\operatorname{div})$. Moreover,because $u \times n = u_h \times n$ is a piecewise polynomial on each face F of Ω, by Lemma 3.3.11, we know $u \times n \in H^t(\Gamma)$ for any $0 < s < 1/2$. It then follows from Lemma 3.3.12 that $u \in H^t(\Omega)$ for $t > 0$ sufficiently small. Besides, we have $\operatorname{curl} u = \operatorname{curl} u_h \in L^\infty(\Omega)$. Therefore, by Lemma 3.3.10, πu is defined.

It remains to establish the estimate. We first observe that $u_h - \pi u$ is both curl-free and trace-free. Indeed, the trace-free part comes from the fact that the canonical projection preserves vanishing traces:

$$\begin{aligned}(u - u_h) \times n = 0 \implies \pi(u - u_h) \times n = 0& \\ \implies (\pi u - \pi u_h) \times n = 0 \implies (\pi u - u_h) \times n = 0.&\end{aligned}$$

The curl-free part is a result of commutativity of the canonical projections with curl operator:

$$\begin{aligned}\operatorname{curl}(u_h - u) = 0 \implies \pi \operatorname{curl}(u_h - u) = 0& \\ \implies \operatorname{curl} \pi(u_h - u) = 0 \implies \operatorname{curl}(u_h - \pi u) = 0.&\end{aligned}$$

Thus by definition, $u_h - \pi u \in \mathring{\mathfrak{Z}}_h^1 \subset \mathring{\mathfrak{Z}}^1$, and hence

$$\langle u - u_h, u_h - \pi u\rangle = \langle u, u_h - \pi u\rangle + \langle u_h, u_h - \pi u\rangle = 0$$

By the Pythagorean theorem, (3.22) follows. The lemma is proved. $\square$

Now we are ready to prove Theorem 3.3.9 for the case $k = 1$.

Proof of Theorem 3.3.9 for $k = 1$. For any such u_h, we define

$$u = u_h - P u_h,$$

where P is the L^2 projection from $L^2\Lambda^1$ to $\mathring{\mathfrak{Z}}^1$.

By Lemma 3.3.13, we know that πu is defined, and

$$\|u_h\| \le \|u - u_h\| + \|u\| \le \|u - \pi u\| + \|u\|.$$

By Lemma 3.3.4 and Lemma 3.3.13, we have

$$\|u\| \le C(\|\operatorname{curl} u\| + \|u \times n\|_\Gamma) \le C(\|\operatorname{curl} u_h\| + \|u_h \times n\|_\Gamma)$$

Thus it suffices to prove

$$\|u - \pi u\| \le C(\|\operatorname{curl} u_h\| + \|u_h \times n\|_\Gamma). \tag{3.23}$$

We will prove this by a scaling argument, as below.

Denote $\hat{T}$ the reference tetrahedron. On each tetrahedron T,

$$\begin{aligned} \|u - \pi u\|_T^2 &\le Ch^3 \|\hat{u} - \hat{\pi}\hat{u}\|_{\hat{T}}^2 \\ &= Ch^3 \|(I - \hat{\pi})(\hat{u} - v)\|_{\hat{T}}^2 \\ &\le Ch^3 (\|\hat{u} - v\|_{H^{t+1/2}(\hat{T})}^2 + \|\operatorname{curl}(\hat{u} - v)\|_{L^p(\hat{T})}^2), \end{aligned} \tag{3.24}$$

for any v in the finite element space in $\hat{T}$. Since $\operatorname{curl}(\hat{u} - v)$ is a piecewise polynomial (up to some degree), we can use equivalence of norms and obtain

$$\|\operatorname{curl}(\hat{u} - v)\|_{L^p(\hat{T})}^2 \le C\|\operatorname{curl}(\hat{u} - v)\|_{H^{-1}(\hat{T})}^2 \le C\|\hat{u} - v\|_{L^2(\hat{T})}^2 \le C\|\hat{u} - v\|_{H^{t+1/2}(\hat{T})}^2.$$

Thus (3.24) yields

$$\|u - \pi u\|_T^2 \le Ch^3 \inf_v \|\hat{u} - v\|_{H^{t+1/2}(\hat{T})}^2.$$

By [18, Theorem 6.1], we have

$$\|u - \pi u\|_T^2 \le Ch^3 |\hat{u}|_{H^{t+1/2}(\hat{T})}^2.$$

Next we scale back to T:

$$|\hat{u}|_{H^{t+1/2}(\hat{T})}^2 \le Ch^{2(t+1/2)-3} |u|_{H^{t+1/2}(T)}^2.$$

The last two estimates yield

$$\|u-\pi u\|_T^2 \le Ch^{2t+1}|u|^2_{H^{t+1/2}(T)}.$$

Summing over all T, we obtain

$$\|u-\pi u\|_\Omega \le Ch^{t+1/2}|u|_{H^{t+1/2}(\Omega)}. \tag{3.25}$$

For the $H^{t+1/2}$ semi-norm, we apply Lemma 3.3.12, and use the fact that $\operatorname{div} u = 0$ from Lemma 3.3.13 to derive

$$\|u\|_{H^{t+1/2}} \le C(\|u\| + \|\operatorname{curl} u\| + \|u \times n\|_{H^t(\Gamma)}).$$

By continuous Poincaré inequality (cf. Lemma 3.3.4), we can bound $\|u\|$ in above and derive

$$\|u\|_{H^{t+1/2}} \le C(\|\operatorname{curl} u\| + \|u \times n\|_{H^t(\Gamma)}).$$

It remains to bound the last term, for which we use the inverse estimate from Lemma 3.3.11, as below.

$$\|u\times n\|_{H^t(\Gamma)} \le C\sum_F \|u\times n\|_{H^t(F)} \le h^{-t}\sum_F C_F\|u\times n\|_F \le Ch^{-t}\|u\times n\|_\Gamma.$$

The last two estimates and (3.25) yield that

$$\|u-\pi u\| \le Ch^{1/2}(\|\operatorname{curl} u\| + \|u\times n\|_\Gamma).$$

By Lemma 3.3.13, the last estimate implies (3.3.9). Hence we proved the theorem for case $k=1$. $\square$

It remains to prove Theorem 3.3.9 for $k=2$. The idea is similar to the $k=1$ case. In the proof, the canonical projection π (into Λ_h^2 for $k=2$) also plays an important role. Similar to $k=1$, π is not a bounded operation on $H(\mathrm{div})$ or $\mathscr{H}(\mathrm{div})$. The following lemma from [26, Lemma 5.15], which analogies Lemma 3.3.10 but weakens the assumptions, provides a sufficient condition to define a continuous projection.

Lemma 3.3.14. *Assume $\mathcal{T}=\{T\}$ is a triangulation of the domain Ω. The canonical projection πu of $u \in H(\mathrm{div})$ into Λ_h^2 is defined and bounded, if $u \in (H^{1/2+\delta}(T))^3$ for some $\delta > 0$.*

Similar to Lemma 3.3.12, We also have a lemma that gives our function regularity more than $1/2$ in Ω if it has enough regularity on Γ.

Lemma 3.3.15. *Assume $\Omega \in \mathbb{R}^3$ is a polyhedral domain. There exist $C > 0$ and $0 < s < 1/2$, depending on Ω only, such that for all $u \in H(\mathrm{curl}) \cap H(\mathrm{div})$ satisfying $u \cdot n \in H^t(\Gamma)$ for all faces F of Ω, where $0 < t < s$, one has $u \in H^{t+1/2}$, and*

$$\|u\|_{H^{t+1/2}} \le C\left(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|u \cdot n\|_{H^t(\Gamma)}\right).$$

Proof. Consider the Neumann problem

$$\begin{cases} -\Delta\rho &= f, \quad \text{in } \Omega, \\ \frac{\partial\rho}{\partial n} &= u \cdot n \in H^t(\Gamma), \quad \text{on } \Gamma, \end{cases}$$

where $f = \frac{1}{Vol(\Omega)} \int_\Gamma u \cdot n$. By [26, Theorem 3.18], ρ is in $H^{3/2+\epsilon}$ for some $0 < \epsilon < 1/2$.

Let $U = \operatorname{grad} \rho$, then

$$\operatorname{curl} U = 0, \quad \operatorname{div} U = -f, \quad U \cdot n = u \cdot n.$$

Therefore, $u - U$ is in the space $H(\mathrm{curl}) \bigcap \mathring{H}(\mathrm{div})$. Thus we know from [1, Proposition 3.7] that there is $0 < \delta_0 < 1/2$, such that for any $0 < \delta < \delta_0$, $u - U$ is in $H^{1/2+\delta}(\Omega)$, and

$$\begin{aligned} \|u - U\|_{H^{1/2+\delta}} &\le C(\|u - U\| + \|\operatorname{curl}(u - U)\| + \|\operatorname{div}(u - U)\|) \\ &\le C(\|u\| + \|U\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|f\|) \end{aligned}$$

By Cauchy-Schwarz inequality, we have $\|f\| \le C\|u \cdot n\|_\Gamma$. Thus we have

$$\|u - U\|_{H^{1/2+\delta}} \le C(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|u \cdot n\|_\Gamma + \|U\|)$$

In particular, for any $0 < t < \min(\epsilon, \delta)$, we know that $U = \operatorname{grad} \rho \in H^{t+1/2}$. Hence $u \in H^{t+1/2}$, and the above estimate yield

$$\|u\|_{H^{t+1/2}} \le C(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\| + \|u \cdot n\|_\Gamma + \|\operatorname{grad} \rho\|_{H^{t+1/2}})$$

Furthermore, by regularity of ρ, we have

$$\|\operatorname{grad} \rho\|_{H^{t+1/2}} \le \|\rho\|_{H^{t+3/2}} \le C(\|f\| + \|u \cdot n\|_{H^{t+1/2}(\Gamma)}) \le C\|u \cdot n\|_{H^{t+1/2}(\Gamma)}.$$

The lemma follows from the two estimates above. $\square$

As before, Lemmas 3.3.11 will be helpful establish necessary regularity of u on Γ, as we will see in the next proof of Theorem 3.3.9 for $k = 2$.

Proof of Theorem 3.3.9 for $k = 2$. The proof is much analogous to the previous case.

For any such u_h, we define

$$u = u_h - Pu_h,$$

where P is the L^2 projection from $L^2\Lambda^2$ to $\mathring{\mathfrak{Z}}^2$.

As before, we will first show that πu is defined, and

$$\|u_h\| \leq \|u\| + C\|u - \pi u\|. \tag{3.26}$$

One can check that

$$\operatorname{div} u_h = \operatorname{div} u, \quad u_h \cdot n = u \cdot n, \quad \operatorname{curl} u = 0. \tag{3.27}$$

Therefore, u satisfies all hypotheses in Lemma 3.3.15, and hence

$$\|u\|_{H^{t+1/2}} \leq C\left(\|u\| + \|\operatorname{div} u\| + \|u \cdot n\|_{H^t(\Gamma)}\right). \tag{3.28}$$

Consequently, u also satisfies the hypotheses in Lemma 3.3.10. Hence, πu is well-defined. Next, we can check $u_h - \pi u \in \mathring{\mathfrak{Z}}^{2,\perp}$ by the following.

$$\begin{aligned}(u - u_h) \cdot n = 0 \implies \pi(u - u_h) \cdot n = 0 \\ \implies (\pi u - \pi u_h) \cdot n = 0 \implies (\pi u - u_h) \cdot n = 0.\end{aligned}$$

$$\begin{aligned}\operatorname{div}(u_h - u) = 0 \implies \pi \operatorname{div}(u_h - u) = 0 \\ \implies \operatorname{div} \pi(u_h - u) = 0 \implies \operatorname{div}(u_h - \pi u) = 0.\end{aligned}$$

Now that we have verified $u_h - \pi u \in \mathring{\mathfrak{Z}}^2$, we know in particular that it is orthogonal to $u - u_h$. By the Pythagorean theorem, we have $\|u - u_h\| \leq \|u - \pi u\|$ and hence (3.26) follows. Thus, to prove the result, it suffices to bound $\|u - \pi u\|$.

By a scaling argument similar to the previous case,[2] one can show that

$$\|u - \pi u\| \leq Ch^{t+1/2}\|u\|_{H^{t+1/2}}.$$

Hence we have

$$\|u_h\| \leq C(\|\operatorname{div} u\| + \|u \cdot n\|_\Gamma + h^{t+1/2}\|u\|_{H^{t+1/2}}). \tag{3.29}$$

[2]The scaling argument here is a little simpler than the previous case. We do not have a $\operatorname{div} u$ term from Lemma 3.3.14, while we had $\operatorname{curl} u$ from Lemma 3.3.10 before.

For the $H^{t+1/2}$ norm, Applying Lemma 3.3.15, and the Poincaré inequality (cf. Lemma 3.3.4), and the inverse estimate (cf. Lemma 3.3.11) one can derive the estimate

$$\|u\|_{H^{t+1/2}} \le C(\|\operatorname{div} u\| + h^{-t}\|u \cdot n\|_\Gamma)$$

and hence prove the desired inequality from (3.29). □

3.3.4 Stability of the discrete problem

With the preceding results, we now have the stability result from Theorem 3.2.9. We note that in this problem, the term $\inf_{v\in V_h^k} \|\mathrm{Proj}_{D(V^{k-1})}u - v\|$ is just $\|\mathrm{Proj}_{\mathring{\mathfrak{H}}_h^k} u\|$. Therefore, the general Theorem 3.2.9 becomes the following particular statement.

Theorem 3.3.16. *Let Ω be a Lipschitz domain with quasi-uniform meshes. There is $C > 0$, dependent on Ω and λ only, such that for (σ, u, p) the solution of Problem 2.5.3, and (σ_h, u_h, p_h) the solution of Problem 3.3.8, it holds that*

$$\begin{aligned}
&\|\sigma - \sigma_h\|_{H\Lambda} + \|u - u_h\|_{\mathscr{H}\Lambda} + \|p - p_h\| \\
&\quad \le C\left(\inf_{\tau\in\mathring{\Lambda}_h^{k-1}} \|\sigma - \tau\|_{H\Lambda} + \inf_{v\in\Lambda_h^k} \|u - v\|_{\mathscr{H}\Lambda} + \inf_{q\in\mathring{\mathfrak{H}}_h^k} \|p - q\| + \|\mathrm{Proj}_{\mathring{\mathfrak{H}}_h^k} u\|\right), \quad (3.30)
\end{aligned}$$

Remark 3.3.17. *Similar to Theorem 7.4 in [3] that if the exact solution is smooth enough, then $\|\mathrm{Proj}_{\mathring{\mathfrak{H}}_h^k} u\| = O\big(h^{2r}\big)$. Thus we have*

$$\|\sigma - \sigma_h\|_{H\Lambda} + \|u - u_h\|_{\mathscr{H}\Lambda} + \|p - p_h\| = O(h^r)$$

3.4 Analysis of the semi-natural Robin BVP

Now we move on to the semi-natural Robin BVP 2.5.5. As before, we want to associate the problem with the abstract Hodge Laplacian for a certain closed Hilbert complex. Consider the complex (W, D) (as in (3.1)) with Hilbert spaces

$$W^{k-1} = L^2\Lambda^{k-1} \times L^2\Lambda^{k-1}(\Gamma), \quad W^k = L^2\Lambda^k, \quad W^{k+1} = L^2\Lambda^{k+1},$$

where W^{k-1} is equipped with the norm $\|(\sigma,\mu)\|^2 = \|\sigma\|^2 + \lambda\|\mu\|_\Gamma^2$, and $D^{k-1} : (\sigma, \mu) \mapsto d^{k-1}\sigma$, $D^k = d^k$. We have the domains

$$V^{k-1} = \Big\{(\sigma, \operatorname{tr}\sigma) \mid \sigma \in \mathscr{H}\Lambda^{k-1}\Big\} \quad \text{and} \quad V^k = H\Lambda^k. \qquad (3.31)$$

These spaces and operators gives a Hilbert complex.

Proposition 3.4.1. *The sequence* (W, D) *with domains* V*, as defined above, is a Hilbert complex.*

We will need the following result. It will serve as an intermediate step in the density argument when proving Proposition 3.4.1. However, the reasoning is not so straightforward. Thus we state it as a lemma.

Lemma 3.4.2. *Let* $\Omega \subset \mathbb{R}^3$ *be a Lipschitz domain. For any* $\epsilon > 0$*,* $k = 0, 1, 2$*, and* $\sigma \in L^2\Lambda^k(\Gamma)$*, there exists* $u \in \mathscr{H}\Lambda^k$*, such that*

$$\|u\| \leq \epsilon, \quad \|\operatorname{tr} u - \sigma\|_\Gamma \leq \epsilon.$$

Proof. We will prove this in two steps. First, we will show there exists some $u_1 \in \mathscr{H}\Lambda^k$, whose trace is close to the given σ. This will be done separately for each k. Then we will show that we can find u with the same trace as u_1 and also with small L^2-norm. We will give one argument for the second part for all k.

If $k = 0$, for such σ, we can first find $\sigma_1 \in H^{1/2}(\Gamma)$ such that $\|\sigma - \sigma_1\| \leq \epsilon$. By the well-known trace theorem that tr is surjective from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$, we can find a $u_1 \in H^1$ with $\|\operatorname{tr} u_1 - \sigma\| \leq \epsilon$.

If $k = 1$, for any such σ, one can find a $\sigma_1 \in H^1\Lambda^1(\Gamma)$ with $\|\sigma - \sigma_1\|_\Gamma \leq \epsilon/2$. It is obvious that both σ and $d_\Gamma\sigma$ have $-1/2$-regularity on Γ:

$$\sigma_1 \in H^1\Lambda^1(\Gamma) \subset H^{-1/2}\Lambda^1(\Gamma), \quad d_\Gamma\sigma_1 \in L^2\Lambda^2(\Gamma) \subset H^{-1/2}\Lambda^2(\Gamma).$$

It follows from [8] that there exists $u_1 \in H(\text{curl})$ that satisfies $\operatorname{tr} u_1 = \sigma_1$, and hence $\|\operatorname{tr} u_1 - \sigma\|_\Gamma \leq \epsilon$.

For the case $k = 2$, let us consider the Neumann problem

$$\begin{cases} -\Delta\rho & = \frac{1}{Vol(\Omega)}\int_\Gamma \sigma, \quad \text{in } \Omega, \\ \frac{\partial\rho}{\partial n} & = \sigma, \quad \text{on } \Gamma. \end{cases}$$

One can check that the data satisfies the standard compatibility condition. Hence there is a solution $\rho \in H^1$. Let $u_1 = \operatorname{grad}\rho$, then $u_1 \in H(\text{div})$ and $\operatorname{tr} u_1 = \frac{\partial\rho}{\partial n} = \sigma$. Thus we have accomplished the first step for all k.

Next, we can define a H^1-functions on Ω, parameterized by $\lambda > 0$, that equal 1 in a neighborhood of Γ, and decays to 0 quickly. For example,

$$g_\lambda(x) = \begin{cases} 1, & \text{if } x \in \Omega \text{ and } \operatorname{dist}(x,\Gamma) < \lambda, \\ \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\operatorname{dist}(x,\Gamma)}{\lambda}\pi\right), & \text{if } x \in \Omega \text{ and } \lambda \leq \operatorname{dist}(x,\Gamma) < 2\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Now we consider $u_\lambda = u_1 g_\lambda$. Choosing $\lambda > 0$ sufficiently small, we finally obtain $\|u_\lambda\| \le \epsilon$ and $\|\operatorname{tr} u_\lambda - \sigma\| = \|\operatorname{tr} u_1 - \sigma\| \le \epsilon$. $\square$

Now we can prove Proposition 3.4.1

Proof of Proposition 3.4.1. It is obvious that $D^k \circ D^{k-1} = 0$. Again, we need to prove two properties of (W, D):

1. The operators D^{k-1} and D^k are closed, i.e., the resulting graph $\{(u, Du)\}$ is a closed set;

2. The domains V^{k-1} and V^k are dense in W^{k-1} and W^k.

For the first property, we just need to check for D^{k-1}, as the closedness of $D^k = d^k$ is well-known. Suppose we have

$$(\sigma_i, \operatorname{tr}\sigma_i, d\sigma_i) \to (\sigma, \mu, u) \in L^2\Lambda^{k-1} \times L^2\Lambda^{k-1}(\Gamma) \times L^2\Lambda^k, \quad \text{as } i \to \infty.$$

We want to show that $\sigma \in \mathscr{H}\Lambda^{k-1}$, $\operatorname{tr}\sigma = \mu$, and $d\sigma = u$. In fact, we have $d\sigma_i \to d\sigma$ in H^{-1} because $\sigma_i \to \sigma$ in L^2. Since we also have $d\sigma_i \to u$ in L^2, we hence have

$$u = d\sigma, \quad \text{and} \quad \sigma_i \to \sigma \quad \text{in } H\Lambda.$$

Therefore, we know $\operatorname{tr}\sigma_i \to \operatorname{tr}\sigma$ in $H^{-1/2}$. We also have $\operatorname{tr}\sigma_i \to \mu$ in L^2. So, $\mu = \operatorname{tr}\sigma$. Hence the first property is proved.

Next, we check the density property. The result for V^k is known, thus we only need to check for V^{k-1}. For any $(\sigma, \mu) \in L^2\Lambda^{k-1} \times L^2\Lambda^{k-1}(\Gamma)$, there exist $\sigma_1 \in C_0^\infty\Lambda^{k-1}$ such that $\|\sigma_1 - \sigma\| \le \epsilon$ by density, and $\sigma_2 \in \mathscr{H}\Lambda^{k-1}$ that satisfies

$$\|\operatorname{tr}\sigma_2 - \mu_1\|_\Gamma \le \epsilon, \quad \text{and} \quad \|\sigma_2\| \le \epsilon,$$

by Lemma 3.4.2. Now we consider $(\sigma_1 + \sigma_2, \operatorname{tr}(\sigma_1 + \sigma_2)) = (\sigma_1 + \sigma_2, \operatorname{tr}\sigma_2) \in V^{k-1}$. It is “close” to (σ, μ):

$$\begin{aligned}
&\|(\sigma_1 + \sigma_2, \operatorname{tr}(\sigma_1 + \sigma_2)) - (\sigma, \mu)\|_{L^2(\Omega)\times L^2(\Gamma)} \\
\le&\|\sigma_1 - \sigma\| + \|\sigma_2\| + \lambda\|\operatorname{tr}\sigma_2 - \mu\|_\Gamma \\
\le&\epsilon + \epsilon + \lambda(\|\operatorname{tr}\sigma_2 - \mu_1\|_\Gamma + \|\mu_1 - \mu\|_\Gamma) \\
\le&(2 + 2\lambda)\epsilon.
\end{aligned}$$

This proves the density property. $\square$

As before, a Poincaré inequality is fundamental to analyze our semi-natural Robin BVP. For the semi-essential BVP before, we proved a compactness property to derive the Poincaré inequality. For the semi-natural BVP, a regular decomposition due to Demlow and Hirani [17] is helpful.

Lemma 3.4.3. *Assume Ω a bounded Lipschitz domain in $\mathbb{R}^3$, and let $k = 0, 1, 2$. Given $v \in H\Lambda^k$, there exists $\phi \in H^1\Lambda^{k-1}$ and $z \in H^1\Lambda^k$, such that*

$$v = d\phi + z, \quad \|\phi\|_{H^1} + \|z\|_{H^1} \leq C\|v\|_{H\Lambda}.$$

We will mainly use the following consequent lemma.

Lemma 3.4.4. *Assume Ω a bounded Lipschitz domain in $\mathbb{R}^3$, and let $k = 0, 1, 2$. Given $v \in H\Lambda^k$, there exists $\rho \in H^1\Lambda^k$ satisfying*

$$d\rho = dv, \quad \|\rho\|_{H^1} \leq C\|dv\|.$$

Proof. We consider L^2 orthogonal complement, denoted by $\mathfrak{Z}^\perp$, of the null space of $d^k : H\Lambda^k \to L^2\Lambda^{k+1}$ in $L^2\Lambda^k$. For any such v, we take its L^2 projection into $\mathfrak{Z}^\perp$, and denote that projection by $v^\perp$. It is obvious that we have $dv = dv^\perp$. We also have the Poincaré inequality $\|v^\perp\| \leq C\|dv^\perp\|$.

Applying Lemma 3.4.3 to $v^\perp$, we write $v^\perp = d\phi + \rho$ accordingly. Taking the exterior derivative of the equation, we verify that $d\rho = dv^\perp = dv$. We also have

$$\|\rho\|_{H^1} \leq C\|v^\perp\| \leq C\|dv^\perp\| \leq C\|dv\|.$$

The lemma is thus proven. $\square$

Lemma 3.4.5. *Let Ω be a connected Lipschitz domain in $\mathbb{R}^3$.*

- *(Poincaré inequality for k.) For all $u \in V^k$ that is orthogonal to the null space of D^k, we have*

$$\|\sigma\| \leq C\|d\sigma\|,$$

  where the constant C depends on Ω and k only.

- *(Poincaré inequality for $k-1$.) For all $(\sigma, \operatorname{tr}\sigma) \in V^{k-1}$ that is orthogonal to the null space of D^{k-1}, we have*

$$\|\sigma\| + \|\operatorname{tr}\sigma\| \leq C\|d\sigma\|,$$

  where the constant C depends on Ω and k only.

Proof. The result for k is well-known because $D^k = d^k$. We hence omit that and only prove the inequality for $k-1$. For any such σ, by Lemma 3.4.4, we have some $\rho \in H^1\Lambda^{k-1}$ that satisfies

$$d\rho = d\sigma \in H\Lambda^k, \quad \|\rho\|_{H^1} \leq C_1\|d\sigma\|. \tag{3.32}$$

Note that $\rho - \sigma \in \mathscr{H}\Lambda^{k-1}$, and is in the null space of d: $d(\rho - \sigma) = 0$. By the theorem hypothesis, we have $\langle(\sigma, \operatorname{tr}\sigma), (\rho - \sigma, \operatorname{tr}(\rho - \sigma))\rangle_{W^{k-1}} = 0$. Thus we can apply the Pythagorean theorem and obtain that $\|(\sigma, \operatorname{tr}\sigma)\|_{W^{k-1}} \leq \|(\rho, \operatorname{tr}\rho)\|_{W^{k-1}}$, or in terms of the L^2-norm,

$$\|\sigma\| + \|\operatorname{tr}\sigma\|_\Gamma \leq C(\|\rho\| + \|\operatorname{tr}\rho\|_\Gamma).$$

We note that, in the last inequality, the norm of the $H\Lambda$-trace of ρ is no greater than that of the H^1-trace. The latter is bounded by the H^1-norm of tr by the trace theorem, and hence by $\|d\sigma\|$ according to (3.32). This gives us the desired result. We summarize this argument by the following[3]:

$$\|\sigma\| + \|\operatorname{tr}\sigma\|_\Gamma \leq C(\|\rho\| + \|\operatorname{tr}_{H\Lambda}\rho\|_\Gamma) \leq C(\|\rho\| + \|\operatorname{tr}_{H^1}\rho\|_\Gamma) \leq C\|\rho\|_{H^1} \leq C\|d\sigma\|.$$

□

By Lemmas 3.2.2 and 3.4.5, we have the following closedness.

Proposition 3.4.6. *The sequence* (W, D) *with domains* V *is a closed Hilbert complex.*

3.4.1 Adjoint complex of the semi-natural Robin BVP

In this section, we determine the operators D_k^* and D_{k+1}^*, and domains V_k^* and V_{k+1}^* of the adjoint complex:

$$W^{k-1} \leftarrow W^k \leftarrow W^{k+1}.$$

Since $D^k = d^k$, we immediately have that

$$D_{k+1}^* = d_{k+1}^* = \delta^{k+1}, \quad \text{and} \quad V_{k+1}^* = \mathring{H}^*\Lambda^k.$$

[3]We use $\operatorname{tr}_{H\Lambda}$ and tr_{H^1} to distinguish the $H\Lambda$- and H^1-traces in the inequality

To determine D_k^* and V_k^*, we apply the definition of adjoint: Given $u \in L^2\Lambda^k$, $u \in V_k^*$ and $D^*u = (\tau, \nu)$ if and only if

$$\langle (\tau, \nu), (\sigma, \mu) \rangle = \langle u, D(\sigma, \mu) \rangle = \langle u, d\sigma \rangle, \quad \forall (\sigma, \mu) \in V^{k-1}.$$

Thus we have

$$\langle \sigma, \tau \rangle + \lambda \langle \mu, \nu \rangle_\Gamma = \langle u, d\sigma \rangle.$$

In particular, for all $\sigma \in \mathring{H}\Lambda^k$, we have

$$\langle \sigma, \tau \rangle = \langle d\sigma, u \rangle$$

from integration by parts. We hence have $u \in \mathscr{H}\Lambda^k$, and $\tau = \delta u$. Thus,

$$\langle \sigma, \delta u \rangle + \lambda \langle \mu, \nu \rangle_\Gamma = \langle u, d\sigma \rangle = \langle \sigma, \delta u \rangle + \int \operatorname{tr} \sigma \wedge \operatorname{tr} * u.$$

So we derive

$$\lambda \int_\Gamma \operatorname{tr} \sigma \wedge \star_\Gamma \nu = \int_\Gamma \operatorname{tr} \sigma \wedge \operatorname{tr} \star u,$$

which indicates $\lambda(\star_\Gamma \nu) = \operatorname{tr} \star u$. Thus, we just determined that

$$D^* u = (\delta u, \nu), \quad \text{where} \quad \nu = (-1)^k \frac{1}{\lambda} \star_\Gamma \operatorname{tr} \star u, \tag{3.33}$$

based on which we must define

$$V_k^* = \left\{ u \in L^2\Lambda^k \mid \operatorname{tr} \star u \in L^2\Lambda^k(\Gamma), \delta u \in L^2\Lambda^{k-1} \right\} = \mathscr{H}^*\Lambda^k.$$

Based on these results, we can determine that the abstract harmonic form for this complex becomes $\mathfrak{H}^k$, as defined in (2.4). By Theorem 3.2.7, we have the well-posedness of the semi-natural Robin BVP.

Theorem 3.4.7 (Well-posedness of the semi-natural Robin BVP). *Assume that $\Omega \in \mathbb{R}^3$ is a Lipschitz domain, and $\lambda > 0$. There exist $C > 0$, depending on Ω and λ only, such that for any $f \in L^2\Lambda^k$, there exists a solution $(\sigma, u, p) \in \mathscr{H}\Lambda^{k-1} \times H\Lambda^k \times \mathfrak{H}^k$ for Problem 2.5.3. Moreover,*

$$\|\sigma\|_{H\Lambda^{k-1}} + \|u\|_{\mathscr{H}\Lambda^k} + \|p\| \leq C\|f\|.$$

3.4.2 Discretization

We consider the finite element space $\Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$, where $\Lambda_h^{k-1} \times \Lambda_h^k$ are one of the following:

$$\mathcal{P}_r^- \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k, \quad \mathcal{P}_r \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k,$$
$$\mathcal{P}_{r+1}^- \Lambda^{k-1} \times \mathcal{P}_r \Lambda^k, \quad \mathcal{P}_{r+1} \Lambda^{k-1} \times \mathcal{P}_r \Lambda^k,$$

and

$$\mathfrak{H}_h^k = \left\{ u \in \Lambda_h^k \mid du = 0, \quad \langle u, d\sigma \rangle = 0 \quad \forall \sigma \in \Lambda_h^{k-1} \right\}.$$

In other words, we use the same fours sets of finite elements as we did for the discrete natural BVP. The discrete problem that we shall analyze is

Problem 3.4.8. *Find $(\sigma, u, p) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$ that satisfies*

$$\begin{aligned} -\langle \sigma, \tau \rangle - \lambda \langle \operatorname{tr} \sigma, \operatorname{tr} \tau \rangle_\Gamma + \langle u, d\tau \rangle &= 0, \quad \forall \tau \in \Lambda_h^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in \Lambda_h^k, \\ \langle u, q \rangle &= 0, \quad \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{3.34}$$

As for the discrete semi-essential Robin BVP, our next goal is to state and prove a discrete Poincaré inequality in the next section. After then, we can apply Theorems 3.2.10 and 3.2.9 to derive our main result.

3.4.3 Discrete Poincaré inequalities

As before, we need discrete Poincaré inequalities for $(k-1)$- and k-forms.

Theorem 3.4.9 (Discrete Poincaré inequalities)**.** *Given $\Omega \in \mathbb{R}^3$ a Lipschitz polyhedral domain and $\mathcal{T}$ a quasi-uniform mesh on it.*

- *(Poincaré inequality for k.) For all $u_h \in V_h^k$ that is orthogonal to the null space of D^k on V_h^k, we have*

  $$\|u_h\| \leq C \|du_h\|,$$

  where the constant C depends on Ω and k only.

- *(Poincaré inequality for $k-1$.) For all $(\sigma_h, \operatorname{tr} \sigma_h) \in V_h^{k-1}$ that is orthogonal to the null space of D^{k-1} on V_h^{k-1}, we have*

  $$\|\sigma_h\| + \|\operatorname{tr} \sigma_h\| \leq C \|d\sigma_h\|,$$

where the constant C depends on Ω and k only.

Proof. The first statement is well-known. We will prove the second inequality only.

The inequality is straightforward for $k=1$. In this case, d is grad. The null space consists of $(\tau_h, \operatorname{tr}\tau)$ such that τ_h is a connected component-wise constant function, which is the same as the null space of d in V^0. Thus, we immediately have the desired result from part two of Lemma 3.4.5.

For the $k=3$ case, we will prove the result in a similar fashion as for Lemma 3.4.5. For any such σ_h, we can find $\rho \in H^1\Lambda^{k-1}$, as described by Lemma 3.4.4, such that $d\rho = d\sigma_h$. We note that this implies

$$d(\pi\rho - \sigma_h) = d(\pi(\rho - \sigma_h)) = \pi d(\rho - \sigma_h) = 0.$$

We thus have that $(\pi\rho - \sigma_h, \operatorname{tr}\pi\rho - \operatorname{tr}\sigma_h)$ is in the null space of D^{k-1}, and hence by hypothesis, we have the following W-orthogonality: $(\pi\rho - \sigma_h, \operatorname{tr}\pi\rho - \operatorname{tr}\sigma_h) \perp_W (\sigma_h, \operatorname{tr}\sigma_h)$. By the Pythagorean theorem, we have $\|(\sigma_h, \operatorname{tr}\sigma_h)\|_W \leq \|(\pi\rho, \operatorname{tr}\pi\rho)\|_W$. In other words,

$$\|\sigma_h\| + \|\operatorname{tr}\sigma_h\| \leq C(\|\pi\rho\| + \|\operatorname{tr}\pi\rho\|). \tag{3.35}$$

We will prove the bound $\|\operatorname{tr}\pi\rho\| \leq \|\operatorname{tr}\rho\|$ on each face F of each element T. For our sets of finite element spaces, $\pi\rho$ is either in a $\mathcal{P}$ space or a $\mathcal{P}^-$ space. If the element is $\mathcal{P}_r\Lambda^2$, $\operatorname{tr}\pi\rho$ is an rth degree polynomial on F. We also have that $(\operatorname{tr}\rho - \operatorname{tr}\pi\rho)$ is L^2-orthogonal to any rth degree polynomial on F from (2.15). Hence the desired bound follows from the Pythagorean theorem. If the element is $\mathcal{P}_r^-\Lambda^2$, $\operatorname{tr}\pi\rho$ is an $(r-1)$st degree polynomial on F. We now have that $(\operatorname{tr}\rho - \operatorname{tr}\pi\rho)$ is L^2-orthogonal to any $(r-1)$st degree polynomial on F from (2.16), which also gives the desired bound by the Pythagorean theorem. Therefore, we have the following estimates:

$$\|\sigma_h\| + \|\operatorname{tr}\sigma_h\|_\Gamma \leq \|\pi\rho\| + \|\operatorname{tr}\rho\|_\Gamma.$$

We apply Lemma 3.3.14 on $\|\pi\rho\|$, the trace theorem on $\|\operatorname{tr}\rho\|_\Gamma$, and finally Lemma 3.4.4 to obtain the desired inequality:

$$\|\sigma_h\| + \|\operatorname{tr}\sigma_h\|_\Gamma \leq C\|\rho\|_{H^1} \leq C\|d\sigma\|.$$

The 2-form case is thus proven.

Finally we prove the result for $k=2$. Similar to the previous case, by Lemma 3.4.4, we can find a $\rho \in H^1\Lambda^1$, such that

$$d\rho = d\sigma_h, \quad \|\rho\|_{H^1} \le C\|d\sigma_h\|.$$

It is clear that $\pi\rho$ is defined by Lemma 3.3.10, because, first, ρ is in $H^1\Lambda^1$, second, $\operatorname{curl}\rho = \operatorname{curl}\sigma_h$ is piecewise polynomial and hence L^∞. Besides, we can check that $(\sigma_h - \pi\rho, \operatorname{tr}(\sigma_h - \pi\rho))$ is in the null space of D^1. In fact,

$$\begin{aligned} d(\rho - \sigma_h) = 0 &\implies \pi(d(\rho - \sigma_h)) = 0 \\ &\implies d\pi(\rho - \sigma_h) = 0 \implies d(\pi\rho - \sigma_h) = 0. \end{aligned}$$

Thus we have the orthogonality

$$(\sigma_h, \operatorname{tr}\sigma_h) \perp (\sigma_h - \pi\rho, \operatorname{tr}(\sigma_h - \pi\rho))$$

from the hypothesis, where the orthogonality is in the W inner product (or $L^2(\Omega) \times L^2(\Gamma)$). By the Pythagorean theorem, we have

$$\|(\sigma_h, \operatorname{tr}\sigma_h)\|_{L^2(\Omega)\times L^2(\Gamma)} \le \|(\pi\rho, \operatorname{tr}\pi\rho)\|_{L^2(\Omega)\times L^2(\Gamma)},$$

therefore,

$$\|\sigma_h\| + \|\operatorname{tr}\sigma_h\| \le C(\|\pi\rho\| + \|\operatorname{tr}\pi\rho\|_\Gamma) \tag{3.36}$$

We note that this estimate is very similart to (3.35). However, for $k = 2$, we do not have the sharp trace estimate $\|\operatorname{tr}\pi\rho\| \le \|\operatorname{tr}\rho\|$ as we did case $k = 1$. To work around, we break $\|\operatorname{tr}\pi\rho\|_\Gamma \le \|\pi\rho\|_\Gamma + \|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_\Gamma$, obtain

$$\|\sigma_h\| + \|\operatorname{tr}\sigma_h\| \le C(\|\pi\rho\| + \|\pi\rho\|_\Gamma + \|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_\Gamma), \tag{3.37}$$

and will apply a scaling argument on the last term. For the remainder of the proof, we are going to bound all three terms on the right-hand side in (3.37) by $\|d\sigma_h\|$.

For the first term $\|\pi\rho\|$, we scale between any T and a reference tetrahedron $\hat{T}$, and apply the trace theorem on $\hat{T}$, as follows.

$$\begin{aligned} \|\pi\rho\|_T^2 &\le Ch^3\|\hat{\pi}\hat{\rho}\|_{\hat{\rho}}^2 \\ &\le Ch^3\|\hat{\rho}\|_{H^1(\hat{T})}^2 \\ &\le Ch^3\left(Ch^{-3}\|\rho\|_T^2 + Ch^{-1}|\rho|_{H^1(T)}^2\right) \end{aligned}$$

Summing over all T, we have

$$\|\pi\rho\|^2 \leq C\left(\|\rho\|^2 + h^2|\rho|_{H^1}^2\right) \leq C\|\rho\|_{H^1}^2 \leq C\|d\sigma_h\|.$$

Next, for the second term $\|\operatorname{tr}\rho\|_\Gamma$, we directly apply the trace theorem in Ω to have

$$\|\operatorname{tr}\rho\|_\Gamma \leq C\|\rho\|_{H^1} \leq C\|d\sigma_h\|.$$

It remains to estimate the last term $\|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_\Gamma$. For each T, we first scale to $\hat{T}$, and apply the trace theorem there.

$$\|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_{\partial T}^2 \leq Ch^2\|\operatorname{tr}\hat{\rho} - \operatorname{tr}\hat{\pi}\hat{\rho}\|_{\partial\hat{T}}^2 \leq Ch^2\|\hat{\rho} - \hat{\pi}\hat{\rho}\|_{H^1(\hat{T})}^2.$$

In $\hat{T}$, we have

$$\|\hat{\rho} - \hat{\pi}\hat{\rho}\|_{H^1(\hat{T})}^2 \leq C|\hat{\rho}|_{H^1(\hat{T})}^2$$

by Bramble-Hilbert lemma. Scaling back to T, we obtain

$$|\hat{\rho}|_{H^1(\hat{T})}^2 \leq Ch^{-1}\|\rho|_{H^1(T)}^2.$$

Combining the last three inequalities, in each T, we have

$$\|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_{\partial T}^2 \leq Ch|\rho|_{H^1(T)}^2.$$

In particular, the last estimate is true for all T satisfying $T\bigcap\Gamma \neq \emptyset$. Summing over all such elements, noting that the union of all such ∂T is a superset of Γ, we thus obtain

$$\|\operatorname{tr}\rho - \operatorname{tr}\pi\rho\|_\Gamma^2 \leq Ch|\rho|_{H^1}^2 \leq Ch\|d\sigma_h\|^2.$$

Thus we have bounded all three terms on the right-hand side of (3.37) by $C\|d\sigma_h\|^2$. The case $k = 2$ is proved. Hence we established the lemma. □

3.4.4 Stability of the Discrete Problem

So far we have proven all hypotheses in Theorem 3.2.9 for our semi-natural Robin BVP. We note that in this case, the term $\inf_{v\in V_h^k}\|\mathrm{Proj}_{D(V^{k-1})}u - v\|$ is just $\|\mathrm{Proj}_{\mathfrak{H}_h^k}u\|$. Therefore, the general Theorem 3.2.9 in our particular case gives the following result.

Theorem 3.4.10. *There is $C > 0$, dependent on Ω and λ only, such that for (σ, u, p) the solution of Problem 2.5.3, and (σ_h, u_h, p_h) the solution of Problem 3.3.8, it holds* *that*

$$\|\sigma-\sigma_h\|_{\mathscr{H}\Lambda}+\|u-u_h\|_{H\Lambda}+\|p-p_h\| \\ \leq C\left(\inf_{\tau\in\Lambda_h^{k-1}}\|\sigma-\tau\|_{\mathscr{H}\Lambda}+\inf_{v\in\Lambda_h^k}\|u-v\|_{H\Lambda}+\inf_{q\in\mathfrak{H}_h^k}\|p-q\|+\|\mathrm{Proj}_{\mathfrak{H}_h^k}u\|\right). \quad (3.38)$$

Remark 3.4.11. *Similar to Theorem 7.4 in [3] that if the exact solution is smooth enough, then* $\|\mathrm{Proj}_{\mathfrak{H}_h^k}u\|=O\big(h^{2r}\big)$*. Thus we have*

$$\|\sigma-\sigma_h\|_{\mathscr{H}\Lambda}+\|u-u_h\|_{H\Lambda}+\|p-p_h\|=O(h^r).$$

3.5 Relation between the complexes and Robin boundary value problems

We have introduced two closed Hilbert complexes, and related them with the two types of Robin BVPs introduced in Chapter 2. As seen before, we took advantage of the FEEC framework in our analysis. The key points are to identify appropriate closed Hilbert complexes and suitable subcomplexes. Validation of various Poincaré inequality is an essential step in our work. We close this chapter by summarizing the interpretation as below.

	semi-essential BVPs			semi-natural BVPs		
i	$k-1$	k	$k+1$	$k-1$	k	$k+1$
W^i	$L^2\Lambda^{k-1}$	$L^2\Lambda^k$	$L^2\Lambda^{k-1}\times L^2\Lambda^k(\Gamma)$	$L^2\Lambda^{k-1}\times L^2\Lambda^{k-1}(\Gamma)$	$L^2\Lambda^k$	$L^2\Lambda^{k+1}$
V^i	$\mathring{H}\Lambda^{k-1}$	$\mathscr{H}\Lambda^k$	-	see (3.31)	$H\Lambda^k$	-
D^i	d^{k-1}	(d^k,tr)	-	$(u,\sigma)\mapsto d^{k-1}u$	d^k	-
V_i^*	-	$H^*\Lambda^k$	see (3.19)	-	$\mathscr{H}^*\Lambda^k$	$\mathring{H}^*\Lambda^{k+1}$
D_i^*	-	δ^k	$(u,\sigma)\mapsto\delta^{k+1}u$	-	see (3.33)	δ^{k+1}
$\mathfrak{H}^i$	-	$\mathring{\mathfrak{H}}^k$	-	-	$\mathfrak{H}^k$	-

Chapter 4

Maxwell's equations with impedance boundary conditions

4.1 Introduction

In this chapter, we develop numerical theory on Maxwell's equations with impedance boundary conditions. Maxwell's equations are a set of four differential equations that represent four physical laws, Gauss' laws for electricity and for magnetism, Faraday's law, and Ampere's law. Those equations provide a concise and elegant way to describe electromagnetism. Without understanding of the electromagnetic world that is governed by the equations, it is impossible to have applications as microwaves, computers, MRI scanners, electricity generator, etc.. Therefore, the study for Maxwell's equations is not just of mathematical interest, but also of great importance in physics and engineering.

Two types of boundary value conditions of Maxwell's equation are useful in practice. The electric boundary condition arises when the electromagnetic wave travels between some medium and a perfect conductor. If the medium is surrounded by a non-perfect conductor that allows the wave to penetrate a small distance, we will have the impedance boundary condition in the model. The mathematical expression of these boundary conditions will be given in the next section. As we will see in this chapter, the weak formulation for Maxwell's equation with the impedance boundary condition requires using the space $\mathscr{H}(\text{curl})$ that was defined and studied in the last two chapters. We will also apply the theories from previous chapters to analyze the original equation and its numerical scheme.

In the next section, we will first introduce Maxwell's equations and their boundary conditions in more detail. In particular, we are interested in time-harmonic electromagnetic waves. In such cases, we will derive complex-valued function representations for the electromagnetic fields, and all equations and boundary conditions will consequently become complex-valued. In Section 4.3, we will follow [26] to analyze the weak formulation of the time-harmonic Maxwell's equations. In the next section, we will prove stability result of the discrete method, and derive convergence of the method. We will skip some proofs in these two sections due to their lengths. We will give those proofs in Section 4.5.

4.2 Maxwell's equations and boundary value conditions

In an electromagnetic field, we denote the electric and magnetic field intensities by $\mathcal{E}$ and $\mathcal{H}$, and denote the electric displacement and magnetic induction by $\mathcal{D}$ and $\mathcal{B}$. All of these are real-valued vector fields in space and time. Maxwell's equations give relations among them, as below.

$$\begin{cases} \frac{\partial \mathcal{B}}{\partial t} + \operatorname{curl} \mathcal{E} & = 0, \\ \operatorname{div} \mathcal{D} & = \rho, \\ \frac{\partial \mathcal{D}}{\partial t} - \operatorname{curl} \mathcal{H} & = -\mathcal{J}, \\ \operatorname{div} \mathcal{B} & = 0. \end{cases} \tag{4.1}$$

Here the scalar field ρ is the charge density, and the vector field $\mathcal{J}$ is the current density. Besides, the four vector fields $\mathcal{E}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{B}$ are related in two pairs, as below.

$$\mathcal{D} = \epsilon \mathcal{E}, \quad \mathcal{B} = \mu \mathcal{H}. \tag{4.2}$$

Here ϵ and μ are called electric permittivity and magnetic permeability, respectively. They are positive semi-definite matrix-valued functions in general. However, in vacuum, they obtain constant values:

$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}, \quad \mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}\,.$$

We see from the Maxwell's equations that $\mathcal{E}$ and $\mathcal{H}$ are in $H(\mathrm{curl})$, and $\mathcal{B}$ and $\mathcal{D}$ are in $H(\mathrm{div})$. Thus we can view $\mathcal{E}$ and $\mathcal{H}$ as proxy fields of 1-forms, and $\mathcal{B}$ and $\mathcal{D}$ as proxy fields of 2-forms, respectively. Thus (4.2) can be viewed as $\mathcal{D} = \star_\epsilon \mathcal{E}$ and $\mathcal{H} = \star_{\mu^{-1}} \mathcal{B}$, where the weighted Hodge star $\star_\alpha$ is just the usual Hodge star with a matrix multiplication. For instance, $\star_\epsilon \mathcal{E} = \star(\epsilon E)$. In terms of computation, thanks to (4.2), we can eliminate two variables from our original Maxwell's equation. The most common choice is rewriting the system in terms of $\mathcal{E}$ and $\mathcal{B}$. This is quite natural as well. Compared to $\mathcal{D}$ and $\mathcal{H}$, the quantities $\mathcal{E}$ and $\mathcal{B}$ are physically easier to measure.

For a bounded media (domain $\Omega \in \mathbb{R}^3$), if $\mathbb{R}^3 \setminus \Omega$ is a perfect conductor, then we have the boundary conditions

$$\mathcal{E} \times n = 0, \quad \text{on } \Gamma.$$

If $\mathbb{R}^3 \setminus \Omega$ is an imperfect conductor, $\mathcal{E} \times n$ is not constantly zero. Instead, there is a positive *impedance function* λ, such that

$$\mathcal{H} \times n - \lambda\, n \times (\mathcal{E} \times n) = 0, \quad \text{on } \Gamma. \tag{4.3}$$

In this chapter, we will focus on the study of the Maxwell's problem with the impedance boundary condition (4.3). For simplicity, we will assume λ is a positive constant.

Next, we move to the particularly interesting case where all vector fields are at a single frequency ω. That is, there are time-independent complex-valued vector fields $E, B, D, H : \Omega \to \mathbb{C}$:

$$\mathcal{E}(x,t) = \mathbf{Re}(e^{-i\omega t} E(x)), \quad \mathcal{B}(x,t) = \mathbf{Re}(e^{-i\omega t} B(x))$$
$$\mathcal{D}(x,t) = \mathbf{Re}(e^{-i\omega t} D(x)), \quad \mathcal{H}(x,t) = \mathbf{Re}(e^{-i\omega t} H(x))$$

This setting brings computational simplification. The time derivatives $\partial/\partial t$ of those electromagnetic fields are now equivalent to multiplying those time-independent fields by $-i\omega$. For instance,

$$\frac{\partial \mathcal{E}(x,t)}{\partial t} = \mathbf{Re}(-i\omega e^{-i\omega t} E(x)).$$

With this property, rewriting the time derivatives of $\mathcal{B}$ and $\mathcal{D}$ in (4.1), applying (4.2), we derive

$$-i\omega B + \operatorname{curl} E = 0, \tag{4.4}$$
$$-i\omega(\epsilon E) - \operatorname{curl}(\mu^{-1} B) = -J.$$

Multiplying the second equation by $-i\omega$ and using the first equation, we thus obtain the following second-order Maxwell's equation

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} E) - \omega^2 \epsilon E = i\omega J. \tag{4.5}$$

Meanwhile, by (4.4) and (4.2), we have $i\omega\mu H = \operatorname{curl} E$, thus the impedance boundary condition (4.3) implies

$$\operatorname{curl} E \times n - i\,\lambda\omega\mu n \times (E \times n) = 0 \quad \text{on } \Gamma. \tag{4.6}$$

We will analyze the mixed form this this Maxwell's problem (4.5)(4.6) and its discretization in the next section.

In time-harmonic case, all the functions are complex-valued. Thus we need to deal with sesquilinear forms. For a complex Hilbert space H, we say $a : H \times H \to \mathbb{C}$ is a sesquilinear form, if

$$\begin{aligned} a(\lambda_1 u_1 + \lambda_2 u_2, v) &= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v), \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}, \forall u_1, u_2, v \in H, \\ a(u, \lambda_1 v_1 + \lambda_2 v_2) &= \bar{\lambda}_1 a(u, v_1) + \bar{\lambda}_2 a(u, v_2), \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}, \forall u, v_1, v_2 \in H. \end{aligned}$$

We also adopt the complex inner product $\langle \cdot, \cdot \rangle$ and the weighted inner product $\langle f, g \rangle_\alpha$ for any real symmetric matrix α in this section. For $f, g : H \to \mathbb{C}^3$, We denote $\langle f, g \rangle = \int_\Omega f \cdot \bar{g}$, and $\langle f, g \rangle_\alpha = \langle \alpha f, g \rangle$.

As convention, we say a sesquilinear form $a : H \times H \to \mathbb{C}$ is *bounded*, if there exists $M > 0$ such that

$$|a(u, v)| \leq M \|u\| \|v\| \quad \forall u, v \in H. \tag{4.7}$$

We say a is *coercive*, if there exists $\delta > 0$, such that

$$\|a(u, u)\| \geq \delta \|u\|^2 \quad \forall u \in H. \tag{4.8}$$

Besides, our analysis in this Chapter relies on some assumptions on the parameters.

Assumption 4.2.1. *The parameter functions λ, ϵ, and μ are positive constants*[1]*.*

Assumption 4.2.2. *We assume that ω^2 is not an eigenvalue of the* curl curl *operator with electric boundary condition. In other words, the problem*

$$\operatorname{curl} \mu^{-1} \operatorname{curl} E = \omega^2 \epsilon E \quad \textit{in } \Omega, \quad E \times n = 0 \quad \textit{on } \Gamma, \tag{4.9}$$

or its equivalent mixed form

$$\langle \operatorname{curl} E, \operatorname{curl} F \rangle_{\mu^{-1}} - \omega^2 \langle E, F \rangle_\epsilon = 0, \quad \forall F \in \mathring{H}(\operatorname{curl})$$

has only the trivial solution $E = 0$ in $\mathring{H}(\operatorname{curl})$.

[1]We assume ϵ and μ to be simplicity. It would not be hard to extend our analysis to more general cases, such as ϵ and μ being symmetric positive definite.

4.3 Analysis of the time harmonic Maxwell problem with impedance boundary condition

In this section, we analyze the numerical solutions to the time-harmonic Maxwell's problem with impedance boundary conditions. We first present the weak formulation. Taking any test function $F \in \mathscr{H}(\mathrm{curl})$, multiplying it with (4.5), integrating by parts, and applying the boundary condition (4.6), we derive the following.

Problem 4.3.1. *For any given $J \in L^2(\Omega, \mathbb{C}^3)$, find $E \in \mathscr{H}(\mathrm{curl})$ that satisfies*

$$\langle \mathrm{curl}\, E, \mathrm{curl}\, F\rangle_{\mu^{-1}} - i\,\lambda\omega\langle E \times n, F \times n\rangle_{\mu,\Gamma} - \omega^2\langle E, F\rangle_{\epsilon} = \langle i\omega J, F\rangle, \quad \forall F \in \mathscr{H}(\mathrm{curl}). \tag{4.10}$$

We will follow [26] to prove the following result.

Theorem 4.3.2. *For any given $J \in L^2(\Omega, \mathbb{C}^3)$, there is a unique solution $E \in \mathscr{H}(\mathrm{curl})$ that solves (4.10).*

The left-hand side of (4.10) can be viewed as a sesquilinear form on $\mathscr{H}(\mathrm{curl}) \times \mathscr{H}(\mathrm{curl})$. Unfortunately, it is not coercive, thus in general, we cannot apply the well-known Lax-Milgram theorem directly on an equation involving a. However, we can consider the following slightly different sesquilinear form

$$a(E, F) = \langle \mathrm{curl}\, E, \mathrm{curl}\, F\rangle_{\mu^{-1}} + \omega^2\langle E, F\rangle_{\epsilon} - i\,\lambda\omega\langle E \times n, F \times n\rangle_{\mu,\Gamma} \tag{4.11}$$

for $(E, F) \in \mathscr{H}(\mathrm{curl}) \times \mathscr{H}(\mathrm{curl})$. Then our problem (4.10) can be equivalently written as

$$a(E, F) - 2\omega^2\langle \epsilon E, F\rangle = i\omega\langle J, F\rangle, \tag{4.12}$$

Because ϵ and μ are assumed to be positive definite, their eigenvalues are positive, bounded and away from zero. Hence we have the following proposition.

Proposition 4.3.3. *The sesquilinear form a given by (4.11) is bounded and coercive.*

Now we briefly sketch how we are going to prove Theorem 4.3.2. The nontrivial part is for the case that $J \in \mathring{\mathfrak{Z}}^{\perp}$ (cf. Theorem 4.3.8). To analyze that particular case, we will establish some operator K in Lemma 4.3.5 to convert (4.12) into the equivalent problem of finding E satisfying

$$(I + K)E = -\frac{i}{2\omega}KJ.$$

In other words, our goal will be to show $I + K$ is invertible. To achieve that, we also establish Lemma 4.3.6 showing this K is in fact compact. Thus we can apply Fredholm alternative, and just need to check $(I+K)E = 0$ admits the trivial solution only, which will not be difficult (cf. Lemma 4.3.7).

We start with proving the following form of Lax-Milgram theorem.

Theorem 4.3.4. *Assume H is a complex Hilbert space, and the sesquilinear form $a : H \times H \to \mathbb{C}$ is bounded and coercive, with constants M and δ as in (4.7) and (4.8). There exists a unique linear operator $K : \mathscr{L}(H, \mathbb{C}) \to H$ such that for any bounded linear functional $f : H \to \mathbb{C}$,*

$$a(x, Kf) = f(x), \quad \forall x \in H. \tag{4.13}$$

Moreover, K is continuous, $\|K\| \leq 1/\delta$.

The proof is mostly applying some fundamental results in functional analysis. We will present it in Section 4.5.1 due to its length.

In the next three lemmas, we are going to introduce an operator $K : L^2(\Omega, \mathbb{C}^3) \to \mathscr{H}(\text{curl})$ and validate some desired properties, as planned earlier.

Lemma 4.3.5. *For the sesquilinear form a defined by 4.11, there exists a continuous linear operator $K : L^2(\Omega, \mathbb{C}^3) \to \mathscr{H}(\text{curl})$, such that for any $f \in L^2(\Omega, \mathbb{C}^3)$, Kf satisfies*

$$a(Kf, F) = -2\omega^2 \langle \epsilon f, F \rangle, \quad \forall F \in \mathscr{H}(\text{curl}). \tag{4.14}$$

Proof. For any $f \in L^2(\Omega, \mathbb{C}^3)$, we have a linear functional

$$g_f : F \mapsto -2\omega^2 \langle F, \epsilon f \rangle.$$

It is bounded:

$$\|g_f\| = \sup_F \frac{2\omega^2 |\langle F, \epsilon f \rangle|}{\|F\|_{\mathscr{H}(\text{curl})}} \leq \sup_F \frac{2\omega^2 \|\epsilon f\| \|F\|}{\|F\|_{\mathscr{H}(\text{curl})}} \leq 2\omega^2 \epsilon \|f\|.$$

Thus, we can define $K : f \mapsto K_1 g_f$, where $K_1 \in \mathscr{L}(\mathscr{H}(\text{curl}), \mathbb{C}) \to \mathscr{H}(\text{curl})$ is the operator from Theorem 4.3.4. We have

$$a(F, Kf) = a(F, K_1 g_f) = g_f(F) = -2\omega^2 \langle F, \epsilon f \rangle,$$

which is equivalent to (4.14). The boundedness of K follows from that of K_1 and g_f immediately. The lemma is thus proven. $\square$

We also need a compactness property. We have the decomposition

$$L^2(\Omega, \mathbb{C}^3) = \mathring{\mathfrak{Z}} \oplus \mathring{\mathfrak{Z}}^\perp,$$

where the orthogonality is in the ϵ-weighted sense. In the next lemma, we show that restricted on $\mathring{\mathfrak{Z}}^\perp$, K is compact into $(L^2)^3$.

Lemma 4.3.6. *Let K be the operator described in Lemma 4.3.5. The operator $K|_{\mathring{\mathfrak{Z}}^\perp}$ maps $\mathring{\mathfrak{Z}}^\perp$ into $\mathring{\mathfrak{Z}}^\perp$. Moreover, it is compact in $\mathring{\mathfrak{Z}}^\perp$.*

Proof. We take a bounded (with respect to L^2-norm) sequence f_n in $\mathring{\mathfrak{Z}}^\perp$, and want to show that $Kf_n \in \mathring{\mathfrak{Z}}^\perp$, and has a convergent subsequence.

In (4.14), we take any test function $F \in \mathring{\mathfrak{Z}}$. Then we have orthogonality $\langle f_n, F\rangle_\epsilon = 0$. Therefore, we have $a(Kf_n, F) = 0$ for the operator K given by Lemma 4.3.5. That is

$$\omega^2\langle \epsilon K f_n, F\rangle + \langle \mu^{-1} \operatorname{curl} K f_n, \operatorname{curl} F\rangle - i\lambda\omega\mu\langle K f_n \times n, F \times n\rangle_\Gamma = 0$$

by the definition (4.11). Because our F is trace-free and curl-free. The above equation is simplified into $\langle Kf_n, F\rangle_\epsilon = 0$. Therefore, we have $Kf_n \in \mathring{\mathfrak{Z}}^\perp$.

A consequence of $Kf_n \in \mathring{\mathfrak{Z}}^\perp$ is that Kf_n is div-free, as we know that $\mathring{\mathfrak{Z}}^\perp$ for the *curl* operator is just the range of its adjoint, curl(H(curl)). Moreover, by Lemma 4.3.5, $\|Kf_n\|_{\mathscr{H}(\mathrm{curl})}$ is bounded. Hence Kf_n is a bounded sequence in $\mathscr{H}(\mathrm{curl}) \cap H(\mathrm{div})$. By Lemma 3.3.3, the space $\mathscr{H}(\mathrm{curl}) \cap H(\mathrm{div})$ is compactly embedded in the L^2 vector space. Thus Kf_n has a convergent subsequence. Therefore, $K|_{\mathring{\mathfrak{Z}}^\perp}$ is a compact operator in $\mathring{\mathfrak{Z}}^\perp$. □

Lemma 4.3.7. *For any $f \in \mathring{\mathfrak{Z}}^\perp$, the equation*

$$(I + K|_{\mathring{\mathfrak{Z}}^\perp})E = f$$

has a unique solution E in $\mathring{\mathfrak{Z}}^\perp$.

Proof. We just validated the compactness of $K|_{\mathring{\mathfrak{Z}}^\perp}$ in Lemma 4.3.6. Thus by the Fredholm alternative theorem, it suffices to show that the homogeneous equation

$$(I + K|_{\mathring{\mathfrak{Z}}^\perp})E = 0$$

admits the trivial solution only. Let E be a solution, then we have $K|_{\mathring{\mathfrak{Z}}^\perp}E \in \mathscr{H}(\mathrm{curl})$ by Lemma 4.3.5. Thus $E = -K|_{\mathring{\mathfrak{Z}}^\perp}E$ is also in $\mathscr{H}(\mathrm{curl})$. Consequently, for any test $F \in \mathscr{H}(\mathrm{curl})$, one may write

$$a(E, F) + a(KE, F) = a((I + K|_{\mathring{\mathfrak{Z}}^\perp})E, F) = 0.$$

Applying (4.11) and (4.14), we obtain

$$-\omega^2\langle \epsilon E, F\rangle + \langle \mu^{-1}\operatorname{curl} E, \operatorname{curl} F\rangle - i\lambda\omega\mu\langle E\times n, F\times n\rangle_\Gamma = 0.$$

Noting that all parameters are real, taking the imaginary part of the last equation, we can derive $E \times n = 0$. In other words, $E \in \mathring{H}(\text{curl})$. Furthermore, if we restrict the test function $F \in \mathring{H}(\text{curl}) \subset \mathscr{H}(\text{curl})$, the last equation implies

$$\langle \mu^{-1}\operatorname{curl} E, \operatorname{curl} F\rangle = \omega^2\langle \epsilon E, F\rangle, \quad \forall F \in \mathring{H}(\text{curl}). \tag{4.15}$$

We assumed that ω^2 is not an eigenvalue for time harmonic Maxwell problem with electric boundary condition. Therefore, the last equation yields $E = 0$, which completes the proof. □

Now we can state and prove the well-posedness for the special case that $J \in \mathring{\mathfrak{Z}}^\perp$.

Theorem 4.3.8. *For any given $J \in \mathring{\mathfrak{Z}}^\perp$, there is a unique solution $E \in \mathring{\mathfrak{Z}}^\perp$ that solves (4.10).*

Proof. By Theorem 4.3.4, for any such given J, we have $KJ \in \mathscr{H}(\text{curl})$ that satisfies

$$a(KJ, F) = -2\omega^2\langle \epsilon J, F\rangle, \quad \forall F \in \mathscr{H}(\text{curl}).$$

Let $u = -\frac{i}{2\omega}K(\epsilon^{-1}J)$, we can verify that

$$a(u, F) = i\omega\langle J, F\rangle, \quad \forall F \in \mathscr{H}(\text{curl}). \tag{4.16}$$

Moreover, by Lemma 4.3.6, $u \in \mathring{\mathfrak{Z}}^\perp$, as $J \in \mathring{\mathfrak{Z}}^\perp$ by the hypothesis. Thus, we can apply Lemma 4.3.7 on data u to find $E \in \mathring{\mathfrak{Z}}^\perp$ that solves

$$(I + K)E = u. \tag{4.17}$$

From (4.16)(4.17) and Lemma 4.3.5, we can check that for the given data $J \in \mathring{\mathfrak{Z}}^\perp$, our E satisfies

$$a(E, F) = a(u - KE, F) = a(u, F) - a(KE, F) = i\omega\langle J, F\rangle + 2\omega^2\langle E, F\rangle_\epsilon$$

which is equivalent to (4.10). Existence is thus shown.

It remains to show uniqueness of the solution to (4.10). It suffices to show that the homogeneous problem of (4.10) only admits the trivial solution. Assume E is such a solution. We have

$$\langle \operatorname{curl} E, \operatorname{curl} F\rangle_{\mu^{-1}} - i\,\lambda\omega\langle E\times n, F\times n\rangle_{\mu,\Gamma} - \omega^2\langle E, F\rangle_\epsilon = 0, \quad \forall F\in \mathscr{H}(\operatorname{curl}).$$

Similar to the argument in last lemma, we separate real and imaginary parts of the last equation, and derive $E\in \mathring{H}(\operatorname{curl})$. Moreover, we can verify (4.15). By the non-eigenvalue assumption on ω^2. We know that $E=0$. Thus we have uniqueness of the equation. □

The general case follows immediately.

Proof of Theorem 4.3.2. By the Hodge decomposition, any such J can be decomposed as $J = J_0 + J_\perp \in (\operatorname{grad}\mathring{H}^1 \oplus \mathring{\mathfrak{H}}^1)\oplus \mathring{\mathfrak{Z}}^\perp$. Consider (4.10) with data $J_\perp$ in the place of J. The last theorem says we have a solution, which we denote by $E_\perp$, to this problem. Now it is easy to check that $E = E_\perp - \frac{iJ_0}{\omega\epsilon}$ solves (4.10) (with original data J on the right-hand side). Hence we have existence. Uniqueness is valid due to the same argument as in the last theorem. □

4.4 Discretization and analysis

Now that we have well-posedness of Problem (4.10) from Theorem 4.3.2, we can then consider the discrete problem. We denote Λ_h^1 any finite dimensional subspace of $\mathscr{H}(\operatorname{curl})$.

Problem 4.4.1. *For any given $J\in L^2(\Omega,\mathbb{C}^3)$, find $E_h\in\Lambda_h^1$ that satisfies*

$$\langle \operatorname{curl} E_h, \operatorname{curl} F\rangle_{\mu^{-1}} - i\,\lambda\omega\langle E_h\times n, F\times n\rangle_{\mu,\Gamma} - \omega^2\langle E_h, F\rangle_\epsilon = \langle i\omega J, F\rangle, \quad \forall F\in\Lambda_h^1. \tag{4.18}$$

Two lemmas will be helpful to our error analysis. The first lemma is a regularity result due to Amrouche *et al.* [1, Proposition 3.7].

Lemma 4.4.2. *If $\Omega\subset\mathbb{R}^3$ is a Lipschitz polyhedron, then there exists $s>1/2$, such that for any $u\in H(\operatorname{curl})\cap\mathring{H}(\operatorname{div})$ or $u\in\mathring{H}(\operatorname{curl})\cap H(\operatorname{div})$, we have*

$$u\in H^s(\Omega), \quad \textit{and} \quad \|u\|_{H^s}\le C\left(\|u\| + \|\operatorname{curl} u\| + \|\operatorname{div} u\|\right).$$

The second lemma is an analogue of Lemma 5.9 in [3].

Lemma 4.4.3. *For any $p_h\in\mathring{\mathfrak{H}}_h^k$, its L^2-projection p onto $\mathring{\mathfrak{H}}^k$*

$$p_h\in\mathring{\mathfrak{H}}^k, \quad p - p_h\perp\mathring{\mathfrak{H}}^k$$

satisfies

$$\|p - p_h\| \leq \|(I - \Pi)p\|, \quad \|p\| \leq \|p_h\|, \tag{4.19}$$

where $\Pi : \mathring{H}\Lambda^k \to \mathring{\Lambda}_h^k$ *is the smooth projection given by [11, Section 6].*

Proof. Since we have

$$\operatorname{tr} p = \operatorname{tr} p_h = 0, \quad dp = dp_h = 0, \quad p - p_h \perp \mathring{\mathfrak{H}}^k,$$

there exists some $\sigma \in \mathring{H}\Lambda^{k-1}$ that satisfies $p - p_h = d\sigma$. We also note that

$$\langle p_h, d\tau \rangle = \langle p, d\tau \rangle = 0, \quad \forall \tau \in \mathring{\Lambda}_h^k \subset \mathring{H}\Lambda^k.$$

from (continuous and discrete) Hodge decompositions. In particular, for $\tau = \Pi\sigma$, we have $\langle p - p_h, d\Pi\sigma \rangle = 0$. Hence we have

$$\begin{aligned} \|p - p_h\|^2 &= \langle p_h - p, d\sigma \rangle = \langle p_h - p, d(\sigma - \Pi\sigma) \rangle = \langle p_h - p, (I - \Pi)d\sigma \rangle \\ &\leq \|p - p_h\| \|(I - \Pi)d\sigma\| = \|p - p_h\| \|(I - \Pi)(p_h - p)\| = \|p - p_h\| \|(I - \Pi)p\|. \end{aligned}$$

The first inequality in (4.19) follows. The other inequality is immediate from the definition of p. □

We also state a regularity result. We will need to apply the result to escalate regularity of our auxiliary equation when proving the main theorem. The statement is similar to a result due to Costabel *et al.* [16]. The major difference is that we weaken the assumption on smoothness of the boundary, and get $(1/2+\epsilon)$-regularity, rather than H^1-regularity in [16]. However, we will postpone its proof to next section due to its length.

Theorem 4.4.4. *Given any Lipschitz polyhedral domain* $\Omega \subset \mathbb{R}^3$*, there exists* $\epsilon > 0$*, such that the space*

$$V = \left\{ (E, H) \in [H(\operatorname{curl}) \cap H(\operatorname{div})]^2 \mid H \times n + \alpha n \times (E \times n) = 0 \right\}, \tag{4.20}$$

with α *being a nonzero constant, is continuously embedded in* $H^{1/2+\epsilon}(\Omega) \times H^{1/2+\epsilon}(\Omega)$.

We now state and prove our main result in this section, from which the stability of the Problem 4.4.1 follows. We have the following theorem.

Theorem 4.4.5. *Let* Ω *be a Lipschitz polyhedron domain. There exist constants* $s = s(\Omega) \in (0, 1/2)$ *and* $C_1, C_2 > 0$*, such that if* $E \in \mathscr{H}(\operatorname{curl})$ *solves (4.10) and* $E_h \in \Lambda_h^1$

solves (4.18) for some $J \in L^2(\Omega, \mathbb{C}^3)$, then there holds

$$\|E - E_h\|_{\mathscr{H}(\mathrm{curl})} \le \frac{C_1}{1 - C_2 h^s} \inf_{F \in \Lambda_h^1} \|E - F\|_{\mathscr{H}(\mathrm{curl})},$$

for h sufficiently small.

Proof. We denote $e = E - E_h$ the error. Our goal is to bound the $\mathscr{H}(\mathrm{curl})$-norm of e. We recall the definition

$$\|e\|^2_{\mathscr{H}(\mathrm{curl})} = \langle \mathrm{curl}\, e, \mathrm{curl}\, e\rangle_{\mu^{-1}} + \langle e \times n, e \times n\rangle_{\mu,\Gamma} + \langle e, e\rangle_\epsilon. \tag{4.21}$$

We also denote Q the $\mathscr{H}(\mathrm{curl})$-projection into Λ_h^1. There are two major steps in this proof. First, we will establish

$$\|e\|^2_{\mathscr{H}(\mathrm{curl})} \le C \left(\|e\|_{\mathscr{H}(\mathrm{curl})} \inf_{F \in \Lambda_h^1} \|E - F\|_{\mathscr{H}(\mathrm{curl})} + |\langle e, \mathrm{Q}\, e\rangle_\epsilon| \right) \tag{4.22}$$

Next, we will provide the following estimate for $|\langle e, \mathrm{Q}\, e\rangle_\epsilon|$

$$|\langle e, \mathrm{Q}\, e\rangle_\epsilon| \le C h^{s-1/2} \|e\|^2_{\mathscr{H}(\mathrm{curl})} \tag{4.23}$$

with some $1/2 < s < 1$ for h sufficiently small. Once both (4.22) and (4.23) are validated, it immediately follows that

$$\|e\|_{\mathscr{H}(\mathrm{curl})} \le C_1 \inf_{F \in \Lambda_h^1} \|E - F\|_{\mathscr{H}(\mathrm{curl})} + C_2 h^{s-1/2} \|e\|_{\mathscr{H}(\mathrm{curl})}.$$

where both C_1 and C_2 are independent of h. The theorem follows for h sufficiently small.

To validate (4.22), we start with an error equation. For any test function $F \in \Lambda_h^1$, subtracting (4.18) from (4.10), we obtain that

$$-\omega^2 \langle e, F\rangle_\epsilon + \langle \mathrm{curl}\, e, \mathrm{curl}\, F\rangle_{\mu^{-1}} - i\lambda\omega \langle e \times n, F \times n\rangle_{\mu,\Gamma} = 0, \quad \forall F \in \Lambda_h^1. \tag{4.24}$$

Subtracting the last equation from (4.21), we obtain

$$\begin{aligned} \|e\|^2_{\mathscr{H}(\mathrm{curl})} &= \langle \mathrm{curl}\, e, \mathrm{curl}(e - F)\rangle_{\mu^{-1}} + \langle e \times n, e \times n\rangle_{\mu,\Gamma} + \langle e, e\rangle_\epsilon \\ &\qquad + \omega^2 \epsilon \langle e, F\rangle_\epsilon + i\lambda\omega \langle e \times n, F \times n\rangle_{\mu,\Gamma} \\ &= \langle \mathrm{curl}\, e, \mathrm{curl}(e - F)\rangle_{\mu^{-1}} + \langle e \times n, e \times n\rangle_{\mu,\Gamma} + \langle e, e - F\rangle_\epsilon \\ &\qquad + (1 + \omega^2) \langle e, F\rangle_\epsilon + i\lambda\omega \langle e \times n, F \times n\rangle_{\mu,\Gamma}. \end{aligned} \tag{4.25}$$

For the two boundary integral terms in (4.25), we have

$$\begin{aligned}\langle e\times n, e\times n\rangle_{\mu,\Gamma} &+ i\lambda\omega\langle e\times n, F\times n\rangle_{\mu,\Gamma}\\ &= \langle e\times n,(e-F)\times n\rangle_{\mu,\Gamma} + (1+i\lambda\omega)\langle e\times n, F\times n\rangle_{\mu,\Gamma}.\end{aligned}$$

Substituting this into (4.25), we obtain

$$\begin{aligned}\|e\|^2_{\mathscr{H}(\mathrm{curl})} &= \langle \operatorname{curl} e, \operatorname{curl}(e-F)\rangle_{\mu^{-1}} + \langle e, e-F\rangle_\epsilon + (1+\omega^2)\langle e,F\rangle_\epsilon\\ &\quad + \langle e\times n,(e-F)\times n\rangle_{\mu,\Gamma} + (1+i\lambda\omega)\langle e\times n, F\times n\rangle_{\mu,\Gamma}. \quad (4.26)\end{aligned}$$

Again from the error equation (4.24), one finds that

$$\begin{aligned}\langle e\times n, F\times n\rangle_{\mu,\Gamma} &= \frac{\omega^2}{i\lambda\omega}\langle e,F\rangle_\epsilon + \frac{1}{i\lambda\omega}\langle \operatorname{curl} e, \operatorname{curl} F\rangle_{\mu^{-1}}\\ &= \frac{\omega^2}{i\lambda\omega}\langle e,F\rangle_\epsilon - \frac{1}{i\lambda\omega}\langle \operatorname{curl} e, \operatorname{curl}(e-F)\rangle_{\mu^{-1}} + \frac{1}{i\lambda\omega}\|\operatorname{curl} e\|^2_{\mu^{-1}}.\end{aligned}$$

Substituting the last equation in the last term of (4.26), we thus have

$$\begin{aligned}\|e\|^2_{\mathscr{H}(\mathrm{curl})} &= \langle \operatorname{curl} e, \operatorname{curl}(e-F)\rangle_{\mu^{-1}} + \langle e, e-F\rangle_\epsilon + (1+\omega^2)\langle e,F\rangle_\epsilon\\ &\quad + \langle e\times n,(e-F)\times n\rangle_{\mu,\Gamma} - \frac{(1+i\lambda\omega)\omega^2}{i\lambda\omega}\langle e,F\rangle_\epsilon\\ &\quad - \frac{1+i\lambda\omega}{i\lambda\omega}\langle \operatorname{curl} e, \operatorname{curl}(e-F)\rangle_{\mu^{-1}} + \frac{1+i\lambda\omega}{i\lambda\omega}\|\operatorname{curl} e\|^2_{\mu^{-1}}.\end{aligned}$$

Moving the term $-\frac{1+i\lambda\omega}{i\lambda\omega}\|\operatorname{curl} e\|^2_{\mu^{-1}}$ to the left, and re-arranging the other terms on the right, we derive

$$\begin{aligned}\|e\|^2_{\mathscr{H}(\mathrm{curl})} &- \left(\frac{1}{i\lambda\omega}+1\right)\|\operatorname{curl} e\|^2_{\mu^{-1}}\\ &= -\frac{1}{i\lambda\omega}\langle \operatorname{curl} e, \operatorname{curl}(e-F)\rangle_{\mu^{-1}} + \langle e, e-F\rangle_\epsilon\\ &\quad + \langle e\times n,(e-F)\times n\rangle_{\mu,\Gamma} + \left(-\frac{\omega^2}{i\lambda\omega}+1\right)\langle e,F\rangle_\epsilon. \quad (4.27)\end{aligned}$$

We take the magnitudes of both sides, and note that $i\lambda\omega$ is purely imaginary. Thus the magnitude of the left-hand side of (4.27) has the following lower bound

$$\begin{aligned}\left|\|e\|^2_{\mathscr{H}(\mathrm{curl})} - \left(\frac{1}{i\lambda\omega}+1\right)\|\operatorname{curl} e\|^2_{\mu^{-1}}\right| &= \left|\|e\|^2_\epsilon + \|e\times n\|^2_{\mu,\Gamma} - \frac{1}{i\lambda\omega}\|\operatorname{curl} e\|^2_{\mu^{-1}}\right|\\ &\geq \frac{1}{\sqrt{2}}\left(\|e\|^2_\epsilon + \|e\times n\|^2_{\mu,\Gamma} + \frac{1}{|\lambda\omega|}\|\operatorname{curl} e\|^2_{\mu^{-1}}\right)\\ &\geq c\|e\|_{\mathscr{H}(\mathrm{curl})},\end{aligned}$$

where we used $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$ and Assumption 4.2.1. For the right-hand side of (4.27), we apply the triangle inequality and Cauchy-Schwarz inequality to get

$$\|e\|^2_{\mathscr{H}(\mathrm{curl})} \leq C\left(\|e\|_{\mathscr{H}(\mathrm{curl})}\|e-F\|_{\mathscr{H}(\mathrm{curl})} + |\langle e, F\rangle_\epsilon|\right),$$

Now we choose a particular test function $F = \mathrm{Q}\, e$ in the last estimate, and use the fact that

$$\|e - \mathrm{Q}\, e\|_{\mathscr{H}(\mathrm{curl})} \leq C \inf_{G\in\Lambda_h^1} \|e - G\|_{\mathscr{H}(\mathrm{curl})}.$$

Thus (4.22) follows from the last three estimates.

Next we will prove (4.23). To begin with, we recall the Hodge decomposition for $\mathscr{H}(\mathrm{curl})$

$$\mathscr{H}(\mathrm{curl}) = \operatorname{grad} \mathring{H}^1 \oplus \mathring{\mathfrak{H}}^1 \oplus \mathring{\mathfrak{Z}}^{\perp_{\mathscr{H}(\mathrm{curl})}},$$

where $\mathring{\mathfrak{Z}}$ is the space of vector fields $u \in \mathscr{H}(\mathrm{curl})$ such that $\operatorname{curl} u = 0$ in Ω and $u \times n = 0$ on Γ. Thus for $e \in \mathscr{H}(\mathrm{curl})$, we can break it into two parts

$$e = \operatorname{grad}\rho + \psi, \quad \text{where } \rho \in \mathring{H}^1, \psi \perp \operatorname{grad}\mathring{H}^1 \tag{4.28}$$

Likewise, for $\mathrm{Q}\, e \in \Lambda_h^1$, we can apply a discrete Hodge decomposition

$$\Lambda_h^1 = \operatorname{grad}\Lambda_h^0 \oplus \mathring{\mathfrak{H}}_h^1 \oplus \mathring{\mathfrak{Z}}_h^\perp,$$

and write

$$\mathrm{Q}\, e = \operatorname{grad}\rho_h + \psi_h, \quad \text{where } \rho_h \in \mathring{\Lambda}_h^0, \psi_h \perp d\mathring{\Lambda}_h^0. \tag{4.29}$$

We know that $\langle e, \operatorname{grad}\rho_h\rangle_\epsilon = 0$ by taking $F = \operatorname{grad}\rho_h$ in (4.24). Consequently, we have

$$\langle e, \mathrm{Q}\, e\rangle_\epsilon = \langle e, \operatorname{grad}\rho_h + \psi_h\rangle_\epsilon = \langle e, \psi_h\rangle_\epsilon = \langle \operatorname{grad}\rho + \psi, \psi_h\rangle_\epsilon = \langle \operatorname{grad}\rho, \psi_h\rangle_\epsilon + \langle \psi, \psi_h\rangle_\epsilon. \tag{4.30}$$

We will analyze the last two terms to validate (4.23).

We first look at the term $\langle \operatorname{grad}\rho, \psi_h\rangle_\epsilon$. The discrete Hodge decomposition indicates that in (4.29) we can write further

$$\psi_h = e_{h,2} + e_{h,3}, \quad \text{where } e_{h,2} \in \mathring{\mathfrak{H}}_h^1,\ e_{h,3} \perp \operatorname{grad}\mathring{\Lambda}_h^0 \oplus \mathring{\mathfrak{H}}_h^1. \tag{4.31}$$

By Lemmas 4.4.3, We can find $w_2 \in \mathring{\mathfrak{H}}^1$, such that

$$\|w_2\| \leq \|e_{h,2}\|, \quad \|w_2 - e_{h,2}\| \leq \|(I-\Pi)w_2\|. \tag{4.32}$$

We note that the harmonic function w_2 is curl-free, div-free, and trace-free. Thus Lemma 4.4.2 gives that $w_2 \in H^s(\Omega)$, and $\|w_2\|_{H^s} \leq C\|w\|$. Hence

$$\|w_2 - e_{h,2}\| \leq \|(I-\Pi)w_2\| \leq Ch^s\|w_2\|_{H^s} \leq Ch^s\|w_2\| \leq Ch^s\|e_{h,2}\|.$$

For $e_{h,3}$, we let $w_3 = e_{h,3} - P_{\mathring{\mathfrak{Z}}}e_{h,3}$. Thus we can apply lemmas 3.3.13, 3.3.11, and 3.3.12, and use a scaling argument to obtain the following estimate.

$$\begin{aligned}
\|w_3 - e_{h,3}\| &\leq \|(I-\pi)w_3\| \quad \text{(by Lemma 3.3.13)}\\
&\leq Ch^s\|w_3\|_{H^s} \quad \text{(by a scaling argument)}\\
&\leq Ch^s(\|w_3\| + \|\operatorname{curl} w_3\| + \|w_3 \times n\|_{H^{s-1/2}(\Gamma)}) \quad \text{(by Lemma 3.3.12)}\\
&\leq Ch^{1/2}(\|w_3\| + \|\operatorname{curl} w_3\| + \|w_3 \times n\|_{\Gamma}) \quad \text{(by Lemma 3.3.11)}\\
&\leq Ch^{1/2}\left(\|\operatorname{curl} w_3\| + \|w_3 \times n\|_{\Gamma}\right) \quad \text{(by the Poincaré inequality (3.17))}\\
&\leq Ch^{1/2}\left(\|\operatorname{curl} e_{h,3}\| + \|e_{h,3} \times n\|_{\Gamma}\right) \quad \text{(by Lemma 3.3.13).}
\end{aligned}$$

Now that we have established estimates for $\|e_{h,2} - w_2\|$ and $\|e_{h,3} - w_3\|$, recalling the decompositions (4.28)(4.29), we thus have

$$\begin{aligned}
|\langle \operatorname{grad}\rho, \psi_h\rangle_\epsilon| &\leq |\langle \operatorname{grad}\rho, e_{h,2}\rangle_\epsilon| + |\langle \operatorname{grad}\rho, e_{h,3}\rangle_\epsilon|\\
&= |\langle \operatorname{grad}\rho, e_{h,2} - w_2\rangle_\epsilon| + |\langle \operatorname{grad}\rho, e_{h,3} - w_3\rangle_\epsilon| \quad \text{(by (4.28), (4.29), and (4.31))}\\
&\leq Ch^{1/2}\|\operatorname{grad}\rho\|\left(\|\operatorname{curl} e_{h,3}\| + \|e_{h,3} \times n\|_{\Gamma} + \|e_{h,2}\|\right) \qquad (4.33)
\end{aligned}$$

Again by (4.29), we can further bound the last three terms $\|\operatorname{curl} e_{h,3}\|$, $\|e_{h,3} \times n\|_{\Gamma}$, and $\|e_{h,2}\|$ by $\|\mathrm{Q}\, e\|_{\mathscr{H}(\mathrm{curl})}$. Thus (4.33) yields

$$|\langle \operatorname{grad}\rho, \psi_h\rangle_\epsilon| \leq Ch^{1/2}\|\operatorname{grad}\rho\|\|\mathrm{Q}\, e\|_{\mathscr{H}\Lambda} \tag{4.34}$$

We also know $\|\operatorname{grad}\rho\| \leq \|e\| \leq \|e\|_{\mathscr{H}\Lambda}$ by (4.28). Moreover, since Q is an $\mathscr{H}(\mathrm{curl})$ projection, we naturally have $\|\mathrm{Q}\, e\|_{\mathscr{H}\Lambda} \leq \|e\|_{\mathscr{H}\Lambda}$. Hence, (4.34) yields the following estimate:

$$|\langle \operatorname{grad}\rho, \psi_h\rangle_\epsilon| \leq Ch^{1/2}\|e\|^2_{\mathscr{H}\Lambda}. \tag{4.35}$$

Comparing (4.35) with our goal (4.23), we know it remains to analyze the term $|\langle \psi, \psi_h\rangle_\epsilon|$. We will fulfill this by giving a bound to $\|\psi\|$ via a duality argument. Let us consider this auxiliary problem: Find $z \in \mathscr{H}(\operatorname{curl})$ that solves

$$\langle \operatorname{curl} z, \operatorname{curl} \phi \rangle_{\mu^{-1}} - i\lambda(-\omega)\langle z \times n, \phi \times n \rangle_{\mu,\Gamma} - (-\omega)^2 \langle z, \phi \rangle_{\epsilon} = \langle \psi, \phi \rangle, \quad \forall \phi \in \mathscr{H}(\operatorname{curl}). \tag{4.36}$$

By theorem 4.3.2, we know such z is well-defined and satisfies

$$\|z\|_{\mathscr{H}(\operatorname{curl})} \le C\|\psi\|. \tag{4.37}$$

We need to show that z and $\operatorname{curl} z$ are regular enough to define πz, as we do in the following. Now for any $\phi \in \operatorname{grad} \mathring{H}^1$, we observe that z satisfies

$$\langle z, \phi \rangle = \frac{1}{\omega^2} \langle \psi, \phi \rangle = 0,$$

where the second equality is a result of the Hodge decomposition (4.28). So we have $\operatorname{div} z = 0$, which means $z \in H(\operatorname{curl}) \cap H(\operatorname{div}, 0)$. Noticing the strong form of (4.36), which reads

$$\operatorname{curl} \mu^{-1} \operatorname{curl} z - \omega^2 z = \psi \text{ in } \Omega, \quad \operatorname{curl} z \times n + i\lambda\omega\mu n \times (z \times n) = 0 \text{ on } \Gamma,$$

implies $\operatorname{curl} z \in H(\operatorname{curl})$, and hence $\operatorname{curl} z \in H(\operatorname{curl}) \cap H(\operatorname{div})$, we thus have

$$(z, \operatorname{curl} z) \in V, \text{ with } \alpha = i\lambda\omega\mu, \quad \text{as in (4.20).}$$

Therefore, we can apply Theorem 4.4.4 to conclude that z and $\operatorname{curl} z$ are both in $H^s(\Omega)$ for some $s > 1/2$. An immediate consequence is that the canonical projection πz is defined. Furthermore, we have the following H^s-estimates.

$$\begin{aligned} \|z\|_{H^s} &\le C(\|z\| + \|\operatorname{curl} z\| + \|\operatorname{div} z\|) \\ &= C(\|z\| + \|\operatorname{curl} z\|) \le C\|\psi\|, \end{aligned} \tag{4.38}$$

$$\begin{aligned} \|\operatorname{curl} z\|_{H^s} &\le C(\|\operatorname{curl} z\| + \|\operatorname{curl}\operatorname{curl} z\| + \|\operatorname{div}\operatorname{curl} z\|) \\ &= C\|\operatorname{curl} z\| + \|\mu\omega^2 z + \psi\|) \\ &\le (\|z\| + \|\operatorname{curl} z\| + \|\psi\|) \le C\|\psi\|. \end{aligned} \tag{4.39}$$

By [26, Theorem 5.41] and the trace theorem, we can show that

$$\|z - \pi z\|_{\mathscr{H}(\operatorname{curl})} \le Ch^{s-1/2}\|u\|_{H^s(\operatorname{curl})} \le Ch^{s-1/2}\|\psi\|. \tag{4.40}$$

Now we are in a position to give a bound to $\|\psi\|$. We choose the particular test function $\phi = \psi$ in (4.36), and get

$$\|\psi\|^2 = \langle \operatorname{curl} z, \operatorname{curl} \psi \rangle_{\mu^{-1}} - i\lambda\omega \langle z \times n, \psi \times n \rangle_{\mu,\Gamma} - \omega^2 \langle z, \psi \rangle_\epsilon.$$

Using the fact $\operatorname{div} z = 0$ and the decomposition (4.28), we know that

$$\langle z, \psi - e \rangle = 0, \quad \operatorname{curl} e = \operatorname{curl} \psi, \quad \text{and } e \times n = \psi \times n.$$

Hence

$$\|\psi\|^2 = \langle \operatorname{curl} e, \operatorname{curl} z \rangle_{\mu^{-1}} + i\lambda\omega \langle e \times n, z \times n \rangle_{\mu,\Gamma} - \omega^2 \langle e, z \rangle_\epsilon.$$

Besides, we choose $F = \pi z \in \Lambda_h^1$ in the error equation (4.24) to obtain

$$\langle \operatorname{curl} e, \operatorname{curl} \pi z \rangle_{\mu^{-1}} + i\lambda\omega\mu \langle e \times n, \pi z \times n \rangle_{\mu,\Gamma} - \omega^2 \langle e, \pi z \rangle_\epsilon = 0.$$

The last two equations yield

$$\|\psi\|^2 = \langle \operatorname{curl} e, \operatorname{curl}(z - \pi z) \rangle_{\mu^{-1}} - i\lambda\omega\mu \langle e \times n, (z - \pi z) \times n \rangle_{\mu,\Gamma} - \omega^2 \langle e, z - \pi z \rangle_\epsilon. \tag{4.41}$$

The error equation (4.41) and the estimate (4.40) together yield

$$\|\psi\| \le Ch^{s-1/2} \|e\|_{\mathscr{H}(\operatorname{curl})},$$

and as a result,

$$|\langle \psi, \psi_h \rangle| \le Ch^{s-1/2} \|e\|_{\mathscr{H}(\operatorname{curl})} \|\psi_h\| \le Ch^{s-1/2} \|e\|_{\mathscr{H}(\operatorname{curl})} \| \mathrm{Q}\, e \| \le Ch^{s-1/2} \|e\|^2_{\mathscr{H}(\operatorname{curl})}. \tag{4.42}$$

From (4.30), (4.35), and (4.42), we get the estimate (4.23).

Thus, both (4.22) and (4.23) are validated. As we pointed out in the beginning of the proof, the theorem follows for h sufficiently small. $\square$

Remark 4.4.6. *The constant $\frac{C_1}{1 - C_2 h^s}$ converges to C_1 as $h \to 0$. Thus the theorem actually gives*

$$\|E - E_h\|_{\mathscr{H}\Lambda} \le C \inf_{F \in \Lambda_h^1} \|E - F\|_{\mathscr{H}\Lambda}$$

for h sufficiently small.

As a corollary, we can prove the stability of our discrete problem, as below.

Theorem 4.4.7 (Stability of (4.18)). *The problem (4.18) has a unique solution for any given data.*

Proof. This is a finite dimensional problem, thus it suffices to show that the homogeneous problem

$$\langle \operatorname{curl} E_h, \operatorname{curl} F\rangle_{\mu^{-1}} + i\,\lambda\omega\langle E_h \times n, F \times n\rangle_{\mu,\Gamma} - \omega^2\langle E_h, F\rangle_\epsilon = 0, \quad \forall J \in \Lambda_h^1. \tag{4.43}$$

admits only the trivial solution. Suppose E_h solves (4.43). We knew that $E = 0$ is the only solution to the homogeneous continuous problem

$$\langle \operatorname{curl} E, \operatorname{curl} F\rangle_{\mu^{-1}} - i\,\lambda\omega\langle E \times n, F \times n\rangle_{\mu,\Gamma} - \omega^2\langle E, F\rangle_\epsilon = 0, \quad \forall F \in \mathscr{H}(\operatorname{curl}).$$

By Theorem 4.4.5, we have

$$\|E_h\|_{\mathscr{H}(\operatorname{curl})} \leq \frac{C_1}{1 - C_2 h^s} \inf_{F \in \Lambda^1} \|F\|_{\mathscr{H}(\operatorname{curl})} = 0,$$

and therefore $E_h = 0$. □

4.5 Skipped proofs

We will provide the skipped proofs of Theorem 4.3.2 and Theorem 4.4.4.

4.5.1 Proof of Theorem 4.3.2

In order to prove Theorem 4.3.2, we first establish the following result for bounded coercive sesquilinear forms.

Theorem 4.5.1. *Under the same assumption for H and a as in Theorem 4.3.4, there exists a unique bounded operator $A : H \to H$ such that*

$$a(x, y) = \langle x, Ay\rangle, \quad \forall x, y \in H. \tag{4.44}$$

Furthermore, A^{-1} exists, and is bounded by $1/\delta$.

Proof. For any fixed $y \in H$, we consider the map

$$a_y : H \to \mathbb{C}, \quad x \mapsto a(x, y).$$

By continuity of a, it is straightforward to check that a_y is continuous. Thus, by Riesz representation theorem, there exists a unique $z_y \in H$ such that

$$a(x, y) = \langle x, z_y\rangle, \quad \forall x \in H.$$

This fact gives us a well-defined map

$$A : H \to H, \quad y \mapsto z_y.$$

We will show that such A is a bounded linear operator, and bijective on H.

First, the linearity is not difficult to check. For any $y_1, y_2 \in H$, and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\langle x, Ay_1 + Ay_2\rangle &= \langle x, Ay_1\rangle + \langle x, Ay_2\rangle \\ &= a(x, y_1) + a(x, y_2) = a(x, y_1) + a(x, y_2) = a(x, y_1 + y_2), \quad \forall x \in H, \\ \langle x, \lambda y_1\rangle &= \bar{\lambda}\langle x, y_1\rangle = a(x, \lambda y_1), \quad \forall x \in H.\end{aligned}$$

These equations and the well-definedness of A yield the linearity. Besides, we can validate that A is bounded. In fact,

$$\|Ay\| = \sup_{x \in H \setminus \{0\}} \frac{|\langle x, Ay\rangle|}{\|x\|} = \sup_{x \in H \setminus \{0\}} \frac{|a(x, y)|}{\|x\|} \leq M\|y\|, \quad \forall y \in H.$$

The last inequality shows $\|A\| \leq M$.

Now that we have $A \in \mathscr{L}(H)$, we hope to prove A is bijective so that we can apply the inverse mapping theorem to complete the proof. The injectivity of A is rather straightforward. Assume $Ay = 0$ for some $y \in H$. We have $a(x, y) = 0$ for all $x \in H$ by the way that A is defined in. In particular, $a(y, y) = 0$. Applying the coercivity of a, we have $y = 0$, which implies that a is injective.

Next, we shall prove the surjectivity of A. To accomplish that, we will show $\operatorname{Im} A$, the range of A, is closed, and then show that $\operatorname{Im} A^{\perp}$ is trivial. We take any $z \in \overline{\operatorname{Im} A}$. There exist $y_n \in H$ such that $z = \lim_{n\to\infty} Ay_n$. Thus from coercivity of a, and the equation (4.44), we have,

$$\begin{aligned}\delta\|y_m - y_n\|^2 &\leq |a(y_m - y_n, y_m - y_n)| \\ &= |\langle y_m - y_n, A(y_m - y_n)\rangle| \leq \|y_m - y_n\|\|A(y_m - y_n)\|,\end{aligned}$$

for all m and n. This implies the sequence $\{y_n\}$ is Cauchy, because $\{Ay_n\}$ is convergent (and hence Cauchy). We denote $y = \lim_{n\to\infty} y_n$. By continuity of A, we know that $z = Ay$. Therefore, $\operatorname{Im} A$ is closed.

The other property we claimed, i.e., $\operatorname{Im} A^{\perp} = \{0\}$, is easy to check. Indeed, if $x \in \operatorname{Im} A^{\perp}$, there holds $a(x, y) = \langle x, Ay\rangle = 0$ for all $y \in H$ by (4.44). As before, we have $x = 0$ by taking $y = x$ and applying the coercivity of a. Consequently, we know A is surjective, and hence bijective.

Because A is bijective, by the inverse mapping theorem, we know A^{-1} is bounded.

In particular, in (4.44), we have $a(y, y) = \langle y, Ay\rangle$. The coercivity of a and the Cauchy-Schwarz inequality give $\delta\|y\|^2 \leq \|y\|\|Ay\|$. The last inequality implies $\|A^{-1}\| \leq 1/\delta$. This completes the proof. □

Now we can prove the theorem.

Proof of Theorem 4.3.2. The uniqueness is obvious from coercivity of a. We will prove existence and show continuous dependence.

Given any such $f \in \mathscr{L}(H)$, by Riesz representation theorem, one can find a unique $z_f \in H$ satisfying

$$\begin{aligned} f(x) &= \langle x, z_f\rangle, \quad \forall x \in H \\ \|z_f\| &= \|f\|. \end{aligned}$$

By theorem 4.5.1, we have $L \in \mathscr{L}(H)$ such that

$$\begin{aligned} a(x, Lz_f) &= \langle x, z_f\rangle, \quad \forall x \in H, \\ \|L\| &\leq 1/\delta. \end{aligned}$$

Thus we see that the operator $K : f \mapsto Lz_f$ satisfies $a(x, Kf) = f(x)$. It is obvious that this K is linear. Furthermore,

$$\|Kf\| = \|Lz_f\| \leq \|z_f\|/\delta = \|f\|/\delta.$$

Hence the theorem is proven. □

4.5.2 Proof of Theorem 4.4.4

Next, we follow [16] to prove Theorem (4.4.4). In order to do so, we need to recall and establish some lemmas. To begin with, we define $H^s(\Gamma)$ for $s > 1$, since our domain Ω is assumed to be a Lipschitz polyhedron.

Definition 4.5.1. *For a polyhedral domain Ω with Lipschitz boundary Γ, and $s > 1$,*

$$H^s(\Gamma) = \left\{u \in H^1(\Gamma) \mid u|_{\Gamma_i} \in H^s(\Gamma_i),\, \forall i\right\},$$

where Γ_i are the flat faces of Γ.

The first lemma is [25, Corollary 5.5.2] (and hence we omit the proof here). It gives a regularity result for the (scalar) Laplace equation with non-homogeneous boundary condition.

Lemma 4.5.2. *For any Lipschitz polyhedral domain Ω, there exists $s_\Omega \in (0, 1/2)$, depending on Ω only, such that for any ϕ that satisfies*

$$\begin{cases} \Delta\phi = f & \text{in } \Omega, \\ \phi|_\Gamma = g & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in H^s(\Gamma)$ for some $1 < s < 1 + s_\Omega$, we have $\phi \in H^{1/2+s}(\Omega)$. Moreover, the following holds

$$\|\phi\| \leq C(\|f\|_\Omega + \|g\|_{H^s(\Gamma)}).$$

The Laplace-Beltrami operator also admits some regularity (may be less than 2), as described in the next lemma due to Buffa *et al.* [7, Theorem 8].

Lemma 4.5.3. *Assume Γ is the boundary of a Lipschitz polyhedral domain, and ϕ satisfies*

$$\Delta_\Gamma \phi \in H^s(\Gamma), \quad s > -1.$$

Then there exists some $0 < t < s$ such that $\phi \in H^{1+t}(\Gamma)$.

We are now ready to give a proof to the main theorem of this section.

Proof of Theorem 4.4.4. We observe that it suffices to show that $E \in H^{1/2+\epsilon}$, because the boundary condition in the definition (4.20) of V can be written as

$$E \times n + \left(-\frac{1}{\alpha}\right) n \times (H \times n) = 0.$$

First, by Lemma 3.4.3, we can find $w \in (H^1)^3$ and $\rho \in H^1$ that satisfies

$$E = w + \operatorname{grad} \rho. \tag{4.45}$$

By the assumption that $E \in H(\mathrm{div})$ and the property $w \in H^1(\Omega; \mathbb{R}^3)$, we take the divergence of (4.45) and derive

$$\Delta\rho = -\operatorname{div} \operatorname{grad} \rho \in L^2(\Omega) \tag{4.46}$$

We next derive the regularity of $\rho|_\Gamma$ from (4.45). Thanks to [8], we have $\operatorname{div}_\Gamma(H \times n) \in H^{-1/2}(\Gamma)$. Applying the boundary condition in (4.20), we thus have

$$\operatorname{div}_\Gamma(n \times E \times n) \in H^{-1/2}(\Gamma). \tag{4.47}$$

Because $w \in (H^1)^3$, we know $n \times w \times n \in H^{1/2}_{\tan}(\Gamma)$, and further that

$$\operatorname{div}_\Gamma(n \times w \times n) \in H^{-1/2}(\Gamma). \tag{4.48}$$

From (4.45), (4.47), and (4.48), we obtain that

$$\operatorname{div}_\Gamma(n \times \operatorname{grad}\rho \times n) \in H^{-1/2}(\Gamma),$$

and hence

$$\Delta_\Gamma \rho|_\Gamma = -\operatorname{div}_\Gamma \operatorname{grad}_\Gamma \rho|_\Gamma = -\operatorname{div}_\Gamma(n \times (\operatorname{grad}\rho \times n)) \in H^{-1/2}(\Gamma). \tag{4.49}$$

Thus by Lemma 4.5.3, we know there exists $s_\Gamma > 0$, depending on the shape of Γ (or Ω) only, such that

$$\rho|_\Gamma \in H^{1+s_\Gamma}(\Gamma). \tag{4.50}$$

Finally, by Lemma 4.5.2, (4.46), and (4.50), we set $\rho \in H^{3/2+\epsilon}(\Omega)$ for some $\epsilon > 0$, and so $E \in H^{1/2+\epsilon}(\Omega)$ by (4.45). $\square$

Chapter 5

Numerical Examples

In the last chapter, we provide numerical solutions to several problems to verify our theory in the thesis. We use FEniCS [24], an open-source project specialized in solving PDEs using finite element methods, and its Python interface Dolfin 2019.1.0, to implement our methods.

In this chapter, we write CG for the Lagrange element, and N1Curl, N2Div, etc., for Nédélec elements. A superscript 0 denotes imposing essential boundary conditions on the element. Moreover, we put the degree r of our elements after its name. For instance, $(\mathrm{N2Curl}^0, 1)$ stands for the second kind of Nédélec edge element of the lowest order with vanishing boundary traces.

5.1 Vector Laplacian

In this section we will consider the two Robin BVPs in $\Omega = [0, \pi]^3$. We recall the general definition of the domain of Hodge-Laplacian (2.7), and rewrite it for the special case for vector fields in three dimensions:

$$D(\Delta) = \left\{u \in H(\mathrm{curl}) \cap H(\mathrm{div}) \mid \mathrm{curl}\, u \in H(\mathrm{curl}), \mathrm{div}\, u \in H^1\right\}.$$

Now we state our problems in strong form.

Problem 5.1.1. *For $f \in L^2(\Omega, \mathbb{R}^3)$ and $g \in L^2(\Gamma)$, find $u \in D(\Delta) \bigcap \mathscr{H}(\mathrm{div})$ such that*

$$(\mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div})u = f \quad \text{in } \Omega, \tag{5.1}$$

$$(\mathrm{curl}\, u) \times n = 0, \quad \mathrm{div}\, u + u \cdot n = g, \quad \text{on } \Gamma. \tag{5.2}$$

Problem 5.1.2. *For $f \in L^2(\Omega, \mathbb{R}^3)$ and $g \in L^2_{tan}(\Gamma)$, find $u \in D(\Delta) \bigcap \mathscr{H}(\mathrm{curl})$ such* *that*

$$(\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f \quad \textit{in } \Omega, \tag{5.3}$$

$$\operatorname{div} u = 0, \quad \operatorname{curl} u + n \times u \times n = g, \quad \textit{on } \Gamma. \tag{5.4}$$

As we have seen in Section 2.5, for each problem, we can view u as the proxy of either a 1-form or a 2-form. Correspondingly, we can set $\sigma = \operatorname{div} u$ or $\sigma = \operatorname{curl} u$, and obtain the mixed formulation. Note that since Ω is a cube, we have trivial harmonic function spaces. For each of the four discrete problems, we will first present the mixed formulation, then we will mention the finite elements we use, and show the computational rates of convergence in a table. We also append the tables for each element to show errors of L^2 norms of $u - u_h$, $\operatorname{curl}(u - u_h)$ (when applicable), $\operatorname{div}(u - u_h)$ (when applicable), etc.

5.1.1 A 1-form semi-natural Robin BVP

In Problem 5.1.1, we choose the manufactured soution $u = (x(x-\pi)\cos z, 0, 0)$. Viewing u as a 1-form, and setting $\sigma = -\operatorname{div} u = (-2x + \pi)\cos z$, our goal is to solve

$$\begin{aligned} -\langle \sigma, \tau \rangle - \langle \sigma, \tau \rangle_\Gamma + \langle u, \operatorname{grad} \tau \rangle &= \langle g, \tau \rangle_\Gamma \qquad \forall \tau \in V_h^0, \\ \langle \operatorname{grad} \sigma, v \rangle + \langle \operatorname{curl} u, \operatorname{curl} v \rangle &= \langle f, v \rangle \qquad \forall v \in V_h^1. \end{aligned} \tag{5.5}$$

We choose our finite element to be $(\mathrm{CG}, 1; \mathrm{N1Curl}, 1)$ and $(\mathrm{CG}, 2; \mathrm{N2Curl}, 1)$, we see (cf. Table 5.1) that our numerical solution converges with rate 1.

Elements	Rates of Convergence	Numerical Results
$(\mathrm{CG}, 1) \times (\mathrm{N1Curl}, 1)$	1	cf. Table 5.5
$(\mathrm{CG}, 2) \times (\mathrm{N2Curl}, 1)$	2	cf. Table 5.6

Table 5.1: Convergence rates using finite elements to solve Problem 5.1.1 as a 1-form semi-natural Robin BVP.

5.1.2 A 2-form semi-essential Robin BVP

We still consider Problem 5.1.1 with the same manufactured solution. This time we view u as the proxy of a 2-form, and let $\sigma = \operatorname{curl} u = (0, -x(x - pi)\sin z, 0)$. Our mixed formulation is the semi-essential one:

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, \operatorname{curl} \tau \rangle = 0 \quad \forall \tau \in V_h^1, \\ \langle \operatorname{curl} \sigma, v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle + \langle u \cdot n, v \cdot n \rangle_\Gamma = \langle f, v \rangle + \langle g, v \cdot n \rangle \quad \forall v \in V_h^2. \end{aligned} \tag{5.6}$$

We have four sets of elements. The corresponding rates of convergence are 1, given in the following table.

Elements	Rates of Convergence	Numerical Results
$(\mathrm{N1Curl}^0, 1) \times (\mathrm{N1Div}, 1)$	1	cf. Table 5.7
$(\mathrm{N2Curl}^0, 1) \times (\mathrm{N1Div}, 1)$	1	cf. Table 5.8
$(\mathrm{N1Curl}^0, 2) \times (\mathrm{N2Div}, 1)$	1	cf. Table 5.9
$(\mathrm{N2Curl}^0, 2) \times (\mathrm{N2Div}, 1)$	1	cf. Table 5.10

Table 5.2: Convergence rates using finite elements to solve Problem 5.1.1 as a 2-form semi-essential Robin BVP.

5.1.3 A 1-form semi-essential Robin BVP

We do the same thing to Probelm 5.1.2. We take the manufactured solution $u = (\sin(z), 0, 0)$. If we view u as a proxy of 1 form, the problem becomes a semi-essential Robin BVP, with $\sigma = \operatorname{div} u = 0$. The mixed formulation is

$$\begin{aligned} -\langle \sigma, \tau \rangle + \langle u, \operatorname{grad} \tau \rangle &= 0 && \forall \tau \in V_h^0, \\ \langle \operatorname{grad} \sigma, v \rangle + \langle \operatorname{curl} u, \operatorname{curl} v \rangle + \langle u \times n, v \times n \rangle_\Gamma &= \langle f, v \rangle + \langle g \times n, v \times n \rangle_\Gamma && \forall v \in V_h^1. \end{aligned} \tag{5.7}$$

We solve the problem for finite elements $(\mathrm{CG}^0, 1; \mathrm{N1Curl}, 1)$ and $(\mathrm{CG}^0, 2; \mathrm{N2Curl}, 1)$, and generate the following convergence table.

Elements	Rates of Convergence	Numerical Results
$(\mathrm{CG}^0, 1) \times (\mathrm{N1Curl}, 1)$	1	cf. Table 5.11
$(\mathrm{CG}^0, 2) \times (\mathrm{N2Curl}, 1)$	1	cf. Table 5.12

Table 5.3: Convergence rates using finite elements to solve Problem 5.1.2 as a 1-form semi-essential Robin BVP.

5.1.4 A 2-form semi-natural Robin BVP

Finally, we reconsider Probelm 5.1.2 with the same solution viewed as the proxy of a 2 form. Hence Probelm 5.1.2 is a semi-natural Robin BVP, and $\sigma = \operatorname{curl} u = (0, \cos(z), 0)$. The mixed formulation is

$$\begin{aligned} -\langle \sigma, \tau \rangle - \langle \sigma \times n, \tau \times n \rangle_\Gamma + \langle u, \operatorname{curl} \tau \rangle &= \langle g, \tau \rangle_\Gamma && \forall \tau \in V_h^1, \\ \langle \operatorname{curl} \sigma, v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle &= \langle f, v \rangle && \forall v \in V_h^2. \end{aligned} \tag{5.8}$$

Using our four sets of finite elements, we can generate the following convergence table.

5.1.5 Detailed L^2 error norms

We append detailed error norms to end this section.

Elements	Rates of Convergence	Numerical Results
$(\mathrm{N1Curl}, 1) \times (\mathrm{N1Div}, 1)$	1	cf. Table 5.13
$(\mathrm{N2Curl}, 1) \times (\mathrm{N1Div}, 1)$	1	cf. Table 5.14
$(\mathrm{N1Curl}, 2) \times (\mathrm{N2Div}, 1)$	2	cf. Table 5.15
$(\mathrm{N2Curl}, 2) \times (\mathrm{N2Div}, 1)$	2	cf. Table 5.16

Table 5.4: Convergence rates using finite elements to solve Problem 5.1.2 as a 2-form semi-natural Robin BVP.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{grad}(\sigma - \sigma_h)\|$	rate	$\|\sigma - \sigma_h\|_\Gamma$	rate	$\|u - u_h\|$	rate	$\|\operatorname{curl}(u - u_h)\|$	rate
1	4.515	-	8.381	-	9.156	-	4.596	-	4.093	-
2	1.739	1.375	5.615	0.577	4.041	1.179	3.025	0.603	3.968	0.044
3	0.952	0.869	3.668	0.614	1.607	1.330	2.133	0.504	1.758	1.174
4	0.248	1.938	1.891	0.955	0.428	1.906	1.066	0.999	0.931	0.916

Table 5.5: Rates of convergence using $(\mathrm{CG}, 1; \mathrm{N1Curl}, 1)$ to solve Problem 5.1.1.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{grad}(\sigma - \sigma_h)\|$	rate	$\|\sigma - \sigma_h\|_\Gamma$	rate	$\|u - u_h\|$	rate	$\|\operatorname{curl}(u - u_h)\|$	rate
1	0.920	-	3.063	-	1.816	-	2.779	-	4.039	-
2	0.347	1.404	1.586	0.949	0.516	1.813	2.157	0.365	3.819	0.080
3	0.040	3.110	0.375	2.078	0.060	3.091	0.472	2.191	1.706	1.162
4	0.005	2.905	0.097	1.939	0.007	2.981	0.122	1.943	0.866	0.977

Table 5.6: Rates of convergence using $(\mathrm{CG}, 2; \mathrm{N2Curl}, 1)$ to solve Problem 5.1.1.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{curl}(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\operatorname{div}(u - u_h)\|$	rate	$\|(u - u_h) \times n\|_\Gamma$	rate
1	6.344	-	9.040	-	8.297	-	6.512	-	6.878	-
2	4.388	0.531	4.131	1.129	3.126	1.408	3.886	0.744	1.681	2.032
3	2.261	0.956	3.117	0.406	1.853	0.753	1.716	1.178	0.845	0.992
4	1.142	0.985	1.618	0.945	0.962	0.945	0.899	0.931	0.221	1.933

Table 5.7: Rates of convergence using $(\mathrm{N1Curl}, 1; \mathrm{N1Div}, 1)$ to solve Problem 5.1.1.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{curl}(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\operatorname{div}(u - u_h)\|$	rate	$\|(u - u_h) \times n\|_\Gamma$	rate
1	6.325	-	8.998	-	8.297	-	6.512	-	6.878	-
2	1.336	2.242	4.070	1.144	2.975	1.479	3.886	0.744	1.681	2.032
3	0.802	0.736	3.085	0.399	1.902	0.645	1.716	1.178	0.845	0.992
4	0.206	1.961	1.568	0.976	0.973	0.966	0.899	0.931	0.221	1.933

Table 5.8: Rates of convergence using $(\mathrm{N2Curl}, 1; \mathrm{N1Div}, 1)$ to solve Problem 5.1.1.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{curl}(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\operatorname{div}(u - u_h)\|$	rate	$\|(u - u_h) \times n\|_\Gamma$	rate
1	3.237	-	5.485	-	5.149	-	6.463	-	3.719	-
2	0.677	2.255	2.059	1.413	1.986	1.374	3.875	0.737	0.847	2.134
3	0.248	1.448	0.446	2.207	0.423	2.228	1.686	1.200	0.102	3.043
4	0.063	1.971	0.129	1.782	0.109	1.957	0.850	0.987	0.013	2.909

Table 5.9: Rates of convergence using $(\mathrm{N1Curl}, 2; \mathrm{N2Div}, 1)$ to solve Problem 5.1.1.

mesh	$\|\sigma - \sigma_h\|$	rate	$\|\operatorname{curl}(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\operatorname{div}(u - u_h)\|$	rate	$\|(u - u_h) \times n\|_\Gamma$	rate
1	2.517	-	5.485	-	5.150	-	6.463	-	3.719	-
2	0.474	2.408	2.058	1.414	1.987	1.373	3.875	0.737	0.847	2.134
3	0.050	3.228	0.440	2.223	0.423	2.231	1.686	1.200	0.102	3.043
4	0.006	2.956	0.112	1.971	0.109	1.956	0.850	0.987	0.013	2.909

Table 5.10: Rates of convergence using $(\mathrm{N2Curl}, 2; \mathrm{N2Div}, 1)$ to solve Problem 5.1.1.

mesh	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate	$\|(u-u_h)\times n\|_\Gamma$	rate
1	2.067	-	3.043	-	1.960	-
2	1.559	0.407	1.335	1.188	1.194	0.714
3	0.834	0.902	0.690	0.952	0.661	0.851
4	0.442	0.915	0.352	0.969	0.347	0.929

Table 5.11: Rates of convergence using $(\mathrm{CG}^0, 1; \mathrm{N1Curl}, 1)$ to solve Problem 5.1.2. The errors for σ are all zeros.

mesh	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate	$\|(u-u_h)\times n\|_\Gamma$	rate
1	1.886	-	3.025	-	1.830	-
2	0.364	2.371	1.325	1.190	0.363	2.331
3	0.103	1.817	0.677	0.967	0.089	2.022
4	0.025	2.001	0.340	0.994	0.022	1.968

Table 5.12: Rates of convergence using $(\mathrm{CG}^0, 2; \mathrm{N2Curl}, 1)$ to solve Problem 5.1.2. The errors for σ are all zeros.

mesh	$\|\sigma-\sigma_h\|$	rate	$\|\operatorname{grad}(\sigma-\sigma_h)\|$	rate	$\|\sigma-\sigma_h\|_\Gamma$	rate	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate
1	3.588	-	3.939	-	4.688	-	7.467	-	0.000	-
2	1.750	1.035	1.545	1.350	1.706	1.457	2.186	1.771	0.078	-
3	0.881	0.990	0.709	1.123	0.761	1.163	0.803	1.445	0.015	2.367
4	0.433	1.023	0.346	1.031	0.362	1.069	0.345	1.216	0.003	2.296

Table 5.13: Rates of convergence using $(\mathrm{N1Curl}, 1; \mathrm{N1Div}, 1)$ to solve Problem 5.1.2.

mesh	$\|\sigma-\sigma_h\|$	rate	$\|\operatorname{grad}(\sigma-\sigma_h)\|$	rate	$\|\sigma-\sigma_h\|_\Gamma$	rate	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate
1	3.030	-	3.937	-	4.056	-	7.467	-	0.000	-
2	1.017	1.574	1.544	1.350	1.038	1.9657	2.114	1.820	0.078	-
3	0.260	1.965	0.707	1.126	0.283	1.8763	0.791	1.418	0.015	2.367
4	0.066	1.970	0.344	1.039	0.071	1.9899	0.343	1.204	0.003	2.296

Table 5.14: Rates of convergence using $(\mathrm{N2Curl}, 1; \mathrm{N1Div}, 1)$ to solve Problem 5.1.2.

mesh	$\|\sigma-\sigma_h\|$	rate	$\|\operatorname{grad}(\sigma-\sigma_h)\|$	rate	$\|\sigma-\sigma_h\|_\Gamma$	rate	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate
1	0.506	-	1.186	-	0.562	-	1.240	-	0.002	-
2	0.229	1.144	0.257	2.205	0.204	1.459	0.252	2.293	0.009	-2.049
3	0.059	1.939	0.064	1.994	0.052	1.953	0.064	1.978	0.001	3.180
4	0.014	1.995	0.016	1.969	0.013	1.998	0.016	1.998	1e-4	3.111

Table 5.15: Rates of convergence using $(\mathrm{N1Curl}, 2; \mathrm{N2Div}, 1)$ to solve Problem 5.1.2.

mesh	$\|\sigma-\sigma_h\|$	rate	$\|\operatorname{grad}(\sigma-\sigma_h)\|$	rate	$\|\sigma-\sigma_h\|_\Gamma$	rate	$\|u-u_h\|$	rate	$\|\operatorname{curl}(u-u_h)\|$	rate
1	0.465	-	1.186	-	0.555	-	1.223	-	0.002	-
2	0.067	2.777	0.257	2.205	0.072	2.930	0.250	2.290	0.009	-2.049
3	0.008	3.004	0.064	1.995	0.010	2.832	0.063	1.982	0.001	3.180
4	0.001	3.013	0.016	1.995	0.001	2.907	0.015	1.988	1e-4	3.111

Table 5.16: Rates of convergence using $(\mathrm{N2Curl}, 2; \mathrm{N2Div}, 1)$ to solve Problem 5.1.2.

5.2 Time harmonic Maxwell problems

In this section we will solve the discrete time-harmonic Maxwell problem (4.18). We note that the functions and inner products that we considered and analyzed in Chapter 4 are all complex-valued, while complex numbers are not supported in FEniCS. Thus we need to first convert our problem (4.18) to an equivalent form with real-valued functions and inner products for computational purposes.

It will make our construction clearer to begin with the strong equation (4.5) and the boundary condition (4.6), in which we replace the right-hand side with a tangential vector field g for non-homogeneous boundary conditions. For each vector field, we can split it to real and imaginary parts, indicated by subscripts $\square_r$ and $\square_i$. For instance $E = E_r + iE_i$. Thus (4.5) is broken into two equations

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} E_r) - \omega^2 \epsilon E_r = -\omega J_i, \quad \operatorname{curl}(\mu^{-1} \operatorname{curl} E_i) - \omega^2 \epsilon E_i = \omega J_r, \tag{5.9}$$

and the boundary condition (4.6) also yields two pieces

$$\operatorname{curl} E_r \times n + \lambda\omega\mu n \times (E_i \times n) = g_r, \quad \operatorname{curl} E_i \times n - \lambda\omega\mu n \times (E_r \times n) = g_i. \tag{5.10}$$

Next, as routine, we take real-valued test functions F_r and F_i, multiplying with the equations (5.9) respectively, integrate by parts while applying (5.10). We finally obtain the following mixed form

$$\begin{aligned} \langle \mu^{-1} \operatorname{curl} E_r, \operatorname{curl} F_r \rangle - \omega^2 \epsilon \langle E_r, F_r \rangle + \omega\lambda \langle E_i \times n, F_r \times n \rangle_\Gamma \\ = -\omega \langle J_i, F_r \rangle - \langle g_r, n \times (F_r \times n) \rangle_\Gamma, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \langle \mu^{-1} \operatorname{curl} E_i, \operatorname{curl} F_i \rangle - \omega^2 \epsilon \langle E_i, F_i \rangle - \omega\lambda \langle E_r \times n, F_i \times n \rangle_\Gamma \\ = \quad \omega \langle J_r, F_r \rangle - \langle g_i, n \times (F_i \times n) \rangle_\Gamma, \end{aligned} \tag{5.12}$$

Here all inner products are the real integrals. In particular, it our trial functions E_r and E_i, and test functions F_r and F_i are all from our finite element space, then this two-equation system yields our numerical scheme. It can be checked (and hence we omit the proof) that this system is equivalent to our original one, i.e.,

Theorem 5.2.1. *Given the domain Ω and data $J \in L^2(\Omega, \mathbb{C}^3)$ and $g \in L^2_{tan}(\Gamma)$. For any $E = E_r + iE_i$, with $E \in \mathscr{H}(\operatorname{curl})$ complex-valued and E_r and E_i real-valued, E solves (4.18) if and only if $(E_r, E_i) \in \mathscr{H}(\operatorname{curl}) \times \mathscr{H}(\operatorname{curl})$ (real-valued) solves (5.11) and (5.12).*

5.2.1 Example 1

In our first example, we consider the cubic domain $\Omega = [0,1]^3$, parameters $\epsilon = 1$, $\mu = 1$, and $\omega = 0.1$, the exact solution $E = (y, z, x + y + z)$, and the right-hand side terms

$$J = i\,\omega\epsilon E, \quad g = -\operatorname{curl} E \times n + i\omega\lambda n \times (E \times n). \tag{5.13}$$

We start with $1 \times 1 \times 1$-mesh, and each time uniformly refine the mesh. On the first 4 meshes, we solve our problem with the $H(\text{curl})$ Nedelec space with degree 1. According to Theorem 4.4.4, the $\mathscr{H}(\text{curl})$-norm of the error,

$$\|E_h - E\|_{\mathscr{H}(\text{curl})} = \left(\|E_{h,r} - E\|^2_{\mathscr{H}(\text{curl})} + \|E_{h,i} - 0\|^2_{\mathscr{H}(\text{curl})}\right)^{1/2},$$

should converge to 0 with order 1. Indeed we have such numerical result, as shown in Table 5.17. Besides this result, we make a few more observations. We see that the L^2 convergence of the error and the tangential trace of the error are both approximately of order 1, which is optimal as well. However, curl of the error approaches 0 with a higher rate. This is not surprising. Calculation gives a constant vector $\operatorname{curl} E = (0, -1, -1)$, which lies in the finite element space.

Mesh	$\|err\|$	rate	$\|\operatorname{curl}(err)\|$	rate	$\|err \times n\|_\Gamma$	rate	$\|err\|_{\mathscr{H}(\text{curl})}$	rate
1	0.713	-	0.020	-	1.143	-	1.347	-
2	0.465	0.615	0.009	1.369	0.637	0.843	0.789	0.772
3	0.259	0.847	0.003	1.384	0.332	0.941	0.421	0.907
4	0.132	0.966	0.001	1.570	0.168	0.980	0.214	0.975

Table 5.17: Unit cube meshes, element = N1curl, deg = 1

We can try raising the degree of the element, or switch to other elements, e.g., the 2nd kind Nedelec element. In either case, we note that the finite element space will contain all linear vector fields. So we shall expect to produce the exact solution, which is linear, by our numerical scheme. This is verified by computation.

5.2.2 Example 2

In this example, we work with the same unit cube domain with the same 4 meshes, but with a quadratic exact solution

$$E = (y^2, z^2, x^2 + y^2 + z^2),$$

and resulting right-hand side terms, which are still given by (5.13). As before, we first solve with N1curl with degree 1, and obtain 1st order convergence for the $\mathscr{H}(\text{curl})$-norm of the error (cf. Table 5.18). This time, all break-down errors decays with order one, including the curl-part, because now $\operatorname{curl} E$ is linear, and is not in the finite element space.

Mesh	$\|err\|$	rate	$\|\operatorname{curl}(err)\|$	rate	$\|err \times n\|_\Gamma$	rate	$\|err\|_{\mathscr{H}(\mathrm{curl})}$	rate
1	0.739	-	0.753	-	1.190	-	1.590	-
2	0.497	0.573	0.381	0.982	0.687	0.791	0.930	0.774
3	0.278	0.835	0.197	0.945	0.362	0.924	0.498	0.901
4	0.143	0.960	0.103	0.940	0.184	0.973	0.255	0.963

Table 5.18: Unit cube meshes, element = N1curl, deg = 1

Next, we raise the degree of our space N1curl to $r = 2$, and also try switching to N2curl with degree 1. In neither case is the quadratic exact solution contained in the finite element space, so we expect to observe 2nd order convergence for 2nd degree N1curl element, while still expect to see 1st order convergence with 1st degree N2curl element (cf. Tables 5.19 and 5.20 for details). The first part is not hard to interpret. As we raise the degree, the convergence rates of $\|e\|$, $\|\operatorname{curl} e\|$, and $\|e \times n\|$ all should increases by 1, as the finite element space, its curl, and its tangential trace now all contain one degree higher of complete polynomials. As a matter of fact, we observe even higher convergence rate for $\|\operatorname{curl} e\|$. The reason is similar to the first example. For our particular exact solution E, $\operatorname{curl} E$ is linear, while our element space contains all linear functions. Thus it is possible to have rate higher than 2. As for the 1st-degree N2curl element, the convergence rate for $\|\operatorname{curl} e\|$ does not improve, compared with that for 1st-degree N1curl element. This is a result due to the general fact that

$$\operatorname{curl}(r\text{th-degree N1curl space}) = \operatorname{curl}(r\text{th-degree N2curl space}).$$

In particular, the curl-image of our N2curl space does not contain all piecewise linear vector fields, as before. Hence we again have 1st-order convergence for $\|\operatorname{curl} e\|$. Consequently, $\|e\|_{\mathscr{H}(\mathrm{curl})}$ decays linearly.

Mesh	$\|err\|$	rate	$\|\operatorname{curl}(err)\|$	rate	$\|err \times n\|_\Gamma$	rate	$\|err\|_{\mathscr{H}(\mathrm{curl})}$	rate
1	0.165	-	4e-3	-	0.236	-	0.288	-
2	0.046	1.840	1e-3	2.204	0.063	1.893	0.078	1.875
3	0.011	1.957	2e-3	2.439	0.016	1.954	0.020	1.955
4	0.002	2.022	3e-5	2.478	0.004	1.978	0.005	1.993

Table 5.19: Unit cube meshes, element = N1curl, deg = 2

Mesh	$\|err\|$	rate	$\|\operatorname{curl}(err)\|$	rate	$\|err \times n\|_\Gamma$	rate	$\|err\|_{\mathscr{H}(\text{curl})}$	rate
1	0.187	-	0.752	-	0.307	-	0.834	-
2	0.049	1.922	0.381	0.982	0.081	1.921	0.392	1.087
3	0.012	1.964	0.197	0.944	0.021	1.924	0.199	0.977
4	0.003	1.975	0.103	0.940	0.005	1.931	0.103	0.948

Table 5.20: Unit cube meshes, element = N2curl, deg = 1

References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[2] D. N. Arnold, *Finite element exterior calculus*, vol. 93 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer., 15 (2006), pp. 1–155.

[4] ——, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.), 47 (2010), pp. 281–354.

[5] J. H. Bramble, J. E. Pasciak, and J. Xu, *The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms*, Math. Comp., 56 (1991), pp. 1–34.

[6] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[7] A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Math., 92 (2002), pp. 679–710.

[8] A. Buffa, M. Costabel, and D. Sheen, *On traces for* $\mathbf{H}(\mathbf{curl},\Omega)$ *in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[9] S. H. Christiansen, *Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension*, Numer. Math., 107 (2007), pp. 87–106.

[10] ——, *Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension*, Numer. Math., 107 (2007), pp. 87–106.

[11] S. H. Christiansen and R. Winther, *Smoothed projections in finite element exterior calculus*, Math. Comp., 77 (2008), pp. 813–829.

[12] E. T. Chung, P. Ciarlet, Jr., and T. F. Yu, *Convergence and superconvergence of staggered discontinuous Galerkin methods for the three-dimensional Maxwell's equations on Cartesian grids*, J. Comput. Phys., 235 (2013), pp. 14–31.

[13] P. Ciarlet, Jr. and E. Jamelot, *Continuous Galerkin methods for solving the time-dependent Maxwell equations in 3D geometries*, J. Comput. Phys., 226 (2007), pp. 1122–1135.

[14] B. Cockburn, F. Li, and C.-W. Shu, *Locally divergence-free discontinuous Galerkin methods for the Maxwell equations*, J. Comput. Phys., 194 (2004), pp. 588–610.

[15] M. Costabel, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, Math. Methods Appl. Sci., 12 (1990), pp. 365–368.

[16] M. Costabel, M. Dauge, and S. Nicaise, *Corner singularities of Maxwell interface and eddy current problems*, in Operator theoretical methods and applications to mathematical physics, vol. 147 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2004, pp. 241–256.

[17] A. Demlow and A. N. Hirani, *A posteriori error estimates for finite element exterior calculus: the de Rham complex*, Found. Comput. Math., 14 (2014), pp. 1337–1371.

[18] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp., 34 (1980), pp. 441–463.

[19] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numer., 11 (2002), pp. 237–339.

[20] R. Hiptmair, A. Moiola, and I. Perugia, *Stability results for the time-harmonic Maxwell equations with impedance boundary conditions*, Math. Models Methods Appl. Sci., 21 (2011), pp. 2263–2287.

[21] M. Huber, A. Pechstein, and J. Schöberl, *Hybrid domain decomposition solvers for the Helmholtz and the time harmonic Maxwell's equation*, in Domain decomposition methods in science and engineering XX, vol. 91 of Lect. Notes Comput. Sci. Eng., Springer, Heidelberg, 2013, pp. 279–287.

[22] T. Huttunen, M. Malinen, and P. Monk, *Solving Maxwell's equations using the ultra weak variational formulation*, J. Comput. Phys., 223 (2007), pp. 731–758.

[23] Y. Jiang, P. Sakkaplangkul, V. A. Bokil, Y. Cheng, and F. Li, *Dispersion analysis of finite difference and discontinuous Galerkin schemes for Maxwell's equations in linear Lorentz media*, J. Comput. Phys., 394 (2019), pp. 100–135.

[24] A. Logg, K.-A. Mardal, G. N. Wells, et al., *Automated Solution of Differential Equations by the Finite Element Method*, Springer, 2012.

[25] A. Moiola, *Trefftz-discontinuous Galerkin methods for time-harmonic wave problems*, eth dissertation, ETH Zürich, Switzerland, 2012.

[26] P. Monk, *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[27] J.-C. Nédélec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[28] ——, *A new family of mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 50 (1986), pp. 57–81.

[29] N. C. Nguyen, J. Peraire, and B. Cockburn, *Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations*, J. Comput. Phys., 230 (2011), pp. 7151–7175.

[30] J. Schöberl, *A posteriori error estimates for Maxwell equations*, Math. Comp., 77 (2008), pp. 633–649.