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Periodic Table of the Finite Elements

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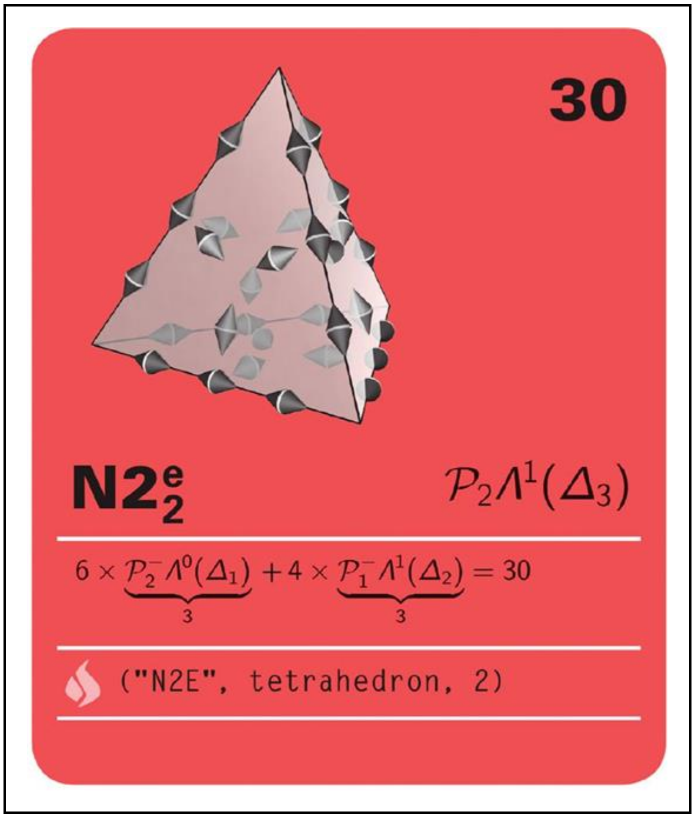

The finite element method is one of the most powerful and widely applicable techniques for the numerical solution of partial differential equations and, therefore, for the simulation of the physical world. First proposed by engineers in the 1950s as a practical numerical method for predicting the deflection and stress of structural components of aircraft, the method has since been continuously extended and refined. It is now used in almost all application areas modeled by PDEs: solid and fluid dynamics, electromagnetics, biophysics, and even finance, to name just a few.

Finite element methods approximate solutions with piecewise polynomials; the first finite element methods were based on the simplest sorts of piecewise polynomials: continuous piecewise linear functions on triangles, and continuous piecewise bilinear functions on squares. Over the years, to extend the stability, accuracy, and applicability of the method, more complex finite element spaces have been introduced, analyzed, implemented, and applied. In addition to the common and the tensor-product Lagrange elements, these include the serendipity elements, Nédélec elements of various types, the Raviart–Thomas elements, and the Brezzi–Douglas–Marini elements. Even to specialists, the resulting collection can seem a disorganized zoo of possibilities.

Fortunately, much as the chemical elements can be arranged in a periodic table based on their electron structure and recurring chemical properties, a broad assortment of finite elements can be arranged in a table that clarifies their properties and relationships. This arrangement, which is based on expression of the finite element function spaces in the language of differential forms, is one of the major outcomes of the theory known as finite element exterior calculus [4, 5], or FEEC. Just as the arrangement of the chemical elements in a periodic table led to the discovery of new elements, the periodic table of finite elements has not only clarified existing elements but also highlighted holes in our knowledge and led to new families of finite elements suited for certain purposes.

A poster displaying this organized presentation of the principal finite elements is included with this issue of *SIAM News*. It is reduced from the full-size poster, which can be obtained at http://femtable.org. The poster was developed by us with the help of graphic designer Mattias Schläger and the support of Simula

Research Laboratory. Harish Narayanan adapted it to the web.

Explanation of the Poster

In our explanation, we assume some familiarity with differential forms on domains in $\mathbb{R}^n$. Readers looking for a brief introduction or review may skip ahead to the final section on differential forms now.

The most prominent aspect of the poster is the arrangement of 108 colored boxes, each corresponding to a finite element space. They form the periodic table of finite elements or, more accurately, a finite section of the table, which is infinite. The element boxes are arranged in four groups, corresponding to the four primary families of finite element spaces, the left-hand two based on simplicial meshes, the right-hand two on cubical or box meshes. The family names, in the notation of FEEC, are $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ for simplices, and $\mathcal{Q}_r^- \Lambda^k$ and $\mathcal{S}_r \Lambda^k$ for cubes—for short, the $\mathcal{P}^-$, $\mathcal{P}$, $\mathcal{Q}^-$, and $\mathcal{S}$ families.

Each family contains elements for differential forms of all possible degrees k, from 0 up to the space dimension n, as indicated by the Λ^k incorporated into the notation for the family. The form degree determines the coloring, with green boxes used for 0-forms and blue boxes for n-forms, the scalar elements. In 3D, red is used for 1-forms and yellow for 2-forms ($(n-1)$-forms), the vector elements. In 2D, 1-forms and $(n-1)$-forms coincide, and are displayed in orange. These correspond to two ways to identify a vector field (v_1, v_2) with a 1-form—as $v_1 \mathrm{d}x^1 + v_2 \mathrm{d}x^2$ or as $v_1 \mathrm{d}x^2 - v_2 \mathrm{d}x^1$—which accounts for the double element diagrams in the orange boxes.

The subscript r in the finite element family name refers to the polynomial degree of the element. The elements exist in any number of space dimensions $n \geq 1$ and for any value of polynomial degree $r \geq 1$, but the poster displays only the lower-order elements ($r = 1, 2,$ and 3) in low dimensions ($n = 1, 2,$ and 3). As we move down the table, the space dimension increases, and for each space dimension the polynomial degree increases. A few elements appear in more than one family and so are repeated on the table.

In summary, each element included in the table corresponds to a choice of

- element family $\mathcal{P}^-$, $\mathcal{P}$, $\mathcal{Q}^-$, or $\mathcal{S}$
- space dimension $n \geq 1$;
- differential form degree k, with $0 \leq k \leq n$;
- polynomial form degree $r \geq 1$.

For example, the element box in Figure 1, which appears in the 8th row, 6th column of the periodic table, corresponds to the choice $n = 3, k = 1, r = 2$ in the $\mathcal{P}$ family. This is an element introduced by Nédélec in 1986 [11], commonly called the Nédélec second kind edge element of degree 2. The common name is reflected in the element box in the symbol assigned to the element, in this case $\mathbf{N2_2^e}$. Next to the element symbol in the box appears the notation $\mathcal{P}_2 \Lambda^1(\Delta_3)$, which is the FEEC notation for the *shape function* space of the element. Finite element functions are piecewise polynomials, and the shape functions are the polynomial pieces; that is, they are the restrictions of the functions in the global finite element space to a single element. The shape functions for $\mathcal{P}_r \Lambda^k$ are differential k-forms whose coefficients are polynomials of degree at most r. Because a differential 1-form can be viewed as a vector field, the shape function space $\mathcal{P}_2\Lambda^1(\Delta_3)$ consists of vector fields on a tetrahedron Δ_3 for which each of the three components is a polynomial of degree at most 2. The space of such polynomials in 3D has dimension 10 (count the monomials: $1, x, y, z, x^2, xy, \ldots, z^2$), so $\dim \mathcal{P}_2\Lambda^1(\Delta_3) = 3 \times 10 = 30$. That dimension appears as the "atomic number" of the element at the upper right of the element box in Figure 1. It is a special case of the general formula

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

This formula, and similar ones for the other three families, appear in the gray family-description boxes near the top of the poster. Tables of values of the dimensions, for n up to 4 and r up to 7, appear at the bottom of the poster. In addition to the dimension and nomenclature for the elements, the element boxes display information about the degrees of freedom of the elements, and how the elements can be used in computer code.

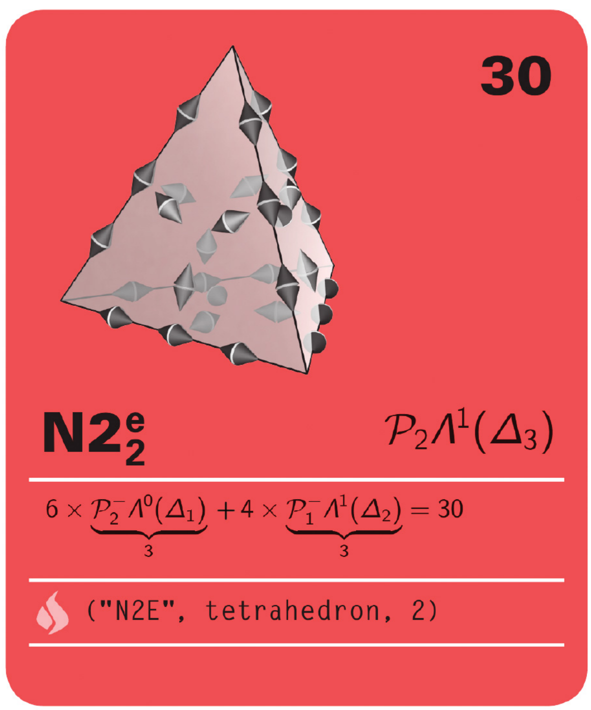

Figure 1. *An element box from the periodic table. This is the Nédélec second kind edge element of degree 2 or, in more mathematical nomenclature,* $\mathcal{P}_2\Lambda^1(\Delta_3)$.

Degrees of freedom. In addition to the shape functions of a finite element, we must specify its degrees of freedom (DOFs). These are a unisolvent set of functionals on the shape functions, with each functional associated to a face of some dimension (e.g., a vertex, edge, 2-face). The DOFs specify how the polynomial pieces are pasted together in a manner that can be efficiently implemented. In constructing the global finite element space from the polynomial pieces (the shape functions), the associated DOFs are constrained to take the same values whenever two elements share a common face. In this way the choice of DOFs imposes a certain degree of continuity on the finite element space.

As shown in Figure 1, for the $\mathcal{P}_2\Lambda^1$ element, 3 DOFs are associated to each of the 6 edges of the tetrahedron, and another 3 to each of its 4 faces, but none are associated to the vertices or to the tetrahedron itself. Because the full set of DOFs is a basis for the dual space of the space of shape functions, their number must equal the dimension of the shape function space, which is reflected in the calculation $6 \times 3 + 4 \times 3 = 30$. The equation

$$6 \times \underbrace{\mathcal{P}_2^- \Lambda^0(\Delta_1)}_{3} + 4 \times \underbrace{\mathcal{P}_1^- \Lambda^1(\Delta_2)}_{3} = 30 \tag{1}$$

displayed in the element box for the $\mathcal{P}_2\Lambda^1$ element includes this information, along with additional information about the DOFs.

For all of the elements in the four families, the DOFs are weighted moments. More precisely, the DOFs associated to a face f of dimension d are functionals acting on a k-form u as

$$u \mapsto \int_f (\mathrm{tr}_f u) \wedge q \tag{2}$$

for an appropriate set of weight functions q. The weight functions are differential $(d-k)$-forms on f; that is, $q \in \Lambda^{d-k}(f)$. Then, because the trace $\mathrm{tr}_f u$ of a differential k-form u belongs to $\Lambda^k(f)$, the integrand $(\mathrm{tr}_f u) \wedge q \in \Lambda^{k+(d-k)}(f) = \Lambda^d(f)$, and so the integral makes sense. Notice that no DOFs are associated to faces of dimension $d < k$.

The choice of weighting functions is described on the poster for each of the four families. For example, the DOFs for the element $\mathcal{P}_r\Lambda^k(\Delta_n)$ are given by (2) with q belonging to the space $\mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f)$ (or more properly to a basis of that space). When $r + k - d < 1$, this space is to be interpreted as vanishing, and so there are DOFs only on the faces of dimension d with $k \le d \le r + k - 1$. Returning to our example with $k = 1, r = 2$, we see that there are indeed DOFs only on the faces of dimensions 1 and 2 (edges and triangles). On an edge f, the weights are given by the 3-dimensional space $\mathcal{P}_2^- \Lambda^0(f)$, while on a triangular face, the weights come from $\mathcal{P}_1^- \Lambda^1(f)$, which is again 3-dimensional. This is all captured in (1), which appears in the element box. Interestingly, the weight functions for the DOFs used to specify an element in the $\mathcal{P}$ family come from the shape function spaces of the $\mathcal{P}^-$ family, and vice versa.

Using the elements in computer code. The final component of the element box in Figure 1 is a snippet of computer code: `("N2E", tetrahedron, 2)`. This can be used to solve PDEs with the $\mathcal{P}_r\Lambda^k(\Delta_n)$ element in the FEniCS finite element software environment [8, 9, 10]. The element is instantiated there by calling `element = FiniteElement("N2E", tetrahedron, 2)`, which translates the element symbol $\mathbf{N2}_{\mathbf{2}}^{\mathbf{e}}$ into FEniCS syntax. Alternatively, FEniCS allows a syntax directly from the FEEC notation: `element = FiniteElement("P", tetrahedron, 2, 1)`.

Finite element spaces and their continuity. For each finite element in the table (choice of family, n, k, and r) and any n-dimensional mesh of simplices (for $\mathcal{P}^-$ or $\mathcal{P}$) or cubes (for $\mathcal{Q}^-$ or $\mathcal{S}$), we obtain a finite element space. This is a space of piecewise polynomial differential k-forms—that is, a space of scalar functions for $k = 0$ or n, a space of vector fields for $k = 1$ or $n-1$.

The DOFs ensure that when two elements share a face, the traces of the corresponding shape functions agree. For 1-forms, for example, this means that the tangential components are continuous across faces, while for 2-forms in 3D it means the normal components are. This is reflected in the choice of symbol used to signify the DOFs in the element diagram. Continuity of the traces is exactly what is required to ensure that the finite element k-forms belong to the domain of the kth exterior derivative. For 0-forms, this means that the function is square integrable together with its gradient (which is the exterior derivative for 0-forms). Thus, the $k = 0$ spaces are spaces of H^1 finite elements, used, for example, to solve scalar second-order elliptic PDE problems, like the Poisson equation. These finite elements are continuous from element to element. By contrast, for $k = n$, no DOFs are specified on any faces of dimension $< n$, so no interelement continuity is imposed. These are L^2 finite elements, which are the basis of the discontinuous Galerkin methods. The case $k = n-1$ gives $H(\text{div})$ finite elements, piecewise polynomial vector fields whose normal component is continuous across element faces. These elements are very important in modern finite element methods, being used, for example, to solve the Darcy flow equations. In three dimensions, there remains the case of $k = 1$, which gives $H(\text{curl})$ finite elements, crucial in electromagnetics.

Differential Forms and Exterior Calculus

The organization and understanding of the periodic table depend heavily on differential forms and the primary operations on them, briefly discussed below.

Differential forms and exterior calculus unify numerous concepts of multivariable calculus in a fashion that applies to arbitrary manifolds. In exterior calculus, the fundamental object is a *differential k-form*, where the integer k ranges from 0 to the dimension n of the domain or manifold. 0-forms and 3-forms capture two different roles of scalar functions in trivariate calculus, while 1-forms and 2-forms give two viewpoints on vector fields. For example, 1-forms are the integrands of line integrals, and 2-forms are fluxes, which can be integrated over surfaces. Although Élie Cartan had not yet developed exterior calculus, Maxwell emphasized this distinction, writing that "physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area." Turning to scalar quantities, 0-forms are point functions whose gradients are 1-forms, while 3-forms are densities that can be integrated over spatial regions. In view of Stokes' theorem, it is not surprising that the curl of a 1-form results in a 2-form, while Green's theorem implies that the divergence of a 2-form gives a 3-form. All the relevant integrals (point evaluation, line integral, surface integral, and volume integral) are unified in the exterior calculus, and all three basic differential operators (grad, curl, and div) are subsumed in the exterior derivative d. Similarly, the various scalar and vector products of three-dimensional geometry are different cases of the wedge product of a k-form v and a j-form w, resulting in a $(j + k)$-form $v \wedge w$.

More precisely, a differential k-form is simply a function v that assigns to each point x of a manifold Ω an *algebraic* k-form on the tangent space $T_x\Omega$; that is, an alternating k-linear map $v_x : T_x\Omega \times \cdots \times T_x\Omega \to \mathbb{R}$. When Ω is a domain in $\mathbb{R}^n$, v is a function of $(k+1)$ variables. The first is the point x belonging to Ω, and the remaining ones are vectors belonging to $\mathbb{R}^n$. As a function of the final k variables, v is required to be linear and alternating, while as a function of x it is required only to possess some desired degree of smoothness $(C^\infty, C^0, L^p, \dots)$. In the special case $k = 0$, differential 0-forms are just real-valued functions on Ω. The space of differential k-forms on Ω is denoted $\Lambda^k(\Omega)$ (typically with C^∞ smoothness understood).

When the domain Ω is a subset of $\mathbb{R}^n$, differential forms can be viewed concretely through their coordinate representation. Let $\mathrm{d}x^i : \mathbb{R}^n \to \mathbb{R}$ denote the linear functional taking a vector to its ith coordinate. Then an algebraic 1-form (i.e., a linear functional) on $\mathbb{R}^n$ can be expressed as $\sum_{i=1}^n v_i \mathrm{d}x^i$ for some coefficients $v_i \in \mathbb{R}$. Allowing the v_i to depend on x, we obtain a differential 1-form. A basis for algebraic k-forms with $k > 1$ is obtained by taking the alternating part of the tensor product of k of the $\mathrm{d}x^i$. These are denoted by $\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$, and so a differential k-form can be uniquely expressed as

$$v = \sum_\sigma v_\sigma \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k},$$

where the sum is over increasing sequences $1 \le \sigma_1 < \cdots < \sigma_k \le n$ and the $\binom{n}{k}$ coefficients v_σ are real-valued functions.

The *exterior derivative* of a differential k-form v is the $(k+1)$-form

$$\mathrm{d}v = \sum_\sigma \sum_{j=1}^n \frac{\partial v_\sigma}{\partial x^j} \, \mathrm{d}x^j \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k},$$

while for a k-dimensional submanifold $\omega \subset \Omega$, the integral

$$\int_\omega v = \sum_\sigma \int_\omega v_\sigma \, \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

is a real number, defined up to a sign that is fixed by the choice of an orientation of ω.

Returning to the case of $\mathbb{R}^3$, a scalar function v can be viewed as either a 0-form (itself) or a 3-form, $v\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. A vector field $(v1, v2, v3)$ corresponds to either the 1-form $v_1\mathrm{d}x^1 + v_2\mathrm{d}x^2 + v_3\mathrm{d}x^3$ or the 2-form $v_1\mathrm{d}x^2 \wedge \mathrm{d}x^3 - v^2\mathrm{d}x^1 \wedge \mathrm{d}x^3 + v^3\mathrm{d}x^1 \wedge \mathrm{d}x^2$. The wedge product of a 0-form and a k-form is the scalar product; the wedge product of two 1-forms is the cross product; and the wedge product of a 1- and a 2-form is the dot product (in other cases the result is 0). The exterior derivative on k-forms is the gradient, curl, divergence, and zero, respectively, for $k = 0, \ldots, 3$, while the integral corresponds to point evaluation for 0-forms, the (tangential) line integral for 1-forms, the (normal) surface integral for 2-forms, and the volume integral for 3-forms.

The $\mathcal{P}$ family of elements. The $\mathcal{P}_r\Lambda^k(\Delta_n)$ shape functions are the simplest to describe, as they consist of all k-forms on the n-simplex whose coefficients are polynomials of degree at most r. For $k = 0$, this is the space of continuous piecewise polynomials of degree r, the Lagrange elements, which are the most basic finite elements (the case $r = 1$ appeared in the earliest paper on finite elements by Courant [7]). At the other end, $k = n$, the shape function space $\mathcal{P}_r\Lambda^n(\Delta_n)$ again contains all polynomials of degree at most r, but, because the DOFs are all interior to the element, the corresponding assembled finite element space contains all piecewise polynomials of degree r without continuity constraints. The other values of k, which require a more sophisticated choice of DOFs, were discovered much later. The $\mathcal{P}_r\Lambda^1(\Delta_2)$ family was introduced by Brezzi, Douglas, and Marini in 1985 [6] as an H(div) finite element space. The next year, Nédélec generalized their space, giving families of H(curl) and H(div) spaces, $\mathcal{P}_r\Lambda^1(\Delta_3)$ and $\mathcal{P}_r\Lambda^2(\Delta_3)$ [11].

The exterior derivative of the $\mathcal{P}_r\Lambda^k$ finite element space is contained in the space $\mathcal{P}_{r-1}\Lambda^{k-1}$, resulting in a complex (for each r and each simplicial mesh)

$$\mathcal{P}_r\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_{r-n}\Lambda^n,$$

which is a finite element subcomplex of the de Rham complex. Moreover, there are projection maps from the spaces of smooth differential forms in the de Rham complex to the spaces in this finite element subcomplex, defined through the DOFs, and these projection maps commute with the exterior derivative. This property is crucial to the stability of these elements when used in mixed finite element methods, to solve, for example, the Darcy flow equations or Maxwell's equations.

The $\mathcal{P}^-$ family of elements. As suggested by the notation, the space $\mathcal{P}_r^-\Lambda^k(\Delta_n)$ is smaller than $\mathcal{P}_r\Lambda^k(\Delta_n)$<. More precisely,

$$\mathcal{P}_{r-1}\Lambda^k(\Delta_n) \subseteq \mathcal{P}_r^-\Lambda^k(\Delta_n) \subseteq \mathcal{P}_r\Lambda^k(\Delta_n),$$

with equality holding on the right if and only if $k = 0$ and equality holding on the left if and only if $k = n$ (and so for $k = 0$ or n the $\mathcal{P}^-$ family does not provide any elements not already present in the $\mathcal{P}$ family). The exact definition of the $\mathcal{P}^-$ shape functions is

$$\mathcal{P}_r^-\Lambda^k(\Delta_n) = \mathcal{P}_{r-1}\Lambda^k(\Delta_n) + \kappa\,\mathcal{P}_{r-1}\Lambda^{k+1}(\Delta_n),$$

where κ is an operator taking polynomial differential $(k+1)$-forms to polynomial differential k-forms of one degree higher, called the *Koszul differential*. We do not give the definition here, except to interpret the case of $n = 3$, $k = 1$. In this case, the space consists of all vector fields of the form $u(x) = v(x) + w(x) \times x$ where v and w are polynomial vector fields of degree at most $r - 1$. Notice that v represents a 1-form, and w a 2-form. The cross product here is not a wedge product but rather the contraction, or interior product, of a 2-form with a vector field.

As for the $\mathcal{P}_r\Lambda^k$ elements, the $\mathcal{P}_r^-\Lambda^k$ elements form a subcomplex of the de Rham complex with commuting projections, but in this case the polynomial degree r stays fixed, rather than decreasing with k:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^n.$$

The $\mathcal{Q}^-$ family of elements. The simplest and best known finite elements on simplicial meshes are the $\mathbf{P_r}$ elements—that is, continuous piecewise polynomials of degree at most r, also called the tensor product Lagrange elements. These elements appear in the FEEC framework as either $\mathcal{P}_r\Lambda^0$ or $\mathcal{P}_r^-\Lambda^0$ (the two families coincide for 0-forms). The analogous elements for cubical meshes are also well known. The $\mathbf{Q_r}$ finite element functions are continuous piecewise polynomials on a cubical mesh for which the pieces are polynomials of degree at most r in each variable. This space can be obtained in n dimensions starting from the $\mathbf{P_r}$ space in one dimension, and taking the tensor product of n copies of the 1D space.

It is less obvious how to construct an analogue of the $\mathcal{P}_r^-\Lambda^k$ or $\mathcal{P}_r\Lambda^k$ elements on cubes for $k > 0$. The trick is to begin not with a single space in one dimension, but with the finite element de Rham complex

$$\mathcal{P}_r\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_{r-1}\Lambda^1 \text{ or, equivalently, } \mathcal{P}_r^-\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^n,$$

for some 1D mesh. There is a natural way to take the tensor product of n-copies of this *complex*, leading to a de Rham subcomplex in n dimensions [3], in which the spaces are finite element spaces with respect to the tensor product cubic mesh. We denote the subcomplex by

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{\mathrm{d}} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{Q}_r^-\Lambda^1,$$

thereby defining the $\mathcal{Q}_r^-\Lambda^k$ spaces shown in the table.

The $\mathcal{S}$ family of elements. Unlike the other three families, this family was not known before the advent of FEEC, but was discovered in an attempt to complete the periodic table [1, 2]. The spaces of 0-forms are the *serendipity* elements, long popular in two dimensions and also for low-degree cases in three dimensions, because they obtain the same order of accuracy as the $\mathbf{P_r}$ spaces, with significantly fewer degrees of freedom. In the same way, the $\mathcal{S}_r\Lambda^k$ spaces obtain the same order of accuracy as the $\mathcal{Q}_r^-\Lambda^k$ spaces on a cubic mesh, again with fewer degrees of freedom. The $\mathcal{S}_r\Lambda^k$ spaces combine to create a de Rham subcomplex with decreasing polynomial degree.

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