Finite element differential forms on simplices and cubes

Douglas N. Arnold

School of Mathematics, University of Minnesota

Shanghai, May 31, 2012

collaborators: G. Awanou, R. Falk, R. Winther, F. Bonizzoni, D. Boffi

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L^2 de Rham complex

1-D: $H^1 \xrightarrow{\text{grad}} L^2$

2-D: $H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2$ or $H^1 \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$

3-D: $H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$

n-D: $H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^{n-1} \xrightarrow{d} H\Lambda^n$

$$H\Lambda^k := \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}$$

These function spaces arise in many of the most fundamental PDEs. For numerical purposes, we need finite element subspaces that work well together.

DNA-Falk-Winther:
Finite element exterior calculus, homological techniques, and applications, Acta Numerica 2006
Finite element exterior calculus: from Hodge theory to numerical stability, Bulletin of the AMS 2010

Finite element de Rham subcomplexes

We need finite element spaces $\Lambda_h^k \subset H\Lambda^k$ which form a subcomplex admitting commuting projections.

$$\begin{tikzcd}
H\Lambda^0 \arrow[r, "d"] \arrow[d, "\pi_h^0"'] & H\Lambda^1 \arrow[r, "d"] \arrow[d, "\pi_h^1"'] & \cdots \arrow[r, "d"] & H\Lambda^{n-1} \arrow[r, "d"] \arrow[d, "\pi_h^{n-1}"'] & H\Lambda^n \arrow[d, "\pi_h^n"'] \\
\Lambda_h^0 \arrow[r, "d"] & \Lambda_h^1 \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Lambda_h^{n-1} \arrow[r, "d"] & \Lambda_h^n
\end{tikzcd}$$

How should we define the finite element spaces Λ_h^k?

Finite element spaces

A finite element is constructed by assembling three ingredients: Ciarlet '78

- A triangulation $\mathcal{T}$ consisting of polygonal elements T
- For each T, a space of *shape functions* $V(T)$, typically polynomial
- For each T, a set of *DOFs*: a set of functionals on $V(T)$, each associated to a face of T. These must be *unisolvent*, i.e., form a basis for $V(T)^*$.

V_h is *defined* as functions piecewise in $V(T)$ with DOFs single-valued on faces. The DOFs determine (1) the interelement continuity, and (2) a projection operator into V_h.

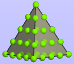
Example: $H\Lambda^0 = H^1$: the Lagrange finite element family

Elements $T \in \mathcal{T}_h$ are simplices in $\mathbb{R}^n$.

Shape fns: $V(T) = \mathcal{P}_r(T)$, some $r \geq 1$.

DOFs:

$$u \mapsto \int_f (\operatorname{tr}_f u) q, \; q \in \mathcal{P}_{r-d-1}(f), \; f \in \Delta(T), \quad d = \dim f$$



- $v \in \Delta_0(T)$: $u \mapsto u(v)$
- $e \in \Delta_1(T)$: $u \mapsto \int_e (\operatorname{tr}_e u) q, \; q \in \mathcal{P}_{r-2}(e)$
- $f \in \Delta_2(T)$: $u \mapsto \int_f (\operatorname{tr}_f u) q, \; q \in \mathcal{P}_{r-3}(f)$
- T: $u \mapsto \int_T uq, \; q \in \mathcal{P}_{r-4}(T)$

Theorem: The number of DOFs $= \dim \mathcal{P}_r(T)$ and they are unisolvent.
The imposed continuity exactly forces inclusion in H^1.

What are the spaces

$$\Lambda_h^k \subset H\Lambda^k$$

analogous to the Lagrange family of elements in the case $k = 0$?

Simplicial finite element differential forms

There are (exactly) *two* families.

First family

Koszul differential

Shape fns: $\mathcal{P}_r^- \Lambda^k(T) := \mathcal{P}_{r-1}\Lambda^k(T) + \kappa \, \mathcal{H}_{r-1}\Lambda^{k+1}(T), \quad r \geq 1$

homogeneous

DOFs: $u \mapsto \displaystyle\int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f), \; f \in \Delta(T), \quad d = \dim f \geq k$

Second family

Shape fns: $\mathcal{P}_r \Lambda^k(T), \quad r \geq 1$

DOFs: $u \mapsto \displaystyle\int_f (\operatorname{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r+k-d}^- \Lambda^{d-k}(f), \; f \in \Delta(T), \quad d = \dim f \geq k$

$$\begin{tikzcd}
 & \mathcal{P}_r^-\Lambda^0 \arrow[r, "d"] & \mathcal{P}_r^-\Lambda^1 \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathcal{P}_r^-\Lambda^n & \\
\text{Lagrange} \arrow[ur] \arrow[dr] & & & & & \text{DG} \arrow[ul] \arrow[dl] \\
 & \mathcal{P}_r\Lambda^0 \arrow[r, "d"] & \mathcal{P}_{r-1}\Lambda^1 \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathcal{P}_{r-n}\Lambda^n &
\end{tikzcd}$$

The Koszul complex

The Koszul differential: $\kappa : \Lambda^k \to \Lambda^{k-1}$

$$\kappa \, dx_i = x_i, \qquad \kappa(\mu \wedge \nu) = (\kappa\mu) \wedge \nu + (-1)^k \mu \wedge (\kappa\nu), \quad \mu \in \Lambda^k, \; \nu \in \Lambda^\ell$$

polynomial de Rham complex

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0$$

Koszul complex

$$0 \leftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \leftarrow 0$$

$$0 \leftarrow \mathcal{P}_r \xleftarrow{\bullet X} (\mathcal{P}_{r-1})^n \leftarrow \cdots \xleftarrow{\times X} (\mathcal{P}_{r-n+1})^n \xleftarrow{X} \mathcal{P}_{r-n} \leftarrow 0$$

Homotopy relation

$$(d\kappa + \kappa d)\,\omega = (r + k)\,\omega, \quad \omega \in \mathcal{H}_r\Lambda^k$$

e.g., $\operatorname{curl}(\vec{x} \times \vec{v}) + \vec{x}\,(\operatorname{div} \vec{v}) = (\deg \vec{v} + 2)\,\vec{v}$

$$\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

The $\mathcal{P}_r^-$ family in 2D

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\,\mathcal{H}_{r-1}\Lambda^{k+1}(T)$$

$\mathcal{P}_r^-\Lambda^0$ Lagrange

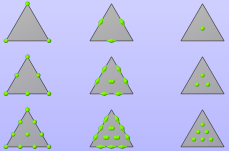
$\mathcal{P}_r^-\Lambda^1$ Raviart-Thomas '76

$\mathcal{P}_r^-\Lambda^2$ DG

$r = 1$

$r = 2$

$r = 3$



The $\mathcal{P}_r^-\Lambda^k$ family in 3D

$\mathcal{P}_r^-\Lambda^0$ Lagrange

$\mathcal{P}_r^-\Lambda^1$ Nédélec '80

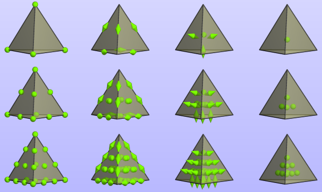
$\mathcal{P}_r^-\Lambda^2$ Nédélec '80

$\mathcal{P}_r^-\Lambda^3$ DG

$r = 1$

$r = 2$

$r = 3$



The $\mathcal{P}_r\Lambda^k$ family in 2D

$\mathcal{P}_r\Lambda^0$ Lagrange

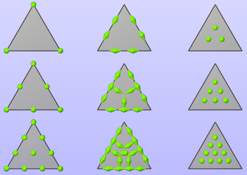
$\mathcal{P}_r\Lambda^1$ Brezzi-Douglas-Marini '85

$\mathcal{P}_r\Lambda^2$ DG

$r = 1$

$r = 2$

$r = 3$



The $\mathcal{P}_r\Lambda^k$ family in 3D

$\mathcal{P}_r\Lambda^0$ Lagrange

$\mathcal{P}_r\Lambda^1$ Nédélec '86

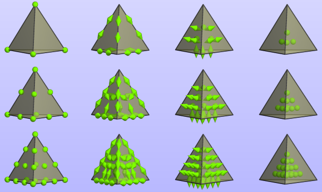
$\mathcal{P}_r\Lambda^2$ Nédélec '86

$\mathcal{P}_r\Lambda^3$ DG

$r = 1$

$r = 2$

$r = 3$



Counting for $\mathcal{P}_r^-\Lambda^k$

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\,\mathcal{H}_{r-1}\Lambda^{k+1}(T)$$
$$\dim \mathcal{P}_r^-\Lambda^k(T) = \dim \mathcal{P}_{r-1}\Lambda^k + \dim \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

Since the Koszul complex

$$\cdots \xleftarrow{\kappa} \mathcal{H}_{r+1}\Lambda^{k-1}(T) \xleftarrow{\kappa} \mathcal{H}_r\Lambda^k(T) \xleftarrow{\kappa} \mathcal{H}_{r-1}\Lambda^{k+1}(T) \xleftarrow{\kappa} \cdots$$

is exact (by the homotopy relation),

$$\dim \kappa\mathcal{H}_r\Lambda^k(T) = \dim \mathcal{H}_r\Lambda^k(T) - \dim \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(T).$$

This enables a backward induction on k to calculate $\dim \kappa\mathcal{H}_r\Lambda^k(T)$. It is then an elementary calculation with binomial identities to verify that

$$\dim \mathcal{P}_r^-\Lambda^k(T) = \#\text{DOFs}$$

Unisolvence for $\mathcal{P}_r^-\Lambda^k$

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\,\mathcal{H}_{r-1}\Lambda^{k+1}(T)$$
$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta(T)$$

Proof of unisolvence: If $u \in \mathcal{P}_r^-\Lambda^k(T)$ and all its DOFS vanish, then $\operatorname{tr}_f u \in \mathcal{P}_r^-\Lambda^k(f)$ and all its DOFs vanish. By induction on dimension, $\operatorname{tr} u$ vanishes on the boundary. So we need to show:

$$u \in \mathring{\mathcal{P}}_r^-\Lambda^k(T), \quad \int_T u\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \implies u = 0$$

A weaker statement is easily shown (by clever choice of q):

$$u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T), \int_T u\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \implies u = 0$$

So we just need to show $u \in \mathcal{P}_{r-1}\Lambda^k(T)$.

- By homotopy relation, $u \in \mathcal{P}_r^-\Lambda^k$, $du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$. So it remains to show that $du = 0$.
- $du \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T)$,
  $\int_T du\wedge p = \pm\int_T u\wedge dp = 0\ \forall p \in \mathcal{P}_{r+k-n}\Lambda^{n-k-1}(T)$.
  Therefore $du = 0$ by the weaker statement (with $k \to k+1$).

Historical notes

- The $\mathcal{P}_1^-\Lambda^k$ complex is in Whitney '57 (Bossavit '88).
- The $\mathcal{P}_r\Lambda^k$ complex is in Sullivan '78.
- Hiptmair gave a uniform treatment of the $\mathcal{P}_r^-\Lambda^k$ spaces in '99.
- The unified treatment and use of the Koszul complex is in DNA-Falk-Winther '06.

Finite element differential forms on cubical meshes

The tensor product construction

Again there are two families (only?). One results from a tensor product construction. (DNA–Boffi–Bonizzoni)
Suppose we have a finite element de Rham subcomplex V on an element $S \subset \mathbb{R}^m$:

$$\cdots \to V^k \xrightarrow{d} V^{k+1} \to \cdots \qquad V^k \subset \Lambda^k(S)$$

and another, W, on another element $T \subset \mathbb{R}^n$:

$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

Shape fns: $\quad (V \wedge W)^k = \bigoplus_{i+j=k} \pi_S^* V^i \wedge \pi_T^* W^j \qquad (\pi_S : S \times T \to S)$

DOFs: $\quad (\eta \wedge \rho)(\pi_S^* v \wedge \pi_T^* w) := \eta(v)\rho(w)$

Finite element differential forms on cubes: the $\mathcal{Q}_r^- \Lambda^k$ family

Start with the simple 1-D degree r finite element de Rham complex, V_r:

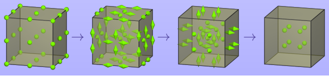
$$0 \to \mathcal{P}_r\Lambda^0(I) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(I) \to 0$$

$$u(x) \quad \to \quad u'(x)\,dx$$

Take tensor product n times: $\quad \mathcal{Q}_r^- \Lambda^k(I^n) := (V_r \wedge \cdots \wedge V_r)^k$

$$\mathcal{Q}_r = \mathcal{P}_r \otimes \mathcal{P}_r, \quad \mathcal{P}_{r-1} \otimes \mathcal{P}_r\, dx_1 + \mathcal{P}_r \otimes \mathcal{P}_{r-1}\, dx_2, \quad \mathcal{P}_{r-1} \otimes \mathcal{P}_{r-1}\, dx_1 \wedge dx_2$$

Raviart-Thomas '76



Nedelec '80

The 2nd family of finite element differential forms on cubes

The $\mathcal{S}_r\Lambda^k(I^n)$ family of FEDFs: (DNA–Awanou '12)

Shape fns:
For a form monomial $m = x_1^{\alpha_i} \cdots x_n^{\alpha_n}\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}$, define $\deg m = \sum \alpha_i$, $\operatorname{ldeg} m = \#\{\, i \,|\, \alpha_i = 1,\ \alpha_i \notin \{\sigma_1, \ldots, \sigma_k\}\,\}$.
Ex: If $m = x_1 x_2 x_3^5 dx_1$, $\deg m = 7$, $\operatorname{ldeg} m = 1$.

$$\mathcal{H}_{r,\ell}\Lambda^k(I^n) = \text{ span of monomials with deg} = r, \text{ ldeg} \geq \ell,$$

$$\mathcal{J}_r\Lambda^k(I^n) = \bigoplus_{\ell \geq 1} \kappa \mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}(I^n),$$

$$\mathcal{S}_r\Lambda^k(I^n) = \mathcal{P}_r\Lambda^k(I^n) \oplus \mathcal{J}_r\Lambda^k(I^n) \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}(I^n).$$

DOFs: $\quad u \mapsto \int_f u \wedge q, \quad q \in \mathcal{P}_{r-2d}\Lambda^{d-k}(f),\ f \in \Delta(I^n)$

Key properties

For any $n \geq 1$, $r \geq 1$, $0 \leq k \leq n$:

Degree property: $\mathcal{P}_r\Lambda^k(I^n) \subset \mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{P}_{r+n-k}\Lambda^k(I^n)$

Inclusion property: $\mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{S}_{r+1}\Lambda^k(I^n)$

Trace property: For each face f of I^n, $\quad \operatorname{tr}_f \mathcal{S}_r\Lambda^k(I^n) = \mathcal{S}_r\Lambda^k(f)$.

Subcomplex property: $d\mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{S}_{r-1}\Lambda^{k+1}(I^n)$

Unisolvence: The indicated DOFs are correct in number and are unisolvent.

Commuting projections: The DOFs determine commuting projections from the de Rham complex to the subcomplex

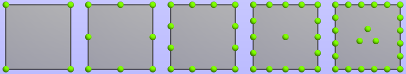
$$\mathcal{S}_r\Lambda^0(I^n) \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1(I^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n(I^n).$$

The case of 0-forms (H^1 elements)

Define sdeg m of a monomial m to be the degree ignoring variables that enter linearly: sdeg $x^3yz^2 = 5$. For a polynomial p, sdeg p is the maximum over its monomials.

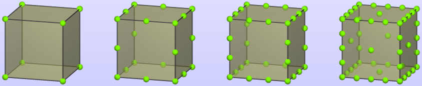
$$\mathcal{S}_r(I^n) = \{\, p \in \mathcal{P}(I^n) \mid \operatorname{sdeg} p \le r \,\}$$ DNA-Awanou '10

1D: $\mathcal{S}_r(I) = \mathcal{P}_r(I)$, 2D: $\mathcal{S}_r(I^2) = \mathcal{P}_r(I^2) + \operatorname{span}[x^r y, xy^r]$ serendipity



$\mathcal{S}_1(I^2)$ $\mathcal{S}_2(I^2)$ $\mathcal{S}_3(I^2)$ $\mathcal{S}_4(I^2)$ $\mathcal{S}_5(I^2)$

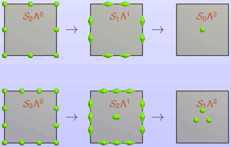
Serendipity 0-forms in more dimensions



Dimensions

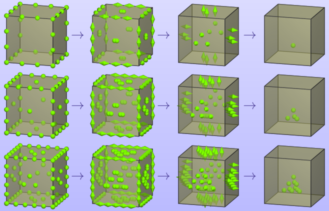
	$\mathcal{P}_r(I^n)$					$\mathcal{S}_r(I^n)$					$\mathcal{Q}_r(I^n)$				
n	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1	2	3	4	5	6	2	3	4	5	6	2	3	4	5	6
2	3	6	10	15	21	4	8	12	17	23	4	9	16	25	36
3	4	10	20	35	56	8	20	32	50	74	8	27	64	125	216
4	5	15	35	70	126	16	48	80	136	216	16	81	256	625	1296

The 2nd cubic family in 2-D



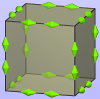
	$\mathcal{S}_r\Lambda^k(I^2)$				
k	1	2	3	4	5
0	4	8	12	17	23
1	8	14	22	32	44
2	3	6	10	15	21

The 2nd cubic family in 3-D



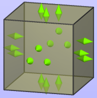
Dimensions and low order cases

	$\mathcal{S}_r\Lambda^k(I^3)$				
k	1	2	3	4	5
0	8	20	32	50	74
1	24	48	84	135	204
2	18	39	72	120	186
3	4	10	20	35	56



$\mathcal{S}_1\Lambda^1(I^3)$

new element



$\mathcal{S}_1\Lambda^2(I^3)$

corrected element

The 3D shape functions in traditional FE language

$\mathcal{S}_r\Lambda^0$: polynomials u such that sdeg $u \le r$

$\mathcal{S}_r\Lambda^1$:

$$(v_1, v_2, v_3) + (x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)) + \operatorname{grad} u,$$

$v_i \in \mathcal{P}_r$, $w_i \in \mathcal{P}_{r-1}$ independent of x_i, sdeg $u \le r + 1$

$\mathcal{S}_r\Lambda^2$:

$$(v_1, v_2, v_3) + \operatorname{curl}(x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)),$$

$v_i, w_i \in \mathcal{P}_r(I^3)$ with w_i independent of x_i

$\mathcal{S}_r\Lambda^3$: $v \in \mathcal{P}_r$

Summary points

- Two families of simplicial FEDF ($\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$) and two familes of cubic FEDF ($\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$)
- Each family contains spaces for all dimensions n, all form degrees $0 \le k \le n$, and all polynomial degrees $r \ge 1$.
- Each family's shape functions and DOFs are given in a unified way, and unisolvence proved for all family members at once.
- Each family is invariant under face traces (shape functions & DOFs).
- Each space has precisely the smoothness needed for inclusion in the energy space $H\Lambda^k$, no more, no less.
- The spaces combine to form de Rham subcomplexes with bounded cochain projections, just what is needed for stable Galerkin methods.
- The exterior calculus framework, including the Koszul complex, brings unity and clarity.
- The right tools for the job!