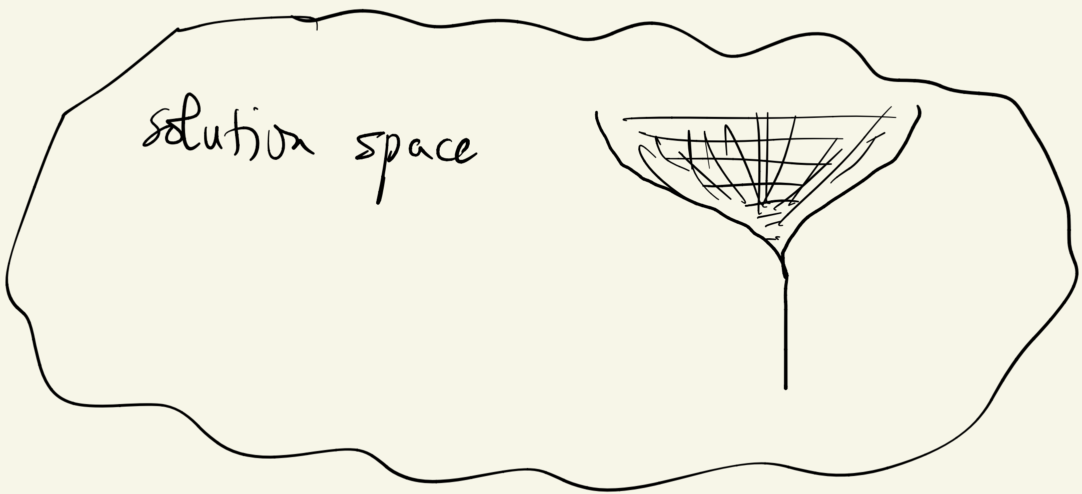
Luke Hilliard

hilli160



$$\forall F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R},$$

Slide 19 of M 9/22

Linearization of

$f'' + 5f' + 3f^2 = 3 \cdot \mathbb{1}$

at $\mathbb{1}$. Similar: $f'' + \frac{1}{5}f' - f^2 = 0 \cdot \mathbb{1}$

$$\boxed{S_F} := \{ Y : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R} \mid \underbrace{\ddot{Y} = F(\dot{Y}, Y)}_{\forall t \in \mathbb{R},\ \ddot{Y}(t) = F(\dot{Y}(t), Y(t))} \}$$

"(long-time) solution space for F"

(Similar: $F_1(x,y) = 3 - 5x + 3y^2$)

Eg if $F_0 : \mathbb{R}^2 \to \mathbb{R}$ is def'd by $\forall x, y \in \mathbb{R}$ $F_0(x,y) = -\frac{1}{5}x + y^2$

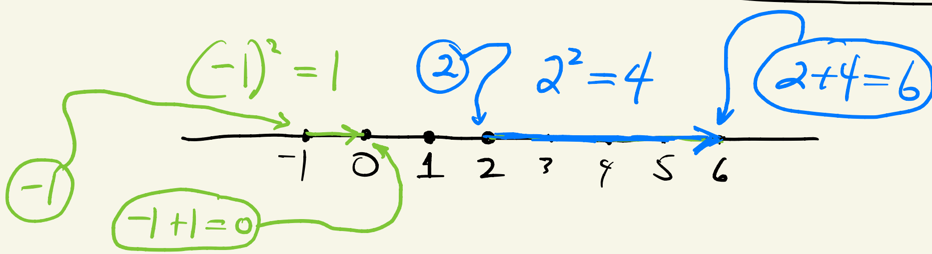
then S_{F_0} is the solution set to $\ddot{Y} = -\frac{1}{5}\dot{Y} + Y^2.$

Say, $\forall t \in \mathbb{R}$, $y(t)$ = position of $\underbrace{\text{particle}}_{\text{traveling in } \mathbb{R},\ \text{mass} = 1}$ at time t

Say $\ddot{y} = -\frac{1}{5}\dot{y} + y^2$. second order ODE nonlinear

---

$$\text{Force} = \underbrace{-\frac{1}{5}\cdot \text{Velocity}}_{\text{drag}} + \underbrace{\text{Position}^2}_{\text{force field}}.$$



etc. "Visualize force field"

Particle in $\mathbb{R}$

$(\text{velocity}, \text{position}) \overset{F}{\longmapsto} \text{force}$

$(\dot{y}, y) \longmapsto F(\dot{y}, y)$

$\forall F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$,

$\boxed{\hat{F}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

is defd by : $\forall x, y \in \mathbb{R}$,

$\hat{F}(x, y) = (F(x,y), x)$.

"special"

C^∞

Eg if $F_0 : \mathbb{R}^2 \to \mathbb{R}$ is defd by $\forall x, y \in \mathbb{R}$, $F_0(x,y) = -\frac{1}{5}x + y^2$

then $\forall x, y \in \mathbb{R}$, $\hat{F}_0(x,y) = (-\frac{1}{5}x + y^2, x)$.

$\forall\ V : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}^2,$

$$\boxed{\hat{S}_V} := \{ (X,Y) : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}^2 \ \text{s.t.}\ \underbrace{(\dot{X},\dot{Y}) = V(X,Y)}_{\forall t\in\mathbb{R},\ (\dot{X}(t),\dot{Y}(t)) = V(X(t),Y(t))} \}$$

(long-time) solution space for V

Eg if $V_0 : \mathbb{R}^2 \to \mathbb{R}$ is def'd by $\forall x,y \in \mathbb{R}$ $V_0(x,y) = (-\frac{1}{5}x + y^2, x)$

then $\hat{S}_{V_0}$ is the solution set to $(\dot{X},\dot{Y}) = (-\frac{1}{5}X + Y^2, X)$.

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$V(x, y) = (\dot{x}, \dot{y})$$

particle in $\mathbb{R}^2$

$$\text{position} \xmapsto{V} \text{velocity}$$

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$F(\dot{y}, y) = \ddot{y}$$

mass=1

particle in $\mathbb{R}$

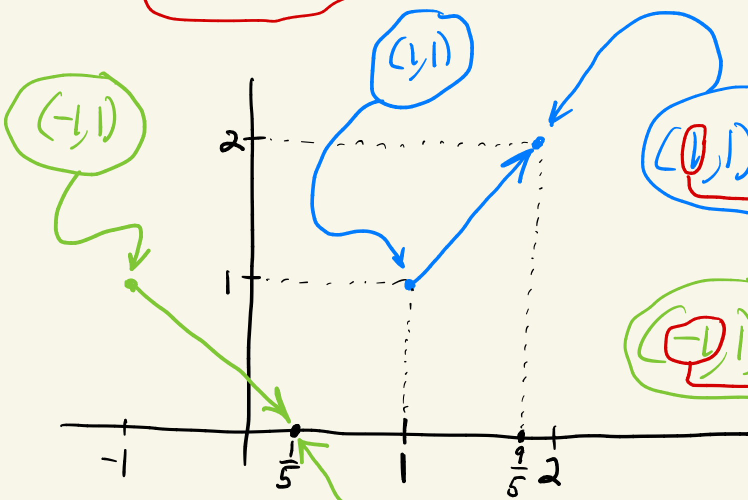
$$(\text{position}, \text{velocity}) \xmapsto{F} \text{force}$$

Eg if $V_0 : \mathbb{R}^2 \to \mathbb{R}$ is defd by $\forall x, y \in \mathbb{R}$, $V_0(x,y) = (-\frac{1}{5}x + y^2, x)$

then $\hat{S}_{V_0}$ is the solution set to $(\dot{X}, \dot{Y}) = (-\frac{1}{5}X + Y^2, X)$.

$V_0(-1, 1) = (\frac{6}{5}, -1)$

$V_0(1, 1) = (\frac{4}{5}, 1)$



$(1,1) + (\frac{4}{5}, 1) = (\frac{9}{5}, 2)$

SPECIAL

rise = x-coord of ftpt

$(-1,1) + (\frac{6}{5}, -1) = (\frac{1}{5}, 0)$

rise = x-coord of ftpt

SPECIAL

etc. "Visualize V_0."

Particle in a wind field
VF

SPECIAL VF

Th $\forall F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$, $\quad S_F \longleftrightarrow \hat{S}_{\hat{F}}$

$$\varphi \longmapsto (\dot{\varphi}, \varphi)$$

$$\varphi \longleftarrow\!\!\shortmid (X, \varphi)$$

$$F_0(x, y) = -\tfrac{1}{5}x + y^2$$

$$\hat{F}_0(x, y) = \left(-\tfrac{1}{5}x + y^2, x\right)$$

$\dot{\varphi} = X$, so $X = \dot{\varphi}$

and $\ddot{\varphi} = \dot{X}$

$$(X, \varphi) \in \hat{S}_{\hat{F}_0} \overset{\text{def}}{\iff} (\dot{X}, \dot{\varphi}) = \left(-\tfrac{1}{5}X + \varphi^2, X\right)$$

$$\Longrightarrow \ddot{\varphi} = \dot{X} = -\tfrac{1}{5}X + \varphi^2 = -\tfrac{1}{5}\dot{\varphi} + \varphi^2$$

$$\Longrightarrow \ddot{\varphi} = -\tfrac{1}{5}\dot{\varphi} + \varphi^2 \overset{\text{def}}{\iff} \varphi \in S_{F_0}$$

Th $\forall V : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}^2$,

$$\hat{S}_V \longrightarrow \mathbb{R}^2$$
$$(X, Y) \longmapsto (X(0), Y(0))$$

is injective. (Bijective if we were doing maximal short-time solns.)

Def Image of this map is $\boxed{C\hat{S}_V}$, the $\boxed{\text{coordinate solution space}}$ of V

$\mathbb{R}^2 \supseteq$

Def $\forall F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$, $\boxed{CS_F} := C\hat{S}_{\hat{F}}$.

"coordinate solution space of F"

$$\hat{F} : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}^2$$
$$(x, y) \longmapsto (F(x,y), x)$$

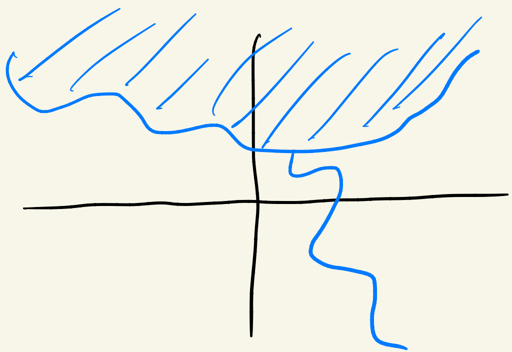
$$\begin{tikzcd} S_F \arrow[r, leftrightarrow] \arrow[rr, bend right] & \hat{S}_{\hat{F}} \arrow[r, hook] & \mathbb{R}^2 \end{tikzcd}$$

same image

image $= CS_F \subseteq \mathbb{R}^2$

Goal Find $F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$

s.t. $CS_F \subseteq \mathbb{R}^2$ is something like



2-dim'l

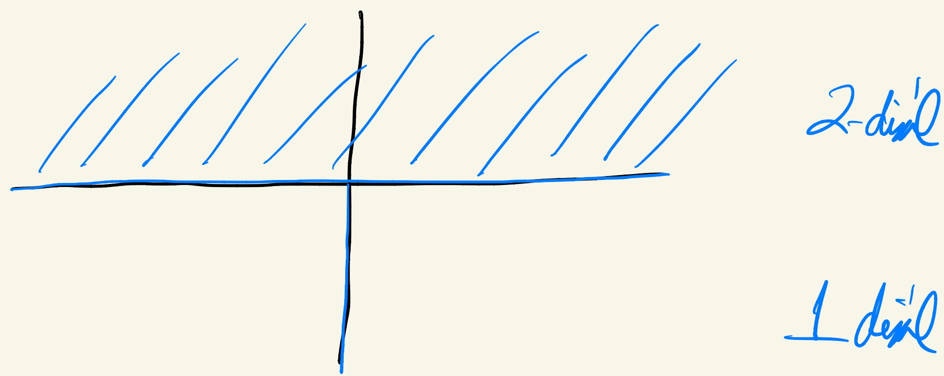
1-dim'l

Too hard...

Easier Goal: Find $G : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}^2$

not necessarily special

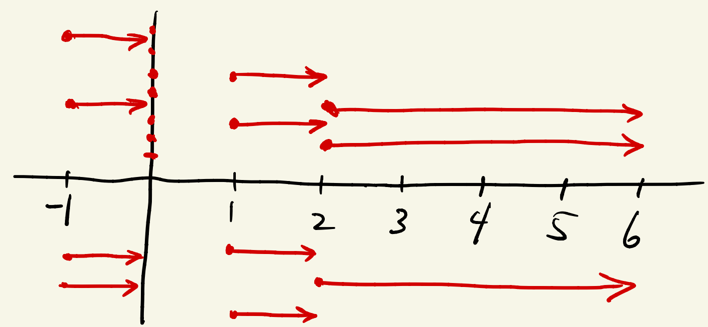
s.t. $\hat{CS}_G$ is



Def $\boxed{G_1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

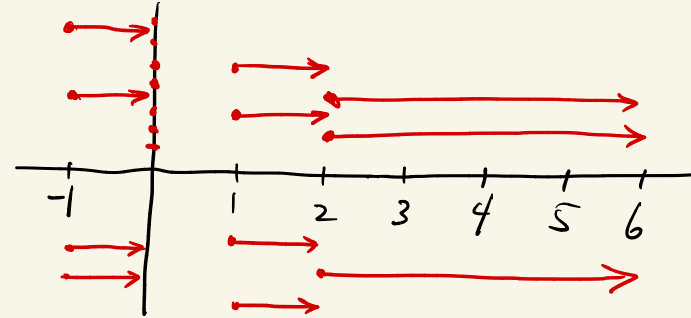
$$\forall x, y \in \mathbb{R}, \quad G_1(x,y) = (x^2, 0).$$

C^∞, <u>not</u> special



<u>Fact</u>: $\hat{CS}_{G_1}$ is the y-axis; 1-dim'l.

$\forall x,y \in \mathbb{R},\ G_1(x,y) = (x^2, 0).$



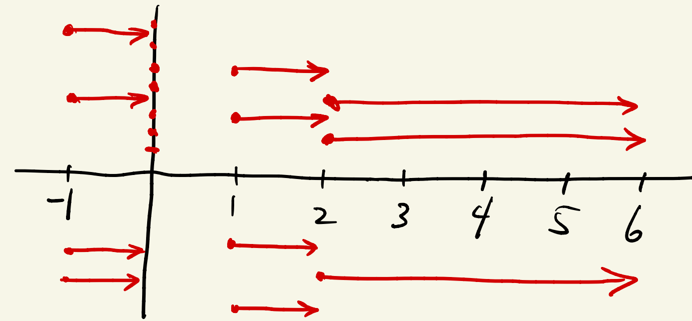
Def $(X,Y): \mathbb{R} \longrightarrow \mathbb{R}^2$ by

$\forall t \in (-\infty; 2),\qquad X(t) = \dfrac{1}{2-t}, \qquad Y(t) = \dfrac{3}{4}.$

$\forall t \in (-\infty; 2),\quad \left(\dot{X}(t), \dot{Y}(t)\right) = \left(\dfrac{1}{(2-t)^2}, 0\right) = G_1(X(t), Y(t)).$

(<u>Fact</u>: (X,Y) is the maximal soln to $(\dot{X}, \dot{Y}) = G_1(X,Y)$ <u>at</u> $\left(\frac{1}{2}, \frac{3}{4}\right)$).

$\forall x,y\in\mathbb{R},\ G_1(x,y)=(x^2,0).$



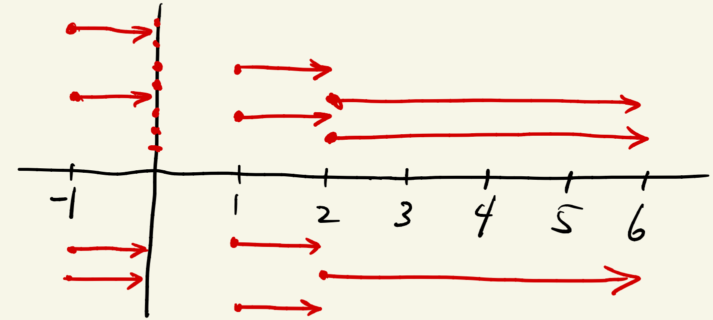
Def $(X,Y):\mathbb{R}\longrightarrow\mathbb{R}^2$ by

$\forall t\in(-2;\infty),\qquad X(t)=\dfrac{1}{-2-t}\quad,\qquad Y(t)=\dfrac{3}{4}.$

$\forall t\in(-2;\infty),\quad \left(\dot X(t),\dot Y(t)\right)=\left(\dfrac{1}{(-2-t)^2},0\right)=G_1(X(t),Y(t)).$

(Fact: (X,Y) is the maximal soln to $(\dot X,\dot Y)=G_1(X,Y)$ at $\left(-\frac{1}{2},\frac{3}{4}\right)$.)

$\forall x,y\in\mathbb{R},\ G_1(x,y)=(x^2,0).$



Def $(X,Y):\mathbb{R}\longrightarrow\mathbb{R}^2$ by

$\forall t\in(-\infty;\infty),\quad X(t)=0,\quad Y(t)=\frac{3}{4}.$

$$\forall t\in(-\infty;\infty),\quad (\dot{X}(t),\dot{Y}(t))=(0^2,0)=G_1(X(t),Y(t)).$$

(<u>Fact</u>: (X,Y) is the maximal soln to $(\dot{X},\dot{Y})=G_1(X,Y)$ <u>at</u> $(0,\frac{3}{4})$.)

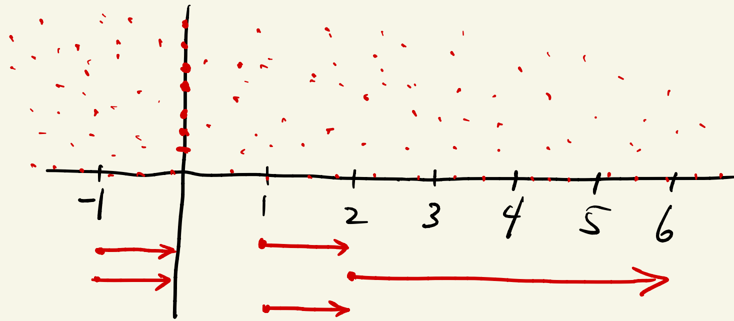
Define $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ by: $\begin{cases} \forall y \geq 0, & \beta(y) = 0 \\ \forall y < 0, & \beta(y) = e^{1/y}. \end{cases}$

<u>Fact</u>: β is C^∞.

Def $\boxed{G_2} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$\forall x, y \in \mathbb{R}$, $\quad G_2(x,y) = (x^2 \cdot (\beta(y)), 0).$

$\forall x, y \in \mathbb{R}, \quad G_2(x,y) = \left(x^2 \cdot (\beta(y)), 0\right).$



G_2 on $y \geq 0$

rescale at (x,y) by $\beta(y)$ to get $G_2(x,y)$

Let $c := \beta(-\frac{3}{4})$.

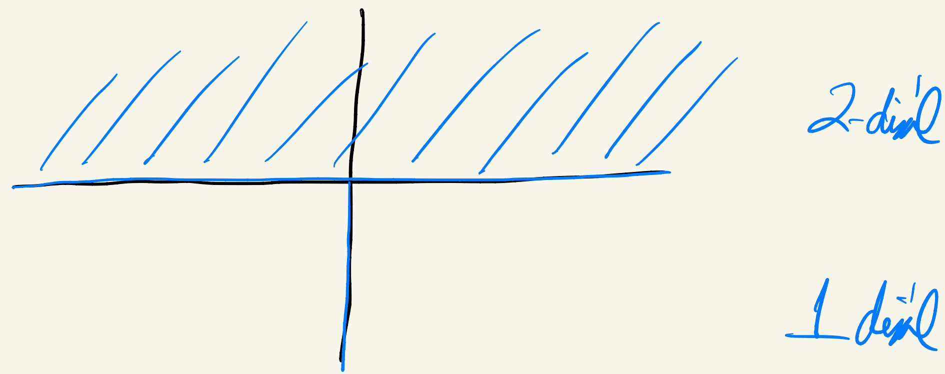
Def $(X, Y): \mathbb{R} \longrightarrow \mathbb{R}^2$ by

$\forall t \in (\frac{2}{c}; \infty), \quad X(t) = \dfrac{1}{2 - ct}, \quad Y(t) = -\dfrac{3}{4}.$

Max soln to $(\dot{X}, \dot{Y}) = G_2(X, Y)$ at $(\frac{1}{2}, -\frac{3}{4})$.

$$\forall x, y \in \mathbb{R}, \qquad G_2(x,y) = \left(x^2 \cdot (\beta(y)),\ 0\right).$$

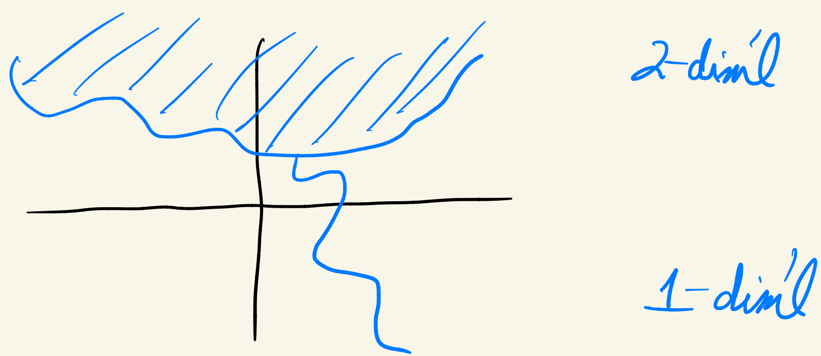
$CS^1_{G_2}$ is:



BACK TO

<u>Goal</u> Find $F : \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}$

s.t. $CS_F^{\subseteq \mathbb{R}^2}$ is something like



Let $\alpha : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$, $x_0, y_0 \in \mathbb{R}$.

Assume $\forall x \neq 0$, $\alpha'(x) \neq 0$

(e.g. $x \longmapsto x^2$ e.g. $3, 4$)

$\forall x \neq 0 \quad 2x \neq 0$

Choose $C \in \mathbb{R}$ s.t. gph of $\alpha + C$ passes through (x_0, y_0).

$\alpha(3) + C = 4$

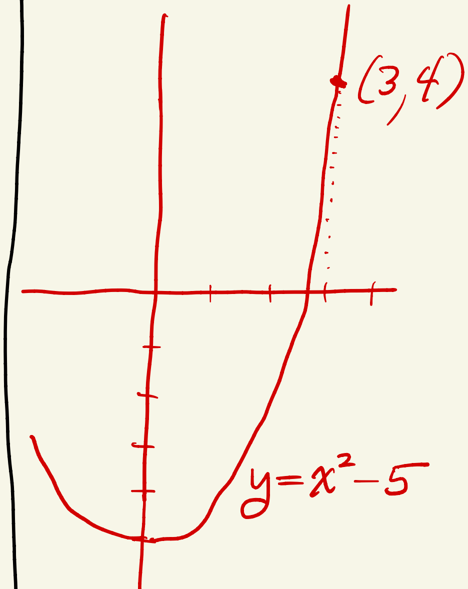
$C = 4 - 3^2 = -5$

$\alpha : \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$, $x_0, y_0 \in \mathbb{R}$, $\alpha + C$ passes through (x_0, y_0)

Find a vector from (x_0, y_0) to $(x_0, y_0) + (\text{run}, \text{rise})$

s.t. vector is tangent to $\alpha + C$ at (x_0, y_0)

<u>and</u> rise $= x_0$ SPECIAL



Do this $\forall x_0, y_0 \in \mathbb{R}$. particle in $\mathbb{R}^2$

position $\mapsto$ velocity

Get a <u>special</u> VF $V : \mathbb{R}^2 \to \mathbb{R}^2$

Special, so $\exists F : \mathbb{R}^2 \to \mathbb{R}$ s.t. $V = \hat{F}$.

particle in $\mathbb{R}$ (velocity, position) $\mapsto$ force

Find a vector from (x_0, y_0) to $(x_0, y_0) + (\text{run}, \text{rise})$

st vector is tangent to $\alpha + C$ at (x_0, y_0)

$\left[(\text{run}, \text{rise}) = (0,0) \ \underline{\text{or}} \ \frac{\text{rise}}{\text{run}} = (\alpha + C)'(x_0) \right.$ and $\text{rise} = x_0$.

$\left. \text{vector is } \underline{\underline{\text{not}}} \text{ vertical} \right]$

Assume $x_0 \neq 0$. (We'll look at $x_0 = 0$ later.)

Then $\alpha'(x_0) \neq 0$ and $\text{rise} \neq 0$. Then $\text{run} \neq 0$. $\frac{\text{rise}}{\text{run}} = \alpha'(x_0)$.

$$\text{run} = \frac{\text{rise}}{\alpha'(x_0)} = \frac{x_0}{\alpha'(x_0)}$$

$$(\text{run}, \text{rise}) = \left(\frac{x_0}{\alpha'(x)}, x_0 \right)$$

$$(\text{run}, \text{rise}) = \left(\frac{x_0}{\alpha'(x)}, x_0\right).$$

Do this $\forall x_0, y_0 \in \mathbb{R}$. Get a <u>special</u> VF $V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$\forall x_0, y_0 \in \mathbb{R}$, $\quad V(x_0, y_0) = \left(\frac{x_0}{\alpha'(x_0)}, x_0\right)$ if $x_0 \neq 0$.

$\forall x, y \in \mathbb{R}$, $\quad V(x, y) = \left(\frac{x}{\alpha'(x)}, x\right)$ if $x \neq 0$.

$$V(x,y) = \left(\frac{x}{\alpha'(x)}, x\right) \text{ if } x \neq 0.$$

$$\alpha(x) = x^2.$$

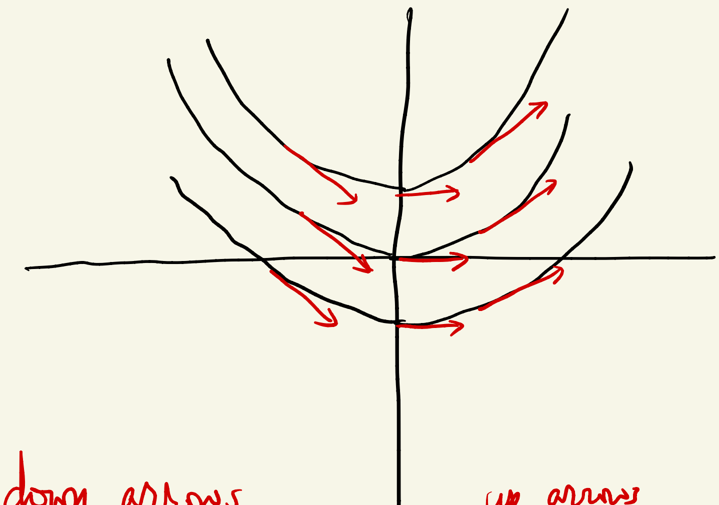
$$V(x,y) = \left(\frac{1}{2}, x\right) \quad \text{if} \quad x \neq 0.$$

$$\forall x \in \mathbb{R}, \quad V(x,y) = \left(\frac{1}{2}, x\right).$$

all arrows have run $\frac{1}{2}$.

Important: $\displaystyle\lim_{x \to 0} \frac{x}{\alpha'(x)}$ exists

rise = x-coord of ftpt



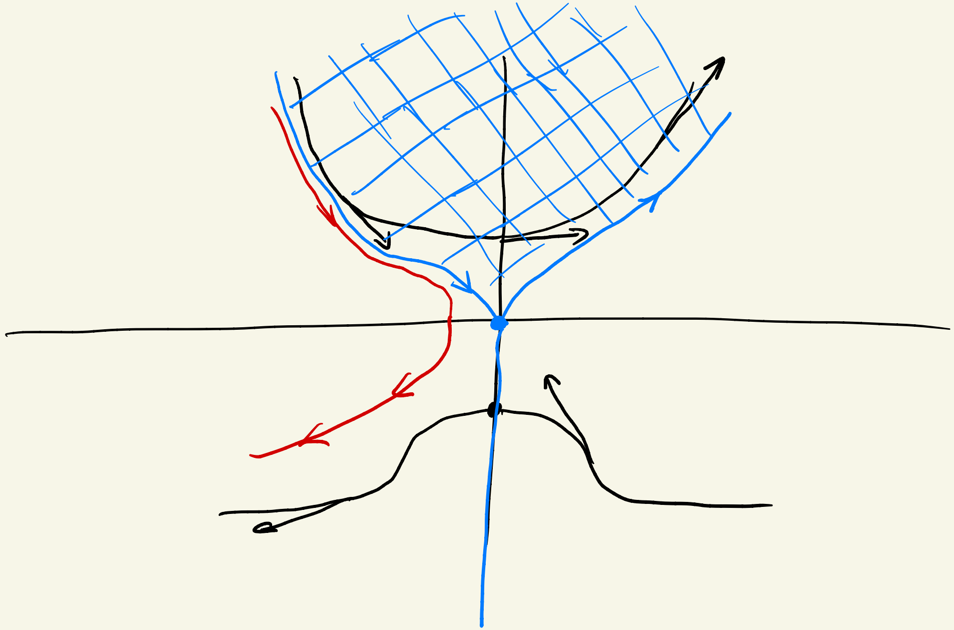
down arrows

up arrows

less & less fall

more & more

horizontal



$y = x^2 + C$

$y = e^{-x^2+C}$

Blue is complete. Rest is not.

Blue $\longleftrightarrow$ solution space to $(\dot{X}, \dot{Y}) = V(X, Y) = (F(X, Y), X)$

$\longleftrightarrow$ solution space to $\ddot{Y} = F(\dot{Y}, Y)$.