FM 5001 Fall 2011, Final Exam

Ending time for in-person students: 8:00 pm on Wednesday 14 December 2011

Time for exam: 1.5 HOURS (ONE AND ONE HALF HOURS)

For PROCTORS of online students:

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Exam must be received by 24 hours after the ending time for in-person students. Thank you.

STUDENT, PLEASE PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.

Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 0031(15), 3 pts.) A matrix $R \in \mathbb{R}^{n \times n}$ is **a rotation matrix** if...

$$R^{-1} = R^t$$

and

$$\det R = 1$$

b. (Topic 0022(16), 3 pts.) Let V and W be two subspaces and let $T : V \to W$ be a linear transformation. The **kernel** of T is $\ker(T) = \cdots$

$$\{ v \in V \mid T(v) = 0 \}$$

c. (Topic 0016(9), 3 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. The k**th order Maclaurin approximation** to f is the polynomial $P : \mathbb{R} \to \mathbb{R}$ such that ...

$$J_0^k f = J_0^k P$$

and

$$\deg P \leqslant k$$

d. (Topic 0029(36), 3 pts.) Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. The **polarization** of Q is the bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that ...

$$\forall v \in \mathbb{R}^n, \qquad Q(v) = B(v, v)$$

and

B is symmetric

e. (Topic 0034(12), 3 pts.) A matrix $M \in \mathbb{R}^{n\times n}$ is **rotationally diagonalizable** if ...

$\exists$ rotation $R \in \mathbb{R}^{n\times n}$

s.t. $R^t M R$ is diagonal

f. (Topic 0023(46), 3 pts.) Two matrices $A, B \in \mathbb{R}^{n\times n}$ are **conjugate** if ...

$\exists$ invertible $C \in \mathbb{R}^{n\times n}$

s.t. $C^{-1} A C = B$

g. (Topic 0032(53), 3 pts.) Let $M \in \mathbb{R}^{n\times n}$ and let a be an eigenvalue of M. Then the a-**eigenspace** of M is...

$\ker(M - aI)$,

where I is the $n \times n$ identity matrix

h. (Topic 0024(12), 3 pts.) Let $M \in \mathbb{R}^{n\times n}$. Then the **exponential** of M is the matrix defined by $e^M = \cdots$

$$\sum_{j=0}^{\infty} \frac{M^j}{j!}$$

II. True or False. (No partial credit.)

a. (Topic 0002(11), 2 pts.) Any compact subset of $\mathbb{R}^n$ is bounded.

T

b. (Topic 034(17), 2 pts.) Any symmetric real matrix is rotationally diagonalizable.

T

c. (Topic 0027(19,24), 2 pts.) For any $A, B \in \mathbb{R}^{n\times n}$, if A and B are conjugate, then $\det(A) = \det(B)$.

T

d. (Topic 0033(10), 2 pts.) Every eigenvalue of an antisymmetric real matrix is a real number.

F

e. (Topic 0017(26), 2 pts.) If a series converges, then any rearrangement of it converges as well.

F

f. (Topic 0033(20), 2 pts.) Any 2×2 Jordan block is diagonalizable.

F

g. (Topic 0036(2), 2 pts.) For any matrix M, there is a nonzero polynomial f such that $F(M) = 0$, where F is the matrix extension of f.

T

h. (Topic 0024(6), 2 pts.) Every nilpotent matrix is invertible.

F

THIS PAGE IS FOR TOTALING SCORES
PLEASE DO NOT WRITE ON THIS PAGE

I.a-d.

I.e-h.

II.a-d.

II.e-h.

III(1).

III(2,3).

III(4).

III(5).

III(6).

III(7).

III(8).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. In this problem, all answers can be expressed using trigonometric functions. You don't need to calculate, *e.g.*, $\sin 3$.

a. (Topic 0019(27), 5 pts.) Compute real numbers a, b, c, d such that $e^{3i} = a + bi$ and $e^{4i} = c + di$.

$a = \cos 3$

$b = \sin 3$

$c = \cos 4$

$d = \sin 4$

b. (Topic 0019(15), 5 pts.) Using a, b, c, d from Part a, expand $(a+bi)(c+di)$, and compute its real part.

$ac - bd = (\cos 3)(\cos 4) - (\sin 3)(\sin 4)$

c. (Topic 0019(27), 5 pts.) Compute the real part of e^{7i}.

$\cos 7$

2. (Topic 0008(10-16), 20 pts.) How many monomials are there of degree = 7 in 15 variables? Write your answer as a product of integers.

$$\binom{7+14}{7} = \binom{21}{7} = \frac{\cancel{21}\cdot\cancel{20}\cdot 19\cdot\cancel{18}\cdot 17\cdot\cancel{16}\cdot 15}{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1} = \frac{\overset{3}{\cancel{21}}\cdot 19\cdot 17\cdot\overset{8}{\cancel{16}}\cdot 15}{\cancel{7}\cdot\cancel{2}}$$

$$= 3\cdot 19\cdot 17\cdot 8\cdot 15$$

3. Let $M := \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ and $N := \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a. (Topic 0023(19), 5 pts.) Compute $M \oplus N$.

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

b. (Topic 0023(20), 10 pts.) Compute $M \otimes N$.

$$\begin{bmatrix} \begin{bmatrix} -4 & -3 \\ -2 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix} \end{bmatrix}$$

4. (Topic 0026(41), 20 pts.) Let $M := \begin{bmatrix} 1 & 6 & 8 \\ 1 & 7 & 6 \\ 0 & 1 & -3 \end{bmatrix}$. Find M^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 8 & 1 & 0 & 0 \\ 1-1 & 7-6 & 6-8 & 0-1 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 6-6 & 8+12 & 1+6 & 0-6 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1-1 & -3+2 & 0+1 & 0-1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 20-20 & 7+20 & -6-20 & 0+20 \\ 0 & 1 & -2+2 & -1-2 & 1+2 & 0-2 \\ 0 & 0 & +1 & -1 & +1 & -1 \end{array}\right]$$

$$\begin{bmatrix} 27 & -26 & 20 \\ -3 & 3 & -2 \\ -1 & 1 & -1 \end{bmatrix}$$

5. (Topic 0026(26), 20 pts.) Find the dimensions of the image and kernel of

$$\begin{bmatrix} 1 & 2 & 4 & 2 & 0 \\ 1{\scriptstyle -1} & 1{\scriptstyle -2} & 2{\scriptstyle -4} & 2{\scriptstyle -2} & 0 \\ 2{\scriptstyle -2} & 3{\scriptstyle -4} & 6{\scriptstyle -8} & 4{\scriptstyle -4} & 0 \\ 3{\scriptstyle -3} & 4{\scriptstyle -6} & 8{\scriptstyle -12} & 6{\scriptstyle -6} & 1 \end{bmatrix}.$$

$$\left[\begin{array}{c|c|c|c|c} 1 & 2-2 & 4-4 & 2 & 0 \\ \hline 0 & +1 & +2 & 0 & 0 \\ \hline 0 & -1+1 & -2+2 & 0 & 0 \\ \hline 0 & -2+2 & -4+4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 2-2 & 0 \\ \hline 0 & 1 & 2-2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

dim image $= 3$, dim kernel $= 2$

6. (Topic 0027(19) and 0027(23) and 0028(42) and 0028(43), 20 pts.) Compute the determinant of

$$A := \begin{bmatrix} 1 & -1^{+1} & 0 & 0 \\ 1 & -1^{+1} & 2 & 0 \\ 2 & 3^{+2} & 6 & 4 \\ 3 & 4^{+3} & 9 & 6 \end{bmatrix}.$$

$$\det A = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 5 & 6 & 4 \\ 3 & 7 & 9 & 6 \end{bmatrix}$$

expand along 1st row

$$= \det \begin{bmatrix} 0 & 2 & 0 \\ 5 & 6 & 4 \\ 7 & 9 & 6 \end{bmatrix}$$

$$= -2 \det \begin{bmatrix} 5 & 4 \\ 7 & 6 \end{bmatrix}$$

$$= (-2)(30 - 28) = (-2)(2)$$

$$= -4$$

7. (Topic 0034(22-36), 25 pts.) Define $Q : \mathbb{R}^2 \to \mathbb{R}$ by $Q(x, y) = 9x^2 + 4xy + 6y^2$. Find a 2×2 rotation matrix R such that $Q \circ L_R$ is a diagonal quadratic form.

$$[Q] = \begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} \qquad \begin{aligned} &\lambda^2 - 15\lambda + 50 \\ &= (\lambda - 5)(\lambda - 10) \end{aligned}$$

$$[Q] - 10I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \qquad 10\text{-eigvect:} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[Q] - 5I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \qquad 5\text{-eigvect:} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$R = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$R^t [Q] R = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

$$(Q \circ L_R)(x, y) = 10x^2 + 5y^2$$

8. (Topic 0024(23), 0032(27), 25 pts.) Let $S = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}$. Find a symmetric matrix $T \in \mathbb{R}^{2\times 2}$ such that $T^2 = S$. *Hint:* Let $R = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$. Then $R^t S R = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}$.

$$S = R \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix} R^t$$

$$T = R \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} R^t$$

$$= \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 9+32 & -12+24 \\ -12+24 & 16+18 \end{bmatrix}$$

$$= \begin{bmatrix} 41/5 & 12/5 \\ 12/5 & 34/5 \end{bmatrix}$$