FM 5001 Fall 2011, Midterm #1
Handout date: Wednesday 19 October 2011
Time for exam: ONE HOUR

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702 Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.
Time to take exam: 1 hour

STUDENT, PLEASE PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 0022(11), 3 pts.) Let V and W be subspaces of Euclidean spaces. A map $L : V \to W$ is **linear** if ....

$\forall a, b \in \mathbb{R}, \quad \forall u, v \in V$

$L(au + bv) = a(L(u)) + b(L(v))$

b. (Topic 0015(13), 3 pts.) $s_1 + s_2 + s_3 + \cdots = s$ means ...

$\lim_{n \to \infty} s_1 + \cdots + s_n = s$

c. (Topic 0002(29), 3 pts.) Let $S \subseteq \overline{\mathbb{R}}$, $b \in \overline{\mathbb{R}}$. We say b is a **lower bound** of S, written $b \le S$, if ...

$\forall s \in S \quad b \le s$

d. (Topic 0002(29), 3 pts.) Let $S \subseteq \overline{\mathbb{R}}$, $b \in \overline{\mathbb{R}}$. We say b is the **infimum** or **glb** of S, written $b = \inf S$, if ...

$[b \le S]$

and $[\forall a \le S, \quad a \le b]$

e. (Topic 0015(4), 3 pts.) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Then the **liminf** of a_j is ...

$\lim_{n \to \infty} \inf_{j \ge n} a_j$

II. True or False. (No partial credit.)

a. (Topic 0017(32), 3 pts.) If a series has only finitely many nonpositive terms, then all of its rearrangements have the same sum.

T

b. (Topic 0020(15), 3 pts.) Every subset of $\mathbb{R}$ is open or closed (or both).

F

c. (Topic 0022(18), 3 pts.) A linear transformation is one-to-one iff its kernel is $\{0\}$.

T

d. (Topic 0016(8), 3 pts.) For any smooth function $f : \mathbb{R} \to \mathbb{R}$, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree ≤ 3 such that $J_0^3 p = J_0^3 f$.

T

e. (Topic 0022(14), 3 pts.) Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Assume, for all $v, w \in \mathbb{R}^n$, that $(A(v)) \cdot w = (B(v)) \cdot w$. Then $A = B$.

T

THE BOTTOM OF THIS PAGE IS FOR TOTALING SCORES
PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1,2).

III(3).

III(4).

III(5).

III(6).

III(7).

III(8ab).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 0008(16), 5 pts.) How many monomials are there of degree exactly 4 in 9 variables? (Express your answer as a product of integers.)

$$\binom{4+8}{4} = \binom{12}{4} = \frac{\cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot 9}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 11 \cdot 45$$

2. (Topic 0009(23-25), 5 pts.) Compute $\int x^2 e^{-x^2/2}\,dx$.

$$\int \underbrace{x}_{\downarrow\, 1} \left(\overbrace{x e^{-x^2/2}}^{-e^{-x^2/2}}\right) dx$$

$$= -xe^{-x^2/2} - \int 1\left(-e^{-x^2/2}\right) dx$$

$$= -xe^{-x^2/2} + \int e^{-x^2/2}\, dx$$

$$= -xe^{-x^2/2} + \sqrt{2\pi}\left[\Phi(x)\right] + C$$

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

a. (Topic 0023(11), 5 pts.) Find a matrix C such that $L_C = L_A \circ L_B$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 8 & 10 & 12 \\ 8 & 10 & 12 \end{bmatrix}$$

b. (Topic 0023(19), 5 pts.) Compute $A \oplus B$. (This is a matrix of scalars, *not* a matrix of matrices.)

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 & 9 \end{bmatrix}$$

c. (Topic 0023(20), 5 pts.) Compute $A \otimes B$. (This *is* a matrix of matrices.)

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{bmatrix}$$

4. (Topic 0015(13), 5 pts.) Let $s := \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \cdots$. Write $s - \frac{s}{2}$ as a sum of a geometric series, and use this to compute s.

$$s = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \cdots$$

$$\frac{s}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{4}{2^5} + \cdots$$

$$s - \frac{s}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$$

$$\|$$

$$\frac{s}{2} \qquad\qquad\qquad 1$$

$$s = 2$$

5. (Topic 0016(22), 10 pts.) Assume that $f'''(x) \le 6$, for all $x \in \mathbb{R}$. Assume that $f(0) = f'(0) = f''(0) = 2$. Among all functions f satisfying those two conditions, find the maximum possible value of $f(3)$.

$\forall x \geqslant 0,$ $\quad f''(x) \leqslant 6x + 2$

$$f'(x) \leqslant 3x^2 + 2x + 2$$

$$f(x) \leqslant x^3 + x^2 + 2x + 2$$

$$f(3) \leqslant 3^3 + 3^2 + 6 + 2$$

6. (Topic 0018(15), 10 pts.) Compute $\lim_{n\to\infty}\left(e^{1/n}+\sin(2/n)\right)^n$.

$$e^x = 1 + x + (\alpha(x))x, \quad \text{some } \alpha(x) \xrightarrow{x\to 0} 0$$

$$\sin x = x + (\beta(x))x, \quad \text{some } \beta(x) \xrightarrow{x\to 0} 0$$

$$e^{1/n} + \sin(2/n) = 1 + \frac{1}{n} + \frac{2}{n} + \frac{\delta_n}{n},$$

$$\text{some } \delta_n \xrightarrow{n\to\infty} 0$$

$$\left[e^{1/n} + \sin(2/n)\right]^n = \left[1 + \frac{3}{n} + \frac{\delta_n}{n}\right]^n$$

$$\Big\downarrow n \to \infty$$

$$e^3$$

7. (Topic 0019(29), 10 pts.) Let $i := \sqrt{-1}$ and let $f(x,y) = |x+iy|^2 + e^{(x+iy)^3}$. (Here x and y are real variables.) Let U and V be, respectively, the real and imaginary parts of $f(x,y)$. Compute U and V as expressions of x and y.

$$(x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$e^{(x+iy)^3} = e^{x^3-3xy^2}\left(\cos(3x^2y - y^3) + i\sin(3x^2y - y^3)\right)$$

$$|x+iy|^2 = x^2 + y^2$$

$$U = x^2 + y^2 + e^{x^3-3xy^2}\left(\cos(3x^2y - y^3)\right)$$

$$V = e^{x^3-3xy^2}\left(\sin(3x^2y - y^3)\right)$$

8. Let $f(x) = (\cos x) + (\sin^2(x/2))$.

a. (Topic 0016(6), 5 pts.) Find the second order Maclaurin approximation of f.

$$1 - \frac{x^2}{2!} + \left(\frac{x}{2}\right)^2$$

$$= 1 - \frac{x^2}{2} + \frac{x^2}{4} = 1 - \frac{x^2}{4}$$

b. (Topic 0018(35), 5 pts.) Compute $\lim_{n\to\infty} [f(5/\sqrt{n})]^n$.

$$f(x) = 1 - \frac{x^2}{4} + (\varepsilon(x))x^2,$$

$$\text{some } \varepsilon(x) \xrightarrow{x \to 0} 0$$

$$f\left(\frac{5}{\sqrt{n}}\right) = 1 - \frac{25/n}{4} + \frac{\delta_n}{n},$$

$$\text{some } \delta_n \xrightarrow{n \to \infty} 0$$

$$\left[f\left(\frac{5}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{25/4}{n} + \frac{\delta_n}{n}\right]^n \xrightarrow{n \to \infty} e^{-25/4}$$