FM 5001 Fall 2011, Midterm #2
Handout date: Wednesday 16 November 2011
**Time for exam: ONE HOUR**

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702 Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.
Time to take exam: 1 hour

STUDENT, PLEASE PRINT NAME: SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

**STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:**

I. Definitions: Complete the following sentences.

a. (Topic 0027(25), 3 pts.) The **determinant** of an $n \times n$ matrix $M$ is denoted $\det(M)$ and is defined by: For all oriented $n$-parallelpipeds $P$, we have ...

$$\mathrm{sv}(MP) = (\det M)(\mathrm{sv}(P)).$$

b. (Topic 0026(14), 3 pts.) Let $S$ be a subspace of a Euclidean space. The **dimension** of $S$ is defined by $\dim(S)$ is the cardinality of ...

any basis of $S$.

c. (Topic 0024(20), 3 pts.) An $n \times n$ matrix $M$ is said to be **orthogonal** if ...

$$MM^t = I,$$

where $I$ is the $n \times n$ identity matrix.

d. (Topic 0024(6), 3 pts.) An $n \times n$ matrix $M$ is said to be **nilpotent** if ...

$\exists$ integer $k \geqslant 1$ s.t. $M^k = 0$,

where $0$ is the $n \times n$ zero matrix.

e. (Topic 0023(45), 3 pts.) Let $A$ and $B$ be $n \times n$ matrices. We say that $B$ is the **inverse** of $A$ if ...

$$AB = BA = I,$$

where $I$ is the $n \times n$ identity matrix.

II. True or False. (No partial credit.)

a. (Topic 0026(33), 3 pts.) If $A$ and $B$ are two matrices, and if $AB$ is an identity matrix, then both $A$ and $B$ are square matrices.

F

b. (Topic 0027(22), 3 pts.) Let $M$ be an $n \times n$ shearing matrix, *i.e.*, a matrix which is equal to the $n \times n$ identity, except that a single off diagonal entry is nonzero. Then $\det(M) = 1$.

T

c. (Topic 0026(7), 3 pts.) If there is an injective linear map $\mathbb{R}^p \to \mathbb{R}^q$, then $p \geq q$.

F

d. (Topic 0025(25-26), 3 pts.) Every elementary matrix is invertible.

T

e. (Topic 0024(6), 3 pts.) If a square matrix is both diagonal and nilpotent, then all of its entries are equal to zero.

T

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PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1,2).

III(3).

III(4).

III(5abc).

III(6).

III. Computations. Some of your answers may involve $\Phi$, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 0006(24), 10 pts.) How many subsets of $\{1, 2, 3, \ldots, 10\}$ have five elements? (Express your answer as a product of positive integers.)

$$\binom{10}{5} = \frac{\overset{2}{\cancel{10}}\cdot 9\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 18\cdot 14$$

2. (Topic 0028(43), 10 pts.) Find the signed volume of the oriented parallelpiped

$$P \quad := \quad (\ (1,3,4)\ ,\ (0,1,-2)\ ,\ (0,0,-1)\ ) \quad .$$

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & -1 \end{bmatrix} = -1$$

3. (Topic 0028(44), 15 pts.) Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. Let $M := \begin{bmatrix} -2 & 4 \\ 3 & -3 \end{bmatrix}$. Let $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the $2 \times 2$ identity matrix. Find a number $\lambda \in \mathbb{R}$ such that $M - \lambda I$ is not invertible.

$$\det(M - \lambda I) = \det \begin{bmatrix} -2-\lambda & 4 \\ 3 & -3-\lambda \end{bmatrix}$$

$$= (-2-\lambda)(-3-\lambda) - 12$$

$$= 6 + 5\lambda + \lambda^2 - 12$$

$$= \lambda^2 + 5\lambda - 6$$

$$= (\lambda + 6)(\lambda - 1)$$

$$[(\lambda = -6) \text{ or } (\lambda = 1)] \Rightarrow [\det(M - \lambda I) = 0]$$

$$\Rightarrow [M - \lambda I \text{ is not invertible}]$$

4. (Topic 0025(44), 10 pts.) Show all fully canonical $3 \times 5$ matrices.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

5. Let $C := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, let $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and let $M := CDC^{-1}$.

a. (Topic 0025(26), 5 pts.) Compute $C^{-1}$.

$$C^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

b. (Topic 0023(15), 5 pts.) Compute $M$.

$$M = CDC^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

c. (Topic 0024(17), 5 pts.) Compute $e^M$.

$$e^M = e^{CDC^{-1}} = Ce^D C^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & -2e \\ 0 & e^2 \end{bmatrix} = \begin{bmatrix} e & -2e+2e^2 \\ 0 & e^2 \end{bmatrix}$$

6. (Topic 0026(41), 10 pts.) Find the inverse of $M := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 13 \\ 2 & 5 & 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4-4 & 8-8 & 13-12 & 0-4 & 1 & 0 \\ 2-2 & 5-4 & 3-6 & 0-2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2-2 & 3+6 & 1+4 & 0 & 0-2 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 1 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9-9 & 5+36 & 0-9 & -2 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 1 & -3+3 & -2-12 & 0+3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 41 & -9 & -2 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 1 & 0 & -14 & 3 & 1 \end{array}\right]$$

$$M^{-1} = \begin{bmatrix} 41 & -9 & -2 \\ -14 & 3 & 1 \\ -4 & 1 & 0 \end{bmatrix}$$