FM 5002 Spring 2012, Final Exam
Ending time for in-person students: 8:00pm on Wednesday 9 May 2012
**Time for exam: 1.5 HOURS (ONE AND ONE HALF HOURS)**

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702 Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.

STUDENT, PLEASE PRINT NAME:

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Turn off all handheld devices, including cell phones.

Show work; a correct answer, by itself, may be insufficient for credit. Arithmetic need not be simplified, unless the problem requests it.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

**STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME**:

I. Definitions: Complete the following sentences.

1. (10 pts, Topic 0038(2).) A **vector field** on $\mathbb{R}^n$ is ...

2. (10 pts, Topic 0040(21).) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth and let $c \in \mathbb{R}$ be a constant. We say that $p \in \mathbb{R}^2$ is a **critical point** for the constrained optimization problem with objective $f$ and constraint $g = c$ if $g(p) = c$ and...

3. (10 pts, Topic 0042(20).) Let $\omega = [(f(x,y))\,dx] + [(g(x,y))\,dy]$ be a one-form in $x$ and $y$. Its **exterior derivative** is given by $d\omega = \cdots$

4. (10 pts, Topic 0045(6).) Let $X$ be a PCRV. We say that $X$ is **deterministic** if ...

5. (10 pts, Topic 0047(12).) Let $X$ and $Y$ be PCRVs. Then $X$ and $Y$ are said to be **independent** if ....

6. (10 pts, Topic 0047(14).) Let $P$, $Q$ and $R$ be events. We say that $P$, $Q$ and $R$ are **jointly independent** if they are pairwise-independent and ...

7. (10 pts, Topic 0049(22).) Let $\Omega := [0,1]$. Let $X$ be a PCRV, and let $\mathcal{P}$ be a partition of $\Omega$ by finite unions of intervals. Then $\mathrm{E}[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in A \in \mathcal{P}$ (and if $A$ is not of zero size), then $(\mathrm{E}[X|\mathcal{P}])(\omega) = \cdots$

8. (10 pts, Topic 0037(56).) The **Hessian** $Hf$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $(Hf)(x,y) = \cdots$

9. (10 pts, Topic 0042(34).) Let $p, q \in \mathbb{C}$ and let $L = (p,q)$ be the directed line segment in $\mathbb{C}$ from $p$ to $q$. Let $\phi : [0,1] \to \mathbb{C}$ be the standard parametrization of $L$. Then $\displaystyle\int_L f(z)\,dz = \cdots$.

10. (10 pts, Topic 0038(3).) Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. A smooth function $c : (a,b) \to \mathbb{R}^n$ is said to be a **flowline** of $V$ if ...

II. True or False. (No partial credit.)

1. (10 pts, Topic 0045(36).) Any two standard PCRVs are identically distributed.

2. (10 pts, Topic 0046(2-9).) Let $M$ be any matrix with real entries. Then both $M^tM$ and $MM^t$ are symmetric and positive semidefinite.

3. (10 pts, Topic 0063(18).) Let $X$ and $Y$ be any two independent PCRVs whose distributions have Fourier transforms $f(t)$ and $g(t)$. Then the Fourier transform of the distribution of $X+Y$ is $[f(t)][g(t)]$.

4. (10 pts, Topic 0047(29-31).) Let $X$ be a PCRV with mean $\mathrm{E}[X]=\mu$ and variance $\mathrm{Var}[X]=\sigma^2$. Then $\mathrm{E}[e^X]=e^{\mu+(\sigma^2/2)}$.

5. (10 pts, Topic 0047(20).) Let $X_1,\ldots,X_n$ be independent PCRVs. Assume, for all integers $j\in[1,n]$, that $\mathrm{E}[e^{X_j}]=2$. Then $E[e^{X_1+\cdots+X_n}]=2^n$.

---

THE BOTTOM OF THIS PAGE IS FOR TOTALING SCORES.
PLEASE DO NOT WRITE BELOW THE LINE.

I.1-5.

I.6-10.

II.

III1.a.

III1.bc.

III2.

III3.

III4.

III5.

III6.

III7.

III8.

III9.

III. Computations. Some of your answers may involve $\Phi$, the cumulative distribution function of the standard normal distribution.

1. Let $C_1, C_2, \ldots$ be an independent sequence of PCRVs such that, for all integers $j \geq 1$, $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. For all integers $n \geq 1$, let $Z_n := (C_1 + \cdots + C_n)/\sqrt{n}$.

a. (10 pts, Topic 0047 (20).) Find a sequence $a_n$ such that, for all integers $n \geq 1$, $\mathrm{E}[e^{Z_n}] = \cosh^n(a_n)$. (Hints: The function $\cosh : \mathbb{R} \to \mathbb{R}$ is defined by $\cosh t := (e^t + e^{-t})/2$. Recall that, if $A$ and $B$ are independent, then $e^A$ and $e^B$ are as well. Recall that, if $P$ and $Q$ are uncorrelated, then $E[PQ] = (E[P])(E[Q])$.)

1. (continued) Recall: $C_1, C_2, \ldots$ is an independent sequence of standard PCRVs such that, for all integers $j \geq 1$, $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. For all integers $n \geq 1$, let $Z_n := (C_1 + \cdots + C_n)/\sqrt{n}$.

b. (20 pts, Topic 0047(28).) Compute $\lim_{n\to\infty} \mathrm{E}[Z_n^6]$.

c. (20 pts, Topic 0047(28).) For all integers $n \geq 1$, let $X_n := (C_1 + \cdots + C_n)/n$. Compute $\lim_{n\to\infty} \mathrm{E}[X_n^6]$.

2. (20 pts, Topic 0040(31).) Find the maximum value of $8x + 27y$ subject to the constraint that $x^4 + y^4 = 97/16$.

3. (15 pts, Topic 0046(43).) Let

$$M := \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}.$$

Find a upper triangular matrix $A$ such that $AA^t = M$ and such that all diagonal entries of $A$ are nonnegative.

4. (15 pts, Topic 0048(26).) Let $A$, $B$ and $C$ be events. Suppose $\Pr[A] = 0.2$. Suppose

$$\Pr[B|A] = 0.4, \quad \Pr[B|(\text{not } A)] = 0.2 \ .$$

Suppose

$$\Pr[C|(B \text{ and } A)] = 0.5, \quad \Pr[C|(B \text{ and } (\text{not } A))] = 0.1 \ .$$

Compute $\text{Odds}[A|(B \text{ and } C)]$.

5. (20 pts, Topic 0044(21).) Compute $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-5)^4 e^{6x} e^{-x^2/2}\, dx$.

6. (20 pts, Topic 0066(9).) For every integer $n \geq 1$, let $X_n \in \sum^n \mathcal{B}_{0.5,d_n}^{0.5,u_n}$. Suppose, for all integers $n \geq 1$, that $\mathrm{E}[X_n] = 0.03$ and $\mathrm{SD}[X_n] = 0.23$. Find $u_n$ and $d_n$, as explicit expressions in $n$.

7. (20 pts, Topic 0049(41).) Define a PCRV $X$ by

$$X(\omega) = \begin{cases} 4, & \text{if } 0 \le x \le 0.25; \\ 8, & \text{if } 0.25 < x \le 1. \end{cases} .$$

Let $\mathcal{P} := \{[0, 0.5), [0.5, 1]\}$. Let $Y := \mathrm{E}[X^2|\mathcal{P}]$. Compute $\mathrm{E}[Y]$.

8. (20 pts, Topic 0050(48).) Stirling's formula asserts that

$$n! \quad \text{is asymptotic to} \quad \sqrt{2\pi n}\,(n/e)^n, \quad \text{as } n \to \infty.$$

Find constants $C$, $k$ and $a > 0$ such that

$$\text{the binomial coefficient } \binom{2n}{n} \quad \text{is asymptotic to} \quad Cn^k a^n, \quad \text{as } n \to \infty.$$

9. (20 pts, Topic 0064(30).) For all integers $n \geq 1$, let

$$p_n := \frac{1}{2} - \frac{1}{3n} \qquad \text{and} \qquad q_n := \frac{1}{2} + \frac{1}{3n}, \qquad \text{and suppose } X_n \in \Sigma^n \mathcal{B}_{q_n}^{p_n}.$$

Assume that $\lim_{n\to\infty} \mathrm{E}[X_n] = 0$ and that $\lim_{n\to\infty} \mathrm{E}[X_n^2] = 1/9$. Compute $\lim_{n\to\infty} \mathrm{E}[X_n^{10} + X_n^9]$.