FM 5002 Spring 2012, Final Exam
Ending time for in-person students: 8:00pm on Wednesday 9 May 2012
Time for exam: 1.5 HOURS (ONE AND ONE HALF HOURS)

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702   Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.

STUDENT, PLEASE PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Turn off all handheld devices, including cell phones.

Show work; a correct answer, by itself, may be insufficient for credit. Arithmetic need not be simplified, unless the problem requests it.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

1. (10 pts, Topic 0038(2).) A **vector field** on $\mathbb{R}^n$ is ...

a smooth function $\mathbb{R}^n \to \mathbb{R}^n$.

2. (10 pts, Topic 0040(21).) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth and let $c \in \mathbb{R}$ be a constant. We say that $p \in \mathbb{R}^2$ is a **critical point** for the constrained optimization problem with objective f and constraint $g = c$ if $g(p) = c$ and...

$\exists \lambda \in \mathbb{R}$ s.t. $(\nabla f)(p) = \lambda \cdot [(\nabla g)(p)]$.

3. (10 pts, Topic 0042(20).) Let $\omega = [(f(x,y))\,dx] + [(g(x,y))\,dy]$ be a one-form in x and y. Its **exterior derivative** is given by $d\omega = \cdots$

$\left([d(f(x,y))] \wedge dx\right) + \left([d(g(x,y))] \wedge dy\right)$.

4. (10 pts, Topic 0045(6).) Let X be a PCRV. We say that X is **deterministic** if ...

$\exists c \in \mathbb{R}$ s.t. $\Pr[X = c] = 1$.

5. (10 pts, Topic 0047(12).) Let X and Y be PCRVs. Then X and Y are said to be **independent** if ....

$\forall S, T \subseteq \mathbb{R}$,

$(X \in S)$ is independent of $(Y \in T)$.

6. (10 pts, Topic 0047(14).) Let P, Q and R be events. We say that P, Q and R are **jointly independent** if they are pairwise-independent and ...

$$\Pr[P \& Q \& R] = (\Pr[P])(\Pr[Q])(\Pr[R]).$$

7. (10 pts, Topic 0049(22).) Let $\Omega := [0,1]$. Let X be a PCRV, and let $\mathcal{P}$ be a partition of Ω by finite unions of intervals. Then $\mathrm{E}[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in A \in \mathcal{P}$ (and if A is not of zero size), then $(\mathrm{E}[X|\mathcal{P}])(\omega) = \cdots$

$$E[X|A].$$

8. (10 pts, Topic 0037(56).) The **Hessian** Hf of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $(Hf)(x,y) = \cdots$

$$\begin{bmatrix} (\partial_{11} f)(x,y) & (\partial_{12} f)(x,y) \\ (\partial_{21} f)(x,y) & (\partial_{22} f)(x,y) \end{bmatrix}.$$

9. (10 pts, Topic 0042(34).) Let $p, q \in \mathbb{C}$ and let $L = (p,q)$ be the directed line segment in $\mathbb{C}$ from p to q. Let $\phi : [0,1] \to \mathbb{C}$ be the standard parametrization of L. Then $\displaystyle\int_L f(z)\, dz = \cdots$.

$$\int_0^1 [f(\phi(t))][\phi'(t)]\, dt.$$

10. (10 pts, Topic 0038(3).) Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. A smooth function $c : (a,b) \to \mathbb{R}^n$ is said to be a **flowline** of V if ...

$$\forall t \in (a,b), \qquad \frac{d}{dt}(c(t)) = V(c(t)).$$

II. True or False. (No partial credit.)

1. (10 pts, Topic 0045(36).) Any two standard PCRVs are identically distributed.

F

2. (10 pts, Topic 0046(2-9).) Let M be any matrix with real entries. Then both M^tM and MM^t are symmetric and positive semidefinite.

T

3. (10 pts, Topic 0063(18).) Let X and Y be any two independent PCRVs whose distributions have Fourier transforms $f(t)$ and $g(t)$. Then the Fourier transform of the distribution of $X+Y$ is $[f(t)][g(t)]$.

T

4. (10 pts, Topic 0047(29-31).) Let X be a PCRV with mean $\mathrm{E}[X] = \mu$ and variance $\mathrm{Var}[X] = \sigma^2$. Then $\mathrm{E}[e^X] = e^{\mu+(\sigma^2/2)}$.

F

5. (10 pts, Topic 0047(20).) Let $X_1, \ldots, X_n$ be independent PCRVs. Assume, for all integers $j \in [1, n]$, that $\mathrm{E}[e^{X_j}] = 2$. Then $E[e^{X_1+\cdots+X_n}] = 2^n$.

T

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PLEASE DO NOT WRITE BELOW THE LINE.

I.1-5.

I.6-10.

II.

III1.a.

III1.bc.

III2.

III3.

III4.

III5.

III6.

III7.

III8.

III9.

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution.

1. Let $C_1, C_2, \ldots$ be an independent sequence of PCRVs such that, for all integers $j \geq 1$, $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. For all integers $n \geq 1$, let $Z_n := (C_1 + \cdots + C_n)/\sqrt{n}$.

a. (10 pts, Topic 0047 (20).) Find a sequence a_n such that, for all integers $n \geq 1$, $\mathrm{E}[e^{Z_n}] = \cosh^n(a_n)$. (Hints: The function $\cosh : \mathbb{R} \to \mathbb{R}$ is defined by $\cosh t := (e^t + e^{-t})/2$. Recall that, if A and B are independent, then e^A and e^B are as well. Recall that, if P and Q are uncorrelated, then $E[PQ] = (E[P])(E[Q])$.)

$$\cosh^n(a_n) = E[e^{Z_n}] = E\left[e^{\sum_{j=1}^{n} C_j/\sqrt{n}}\right]$$

$$= E\left[\prod_{j=1}^{n} e^{C_j/\sqrt{n}}\right] \overset{\checkmark}{=} \prod_{j=1}^{n}\left(E\left[e^{C_j/\sqrt{n}}\right]\right)$$

$$= \prod_{j=1}^{n}\left(\frac{1}{2}e^{1/\sqrt{n}} + \frac{1}{2}e^{-1/\sqrt{n}}\right)$$

$$= \left(\frac{1}{2}e^{1/\sqrt{n}} + \frac{1}{2}e^{-1/\sqrt{n}}\right)^n$$

$$= \left(\cosh\left(1/\sqrt{n}\right)\right)^n = \cosh^n\left(1/\sqrt{n}\right),$$

So $a_n = 1/\sqrt{n}$ works

1. (continued) Recall: $C_1, C_2, \ldots$ is an independent sequence of standard PCRVs such that, for all integers $j \geq 1$, $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. For all integers $n \geq 1$, let $Z_n := (C_1 + \cdots + C_n)/\sqrt{n}$.

b. (20 pts, Topic 0047(28).) Compute $\lim_{n\to\infty} \mathrm{E}[Z_n^6]$.

$$\mathop{\Vert}^{\text{CLT}}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^6 e^{-x^2/2}\, dx$$

$$\Vert$$

$$1 \cdot 3 \cdot 5$$

$$\Vert$$

$$15$$

c. (20 pts, Topic 0047(28).) For all integers $n \geq 1$, let $X_n := (C_1 + \cdots + C_n)/n$. Compute $\lim_{n\to\infty} \mathrm{E}[X_n^6]$.

$$\Vert$$

$$Z_n/\sqrt{n}$$

$$\Vert$$

$$\lim_{n\to\infty} E\left[\left(Z_n/\sqrt{n}\right)^6\right]$$

$$\Vert$$

$$\lim_{n\to\infty} E\left[Z_n^6/n^3\right] = \lim_{n\to\infty} (1/n^3)(E[Z_n^6])$$

$$= (0)(15) = 0$$

2. (20 pts, Topic 0040(31).) Find the maximum value of $8x + 27y$ subject to the constraint that $x^4 + y^4 = 97/16$.

$$(8, 27) = \lambda\,(4x^3, 4y^3),$$

so $\frac{8}{27} = \frac{x^3}{y^3}$, so $\frac{2}{3} = \frac{x}{y}$, so $y = \frac{3}{2}x$.

$$\frac{97}{16} = x^4 + y^4 = x^4 + \left(\frac{3}{2}x\right)^4 = x^4 + \frac{81}{16}x^4$$

$$= \frac{16}{16}x^4 + \frac{81}{16}x^4 = \frac{97}{16}x^4,$$

so $1 = x^4$.

Either $\left[(x=1) \text{ and } \left(y=\frac{3}{2}\right)\right]$

or $\left[(x=-1) \text{ and } \left(y=-\frac{3}{2}\right)\right]$.

At critical points, $8x+27y = \pm\left(8\cdot 1 + 27\cdot\frac{3}{2}\right)$

$$= \pm\left(\frac{16}{2} + \frac{81}{2}\right) = \pm\frac{97}{2}$$

Maximum value is $+\frac{97}{2}$

3. (15 pts, Topic 0046(43).) Let

$$M := \begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix}.$$

Find a upper triangular matrix A such that $AA^t = M$ and such that all diagonal entries of A are nonnegative.

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\begin{bmatrix} 13 & -3 \\ -3 & 1 \end{bmatrix} = M = AA^t = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} x^2+y^2 & yz \\ yz & z^2 \end{bmatrix}$$

$1 = z^2, \quad z \geqslant 0 \quad \therefore \quad z = 1$

$-3 = yz, \quad z = 1 \quad \therefore \quad y = -3$

$13 = x^2 + y^2, \quad y = -3 \quad \therefore \quad x^2 = 13 - 9 = 4$

$x^2 = 4, \quad x \geqslant 0 \quad \therefore \quad x = 2$

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

4. (15 pts, Topic 0048(26).) Let A, B and C be events. Suppose $\Pr[A] = 0.2$. Suppose

$$\Pr[B|A] = 0.4, \quad \Pr[B|(\text{not } A)] = 0.2 \ .$$

Suppose

$$\Pr[C|(B \text{ and } A)] = 0.5, \quad \Pr[C|(B \text{ and } (\text{not } A))] = 0.1 \ .$$

Compute Odds$[A|(B \text{ and } C)]$.

$$= \left(\text{Odds}[A]\right)\left(\frac{0.4}{0.2}\right)\left(\frac{0.5}{0.1}\right)$$

$$= \left(\frac{0.2}{0.8}\right)(2)(5)$$

$$= \left(\frac{1}{4}\right)(10) = \frac{5}{2}$$

5. (20 pts, Topic 0044(21).) Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-5)^4 e^{6x} e^{-x^2/2}\,dx$.

$$\| \; x \mapsto x+6$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x+1)^4 e^{6^2/2} e^{-x^2/2}\,dx$$

$$\|$$

$$e^{36/2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x^4+4x^3+6x^2+4x+1)e^{-x^2/2}\,dx\right]$$

$$\|$$

$$e^{18}\left[\;3+4\cdot 0+6\cdot 1+4\cdot 0+1\right]$$

$$\|$$

$$e^{18}\left[3+6+1\right]$$

$$\|$$

$$10e^{18}$$

6. (20 pts, Topic 0066(9).) For every integer $n \geq 1$, let $X_n \in \sum^n \mathcal{B}_{0.5,d_n}^{0.5,u_n}$. Suppose, for all integers $n \geq 1$, that $\mathrm{E}[X_n] = 0.03$ and $\mathrm{SD}[X_n] = 0.23$. Find u_n and d_n, as explicit expressions in n.

$$n\left(\frac{1}{2}u_n + \frac{1}{2}d_n\right) = 0.03 \quad \therefore \quad \frac{1}{2}u_n + \frac{1}{2}d_n \overset{(*)}{=} \frac{0.03}{n}$$

$$\sqrt{n}\sqrt{\frac{1}{2}\cdot\frac{1}{2}}\,(u_n - d_n) = 0.23 \quad \therefore \quad \frac{1}{2}u_n - \frac{1}{2}d_n \underset{(**)}{=} \frac{0.23}{\sqrt{n}}$$

(*) + (**) yields: $\quad u_n = \dfrac{0.03}{n} + \dfrac{0.23}{\sqrt{n}}$

(*) − (**) yields: $\quad d_n = \dfrac{0.03}{n} - \dfrac{0.23}{\sqrt{n}}$

7. (20 pts, Topic 0049(41).) Define a PCRV X by

$$X(\omega) = \begin{cases} 4, & \text{if } 0 \le x \le 0.25; \\ 8, & \text{if } 0.25 < x \le 1. \end{cases} \quad .$$

Let $\mathcal{P} := \{[0, 0.5), [0.5, 1]\}$. Let $Y := \mathrm{E}[X^2|\mathcal{P}]$. Compute $\mathrm{E}[Y]$.

$$= E[E[X^2|\mathcal{P}]]$$

"Power" Tower Law

$$= E[X^2]$$

$$= \frac{1}{4}\cdot 4^2 + \frac{3}{4}\cdot 8^2$$

$$= 4 + 3\cdot 2\cdot 8$$

$$= 4 + 48$$

$$= 52$$

8. (20 pts, Topic 0050(48).) Stirling's formula asserts that

$$n! \quad \text{is asymptotic to} \quad \sqrt{2\pi n}\,(n/e)^n, \quad \text{as } n \to \infty.$$

Find constants C, k and $a > 0$ such that

$$\text{the binomial coefficient } \binom{2n}{n} \quad \text{is asymptotic to} \quad Cn^k a^n, \quad \text{as } n \to \infty.$$

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} \sim \frac{\sqrt{2\pi(2n)}\,(2n/e)^{2n}}{\left[\sqrt{2\pi n}\,(n/e)^n\right]^2}$$

$$= \frac{\sqrt{4\pi}\cdot\sqrt{n}\cdot 2^{2n}\cdot \cancel{n^{2n}}\cdot \cancel{e^{-2n}}}{2\pi\cdot n\cdot \cancel{n^{2n}}\cdot \cancel{e^{-2n}}}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}\cdot 4^n$$

$$\therefore\ C = \frac{1}{\sqrt{\pi}},\quad k = -\frac{1}{2},\quad a = 4$$

9. (20 pts, Topic 0064(30).) For all integers $n \geq 1$, let

$$p_n := \frac{1}{2} - \frac{1}{3n} \qquad \text{and} \qquad q_n := \frac{1}{2} + \frac{1}{3n}, \qquad \text{and suppose } X_n \in \Sigma^n \mathcal{B}_{q_n}^{p_n}.$$

Assume that $\lim\limits_{n\to\infty} \mathrm{E}[X_n] = 0$ and that $\lim\limits_{n\to\infty} \mathrm{E}[X_n^2] = 1/9$. Compute $\lim\limits_{n\to\infty} \mathrm{E}[X_n^{10} + X_n^9]$.

$$Z_n := 3X_n \xrightarrow[n\to\infty]{\text{TCLT}} Z \quad \text{in distribution,}$$

against continuous exp-bdd

$$\mathrm{E}[X_n^{10} + X_n^9] = \mathrm{E}\left[\left(\frac{Z_n}{3}\right)^{10} + \left(\frac{Z_n}{3}\right)^9\right]$$

$$= \frac{1}{3^{10}}\left(\mathrm{E}[Z_n^{10}]\right) + \frac{1}{3^9}\left(\mathrm{E}[Z_n^9]\right)$$

$$\xrightarrow[n\to\infty]{} \frac{1}{3^{10}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{10} e^{-x^2/2}\,dx\right) + \frac{1}{3^9}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^9 e^{-x^2/2}\,dx\right)$$

$$= \frac{1}{3^{10}}(1\cdot 3\cdot 5\cdot 7\cdot 9) + \frac{1}{3^9}(0)$$

$$= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{3^{10}}$$