FM 5002 Spring 2012, First midterm Exam
Ending time for in-person students: 8:00pm on Wednesday 29 February 2012
Time for exam: 1 HOUR (ONE HOUR)

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702   Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.

STUDENT, PLEASE PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Turn off all handheld devices, including cell phones.

Show work; a correct answer, by itself, may be insufficient for credit. Arithmetic need not be simplified, unless the problem requests it.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 0040(21), 3 pts.) Let $F, g : \mathbb{R}^n \to \mathbb{R}$ be smooth. In the constrained optimization problem to maximize F subject to the constraint $g = 0$, a point $c \in \mathbb{R}^n$ is called a **critical point** for the constrained optimization problem if ...

$$\exists \lambda \in \mathbb{R} \quad \text{s.t.} \quad (\nabla F)(c) = \lambda[(\nabla g)(c)]$$

b. (Topic 0040(32), 3 pts.) Let $g : \mathbb{R}^n \to \mathbb{R}$ be smooth and let $x \in \mathbb{R}^n$ satisfy $g(x) = 0$. We say that x is a **non-smooth point** for the constraint $g = 0$ if ...

$$(\nabla g)(x) = 0$$

c. (Topic 0037(43), 3 pts.) The **gradient** of a smooth $f : \mathbb{R}^3 \to \mathbb{R}$ is the function $\nabla f : \mathbb{R}^3 \to \cdots$ defined by $(\nabla f)(x) = \cdots$. (Fill in BOTH ellipses.)

$\nabla f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(\nabla f)(x) = ((\partial_1 f)(x), (\partial_2 f)(x), (\partial_3 f)(x))$$

d. (Topic 0042(3), 3 pts.) A **directed line segment** in $\mathbb{R}^2$ is ...

an element of $(\mathbb{R}^2)^2$

e. (Topic 0042(14), 3 pts.) A **one-form** in s and t is an expression of the form $P\,ds + Q\,dt$, where P and Q are ...

zero-forms in s and t,

i.e. expressions in s and t

II. True or False. (No partial credit.)

a. (Topic 0038(10), 3 pts.) For any vector field V in $\mathbb{R}$, there is, for some $a < 0$ and $b > 0$, a flowline $c : (a, b) \to \mathbb{R}$ for V footed at 0.

T

b. (Topic 0042(39-41), 3 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function of one variable, and let $\omega := f(x)\,dx$ be the corresponding one-form in x. Then $d\omega = 0$.

T

c. (Topic 0038(58), 3 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth, let $C \in \mathbb{R}$ be a constant and let $Q : \mathbb{R}^2 \to \mathbb{R}$ be quadratic. Assume that the second order Maclaurin approximation to f is $C + Q$. If Q is positive definite, then f has a local maximum at $(0, 0)$.

F

d. (Topic 0037(35), 3 pts.) If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth, then $\partial_1 \partial_2 f = \partial_2 \partial_1 f$.

T

e. (Topic 0042(20), 3 pts.) Let $\omega = p(x, y)\,dx + q(x, y)\,dy$ be a one-form in x and y. Assume that $d\omega = 0$. Then p and q are constants.

F

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PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1).

III(2).

III(3ab).

III(3c).

III(4).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 0042(30), 15 pts.) Let L be the directed line segment in $\mathbb{R}^2$ with starting point $(2,4)$ and ending point $(5,3)$. Compute $\int_L xy\,dx + y\,dy$.

$$\boxed{\begin{aligned} &0 \le t \le 1 \\ &x = 2+3t \\ &dx = 3\,dt \\ &y = 4-t \\ &dy = (-1)\,dt \end{aligned}}$$

$$\|$$

$$\left[\int_0^1 (2+3t)(4-t)(3)\,dt\right] + \left[\int_0^1 (4-t)(-1)\,dt\right]$$

$$\|$$

$$3\left[\int_0^1 (8+10t-3t^2)\,dt\right] - \left[\int_0^1 (4-t)\,dt\right]$$

$$\|$$

$$3\left[8t+5t^2-t^3\right]_{t:\to 0}^{t:\to 1} - \left[4t-\frac{t^2}{2}\right]_{t:\to 0}^{t:\to 1}$$

$$\|$$

$$3\left[8+5-1\right] - \left[4-\frac{1}{2}\right]$$

2. (Topic 0042(22), 15 pts.) Let $R := (3,7) \times (-1,4)$, so R is an open rectangle in the plane. Compute $\int_R xy\, dy \wedge dx$.

$$\|$$

$$-\iint_R xy\, dx\, dy$$

$$\| \text{Fubini}$$

$$-\int_{-1}^{4} \int_{3}^{7} xy \; dx \; dy$$

$$\|$$

$$-\int_{-1}^{4} y \left[\frac{x^2}{2}\right]_{x:\to 3}^{x:\to 7} dy$$

$$\|$$

$$-\int_{-1}^{4} y \left[\frac{49}{2} - \frac{9}{2}\right] dy$$

$$\|$$

$$-20 \int_{-1}^{4} y\, dy = -20 \left[\frac{y^2}{2}\right]_{y:\to -1}^{y:\to 4}$$

$$= -20 \left[\frac{16}{2} - \frac{1}{2}\right]$$

3. Let $f(x, y) = 4 + 3x - 2y + x^2 + 8xy - y^2 + (x^5)(\sin^3 y)$.

a. (Topic 0037(63), 10 pts.) Find the gradient of $f(x, y)$.

$$(\nabla f)(x, y) =$$

$$\left(3 + 2x + 8y + (5x^4)(\sin^3 y),\right.$$

$$\left. -2 + 8x - 2y + (3x^5)(\sin^2 y)(\cos y)\right)$$

b. (Topic 0037(63), 10 pts.) Find the Hessian of $f(x, y)$.

$$(Hf)(x, y) =$$

$$\begin{bmatrix} 2 + (20x^3)(\sin^3 y) & 8 + (15x^4)(\sin^2 y)(\cos y) \\ 8 + (15x^4)(\sin^2 y)(\cos y) & [3x^5][2(\sin y)(\cos^2 y) - (\sin^3 y)] \end{bmatrix}$$

c. (Topic 0037(63), 5 pts.) Find the second order Maclaurin expansion of $f(x, y)$.

$$4 + 3x - 2y + x^2 + 8xy - y^2$$

4. (Topic 0041(30-43), 15 pts.) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the change of variables defined by

$$\phi\,(\,s\;,\;t\,) \quad = \quad (\,4s + 5t + 8\;,\;7 - t\,)\;.$$

Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a bounded function and U is a bounded open set in $\mathbb{R}^2$. Suppose $\underbrace{\int\!\!\int_{\phi(U)} F(x,y)\,dx\,dy}_{\overset{!!}{\cdot\cdot}\; I} = 24$. Compute $\underbrace{\int\!\!\int_U F(\phi(s,t))\,ds\,dt}_{\overset{!!}{\cdot\cdot}\; J}$.

$$I = \iint_U \left[F(\phi(s,t))\right]\left[\left|\underbrace{\det\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}}_{-4}\right|\right] ds\,dt$$

$$= 4J$$

$$J = \frac{I}{4} = \frac{24}{4} = 6$$