FM 5002 Spring 2012, Second midterm exam
Ending time for in-person students: 8:00pm on Wednesday 4 April 2012
Time for exam: 1 HOUR (ONE HOUR)

For PROCTORS of online students:
Email scan to: adams@math.umn.edu
Preferred FAX: 612-624-6702 Alternate FAX: 612-626-2017
Exam must be received by 24 hours after the ending time for in-person students. Thank you.

STUDENT, PLEASE PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Turn off all handheld devices, including cell phones.

Show work; a correct answer, by itself, may be insufficient for credit. Arithmetic need not be simplified, unless the problem requests it.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

STUDENT, PLEASE REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 0045(28), 3 pts.) The **covariance** of two PCRVs X and Y is defined by $\mathrm{Cov}[X,Y] = \cdots$.

$$\frac{(\mathrm{Var}[X+Y]) - (\mathrm{Var}[X]) - (\mathrm{Var}[Y])}{2}$$

b. (Topic 0045(10), 3 pts.) Let X be a PCRV and let $F := \{x \in \mathbb{R} \mid \Pr[X = x] > 0\}$. The **distribution** of X is the function $h : F \to (0,1]$ defined by $h(x) = \cdots$.

$$\Pr[X=x]$$

c. (Topic 0045(28), 3 pts.) Two PCRVs X and Y are **uncorrelated** if ...

$$\mathrm{Cov}[X,Y]=0$$

d. (Topic 0045(36), 3 pts.) A PCRV X is **standard** if ....

$$(E[X]=0) \quad \text{and} \quad (\mathrm{Var}[X]=1)$$

e. (Topic 0042(43), 3 pts.) A function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z \in \mathbb{C}$ if ....

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{(f(z+h)) - (f(z))}{h} \quad \text{exists}$$

II. True or False. (No partial credit.)

a. (Topic 0045(41), 3 pts.) If X and Y are PCRVs with the same distribution, then $\mathrm{Cov}[X,Y] = \mathrm{Var}[X] = \mathrm{Var}[Y]$.

F

b. (Topic 0045(17,37), 3 pts.) If X is a PCRV and $\mathrm{SD}[X] = 0$, then X is deterministic.

T

c. (Topic 0046(2-12), 3 pts.) Any symmetric, positive semidefinite matrix is the variance-covariance matrix of some ordered set of PCRVs.

T

d. (Topic 0047(13), 3 pts.) If X and Y are independent PCRVs, then the distributions of X and Y cannot be the same.

F

e. (Topic 0045(14), 3 pts.) If the joint distribution of (A, B) is the same as that of (X,Y), then the distribution of $A + B$ is the same as that of $X + Y$.

T

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PLEASE DO NOT WRITE BELOW THE LINE

I. abcde

II.abcde

III(1).

III(2,3).

III(4).

III(5).

III(6).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 0044(17-21), 10 pts.) Compute $\int_{-\infty}^{\infty} xe^x e^{-x^2/2}\,dx$.

$x \mapsto x+1$ $\quad \parallel$

$$\int_{-\infty}^{\infty} (x+1)\, e^{1^2/2} e^{-x^2/2}\, dx$$

$$\parallel$$

$$e^{1/2}\left[\int_{-\infty}^{\infty} xe^{-x^2/2}\,dx\right] + e^{1/2}\left[\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right]$$

$$\parallel$$

$$e^{1/2}[0] + e^{1/2}\left[\sqrt{2\pi}\right]$$

$$\parallel$$

$$\sqrt{2\pi e}$$

2. (Topic 0045(49-52), 10 pts.) Let X and Y be PCRVs. Assume $\mathrm{SD}[X] = 3$, $\mathrm{SD}[Y] = 5$ and $\mathrm{Corr}[X,Y] = 0.4$. Find the number s that minimizes $\mathrm{SD}[X - sY]$.

$$\mathrm{Var}[X - sY] = (\mathrm{Var}[X]) - 2s(\mathrm{Cov}[X,Y]) + s^2(\mathrm{Var}[Y])$$

$$= 9 - 2s(3)(5)(0.4) + s^2(25)$$

$$= 25s^2 - 12s + 9$$

$$\frac{d}{ds}(\mathrm{Var}[X - sY]) = 0 \implies 50s - 12 = 0$$

$$\implies s = \frac{12}{50} = \frac{6}{25}$$

3. (Topic 0045(30), 10 pts.) Let $X_1, X_2, X_3, \ldots, X_{100}$ be identically distributed sequence of PCRVs, all with mean μ and standard deviation σ. Assume, for all integers $j, k \in [1, 100]$ that, if $j \neq k$, then $\mathrm{Cov}[X_j, X_k] = 0$. (That is, assume that $X_1, X_2, X_3, \ldots, X_{100}$ are pairwise uncorrelated.) Let $Y := X_1 + \cdots + X_{100}$. Suppose $\mathrm{E}[Y] = 200$ and $\mathrm{SD}[Y] = 50$. Compute μ and σ.

$$\mu = \frac{200}{100} = 2$$

$$\sigma = \frac{50}{\sqrt{100}} = 5$$

4. (Topic 0046(38), 10 pts.) Let X and Y be two uncorrelated standard PCRVs. Choose $a, b, c \in \mathbb{R}$ such that $\mathrm{Var}[aX] = 9$, $\mathrm{Var}[bX + cY] = 50$ and $\mathrm{Cov}[aX, bX + cY] = 21$ and such that $a, c \geq 0$.

$a^2 = 9, \quad a \geqslant 0 \quad \therefore \boxed{a = 3}$

$ab = 21, \quad a = 3 \quad \therefore \boxed{b = 7}$

$b^2 + c^2 = 50, \quad b = 7 \quad \therefore c^2 = 50 - 49 = 1$

$c^2 = 1, \quad c \geqslant 0 \quad \therefore \boxed{c = 1}$

5. (Topic 0042(23), 15 pts.) Let $\omega := 5x\,dy + 3y\,dx$ and let R be the rectangle $(2,5)\times(6,8)$ in $\mathbb{R}^2$. Compute $\int_{\partial R} \omega$.

$\|$Stokes

$$\int_R d\omega = \int_R 5\,dx \wedge dy + 3\,dy \wedge dx$$

$$= \int_R (5-3)\,dx \wedge dy = \int_R 2\,dx \wedge dy$$

$$= \iint_R 2\,dx\,dy = \int_6^8 \int_2^5 2\,dx\,dy$$

$$= \int_6^8 2\cdot(5-2)\,dy$$

$$= 2\cdot(5-2)\cdot(8-6)$$

$$= 2\cdot 3\cdot 2 = 12$$

6. (Topic 0038(49), 15 pts.) Let $M := \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$, let $p := \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and let $q := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. I give you the eigenvalues and eigenvectors of M:

$$Mp = 3p \qquad \text{and} \qquad Mq = 4q \quad .$$

Define a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ by $V(x,y) = L_M(x,y) = (5x+y, -2x+2y)$. The flowline $\gamma : \mathbb{R} \to \mathbb{R}^2$ of V footed at $(1,-2)$ has the form

$$\gamma(t) = (ae^{rt}, be^{st}) \quad ,$$

for some constants a, b, r and s. Find a, b, r and s.

$$\begin{bmatrix} ae^{rt} \\ be^{st} \end{bmatrix} = e^{tM} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = e^{tM} p = e^{3t} p = e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \end{bmatrix}$$

$a = 1$, $r = 3$,

$b = -2$, $s = 3$