Financial Mathematics

Basics of piecewise constant random variables

0045-1.

Let X and Y be identically distributed PCRVs.
Assume $\Pr[X = 3] = 0.1$ and $\Pr[X = 5] = 0.9$.
Let $\rho \in [-1, 1]$.
Assume $\mathrm{E}[XY] = (\mathrm{E}[Y])^2 + \rho(\mathrm{Var}[Y])$

a. Compute $\mathrm{E}[X + Y]$.

b. Compute $\mathrm{Var}[X + Y]$.
Your answer will depend on ρ.

c. Find the value of $\rho \in [-1, 1]$
where $\mathrm{Var}[X + Y]$ is minimized.

d. Find the value of $\rho \in [-1, 1]$
where $\mathrm{Var}[X + Y]$ is maximized.

0045-2. Let X be a PCRV
s.t. $\Pr[X = 1] = 0.2$, s.t. $\Pr[X = 3] = 0.3$.
and s.t. $\Pr[X = 9] = 0.5$.
Find $\mathrm{E}[X]$, $\mathrm{Var}[X]$ and $\mathrm{SD}[X]$.

0045-3. Let X be a binary PCRV
s.t. $\Pr[X = 4] = 0.85$
and s.t. $\Pr[X = 24] = 0.15$.
Find the mean, the variance and
the standard deviation of X.

0045-4. Let $W := \begin{cases} -2, \text{ if } 0 \leq \omega < 0.4 \\ -4, \text{ if } 0.4 \leq \omega \leq 1. \end{cases}$

Let $X := \begin{cases} 3, \text{ if } 0 \leq \omega < 0.4 \\ 6, \text{ if } 0.4 \leq \omega \leq 1. \end{cases}$

Let $Y := \begin{cases} 10^7, \text{ if } 0 \leq \omega \leq 0.4 \\ 10^9, \text{ if } 0.4 < \omega \leq 1. \end{cases}$

Let $Z := \begin{cases} 3, \text{ if } 0 \leq \omega \leq 0.4001 \\ 9, \text{ if } 0.4001 < \omega \leq 1. \end{cases}$

Compute a. $\text{Corr}[W, X]$
b. $\text{Corr}[W, Y]$
c. $\text{Corr}[X, Y]$
d. $\text{Corr}[X, Z]$
e. $\text{Corr}[Y, Z]$

0045-5.

Let C_1 and C_2 be our standard coin flipping random variables.

Let $\rho \in [-1, 1]$.

Find constants $a, b, c, d \in \mathbb{R}$

s.t. if we define

$$X := aC_1 + bC_2$$

and $Y := cC_1 + dC_2$

then we have

$$\mathrm{SD}[X] = 2,$$

$$\mathrm{SD}[Y] = 7$$

and $\mathrm{Corr}[X, Y] = \rho,$

and s.t. at least one of a, b, c, d is zero.

(Some of a, b, c, d will be expressions of ρ.)

0045-6.

Show an example of two PCRVs X and Y
s.t. $\text{Cov}[X, Y] = 0$,
but s.t. X and Y are not independent.

0045-7.

Let X be a PCRV.

a. Let $v := \text{Var}[X]$.
What is $\text{Cov}[-3X, -2X]$?
Your answer will be a (very simple) formula involving v.

b. What is $\text{Corr}[-3X, -2X]$?

0045-8. You manage a portfolio w/ three assets that, today, have per share prices of $\$1$, $\$1$ and $\$1$. Denote their prices, one year from now, by the PCRVs $\$B$, $\$P$ and $\$Q$, respectively. Your market analyst tells you:

$$\mathrm{E}[B] = 1.01, \quad \mathrm{E}[P] = 1.16, \quad \mathrm{E}[Q] = 1.08,$$
$$\mathrm{Var}[B] = 0, \quad \mathrm{Var}[P] = 0.3, \quad \mathrm{Var}[Q] = 0.2,$$
$$\mathrm{Cov}[B, P] = 0, \quad \mathrm{Cov}[B, Q] = 0$$
$$\text{and } \mathrm{Cov}[P, Q] = 0.1.$$

You plan to buy b, p and q shares resp., but you only have $\$5$ to invest, and you wish to achieve a 4% expected return.

Find the portfolio that minimizes variance.

NOTE: b, p and q may be any real numbers; they're NOT constrained to be integers and they're NOT constrained to be positive.