Financial Mathematics

From Stirling's Formula to the Central Limit Theorem

0051-1. $\forall$integers $n \geq 1$, let H_n be the nth histogram, as def'd as defined in this topic.

$\forall$integers $n \geq 1$, $\forall x \in \mathbb{R}$, define $p_n(x)$ as follows:

If x is outside the span of H_n, then $p_n(x) = 0$.

If x is under exactly one bar of H_n,
then $p_n(x)$ is the height of that bar.

If x at the bordeline of two bars,
then $p_n(x)$ is the height of the
leftmost of those two bars.

Let $p(x) := [e^{-x^2/2}]/\sqrt{2\pi}$.

We proved, in this topic, that

$$\forall x \in \mathbb{R}, \qquad \lim_{n \to \infty} p_n(x) = p(x).$$

cont'd below

$\forall$integers $n \geq 1$, $\forall x \in \mathbb{R}$, define $f_n(x)$ as follows:

If x is outside the span of H_n, then $f_n(x) = 0$.

If x is under exactly one bar of H_n,
then $f_n(x)$ is the midpoint of
the base of that bar.

If x at the borderline of two bars,
then $f_n(x)$ is the midpoint of
the base of the leftmost of those two bars.

a. Show, $\forall x \in \mathbb{R}$, that $\lim\limits_{n \to \infty} f_n(x) = x$.

b. Let $g(x) = (e^{2x} - 3)_+$. Show that

$$\mathrm{E}\left[g(D_{2n}/\sqrt{2n})\right] = \int_{-\infty}^{\infty} g(f_n(x))[p_n(x)]\,dx.$$

cont'd below

0051-1 (cont'd). Remark (not assigned):

Let C_n be the usual coin-flipping PCRVs.

Let $D_n := C_1 + \cdots + C_n$.

A version of the CLT says: $\forall$reasonable fn ϕ,

$$\lim_{n\to\infty} \mathrm{E}[\phi(D_{2n}/\sqrt{2n})] = \int_{-\infty}^{\infty} [\phi(x)][p(x)]\, dx.$$

Let's prove this for $\phi(x) = g(x)$:

By the "Dominated Convergence Theorem" (taught in FM 5011), one can prove:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g(f_n(x))[p_n(x)]\, dx$$

$$= \int_{-\infty}^{\infty} \lim_{n\to\infty} g(f_n(x))[p_n(x)]\, dx.$$

That is, $\lim_{n\to\infty} \mathrm{E}\left[g(D_{2n}/\sqrt{2n})\right] = \int_{-\infty}^{\infty} [g(x)][p(x)]\, dx$,

as desired.