Financial Mathematics

Central Limit Theorem

0059-1. Let $C_1, C_2, \ldots$ be the usual sequence of $\{\pm 1\}$-valued PCRVs that model coin-flipping. $\forall$integers $n \geq 1$, let $D_n := C_1 + \cdots + C_n$.

a. Compute $\lim_{n\to\infty} \mathrm{E}[(D_n/\sqrt{n})^4 + (D_n/\sqrt{n})^5]$.

b. Let $f(x) = \begin{cases} 4x, & \text{if } 1 \leq x \leq 7 \\ 0, & \text{otherwise.} \end{cases}$

Compute $\lim_{n\to\infty} \mathrm{E}[f(D_n/\sqrt{n})]$.

c. Compute $\lim_{n\to\infty} \mathrm{E}[(D_n/\sqrt{n})^4 - (D_n/\sqrt{n})^6]$.

d. Compute $\lim_{n\to\infty} \mathrm{E}[(D_n/\sqrt{n})^9 + (D_n/\sqrt{n})^2]$.

0059-2.

a. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{2x-9}e^{-x^2/2}\,dx.$

b. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-1}^{2} e^{2x-9}e^{-x^2/2}\,dx.$

c. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} (e^{2x-9}-2)e^{-x^2/2}\,dx.$

d. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} (e^{2x-9}-2)_+e^{-x^2/2}\,dx.$

e. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} (e^{2ix-9}-2)e^{-x^2/2}\,dx.$

$i=\sqrt{-1}$

0059-3. Let X be a binary PCRV
s.t. $\Pr[X = a] = p$
and $\Pr[X = b] = q$. ($p + q = 1$)

Let $f(t)$ be the Fourier transform of the distribution of X.

a. Compute $E[X^4]$.

b. Compute $E[X^5]$.

c. Compute $f(0)$.

d. Compute $f^{(4)}(0)$, the value at 0 of the fourth derivative of f.

e. Compute $f^{(5)}(0)$, the value at 0 of the fifth derivative of f.

(Answers should be expressions of a, b, p, q.)

0059-4. Let X be a PCRV

s.t. $\Pr[X = a] = p$,

$\Pr[X = b] = q$ $\quad (\quad p + q + r = 1 \quad)$

and $\Pr[X = c] = r$.

Let $f(t)$ be the Fourier transform of the distribution of X.

a. Compute $E[X^4]$.

b. Compute $E[X^5]$.

c. Compute $f(0)$.

d. Compute $f^{(4)}(0)$, the value at 0 of the fourth derivative of f.

e. Compute $f^{(5)}(0)$, the value at 0 of the fifth derivative of f.

(Answers should be expressions of a, b, c, p, q, r.)

0059-5. Let X be a PCRV whose distribution satisfies:

$$\Pr[X = -2] = 0.4$$
$$\Pr[X = 0] = 0.4$$
$$\Pr[X = 4] = 0.2$$

a. Find the generating function of the distribution of X.

b. Find the Fourier transform of the distribution of X.

c. Let $X_1, X_2, \ldots$ be an iid sequence of PCRVs, all with the same distribution as X.

Find the Fourier transform of the distribution of

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}.$$

0059-6. Let X be a PCRV whose distribution satisfies:

$$\Pr[X = -2] = 0.4$$
$$\Pr[X = 0] = 0.4$$
$$\Pr[X = 4] = 0.2$$

The n**th** **raw moment** of X is $\mathrm{E}[X^n]$.

The **moment generating function** is obtained from the generating function after one replaces z by e^s.

Let $\alpha(s)$ be the moment generating fn of the distribution of X.

Let $\beta(t)$ be the Fourier transform of the distribution of X.

a. Find $\alpha(s)$.

b. Find $\alpha'''(0)$.

c. Find the third raw moment of X.

d. Find $\beta'''(0)$.

0059-7. Let X be a binary PCRV s.t. $\Pr[X = a] = p$ and $\Pr[X = b] = q$. ($p + q = 1$)

Let $f(t)$ be the Fourier transform of the distribution of X.

Assume $\mathrm{E}[X] = 0$, *i.e.*, $pa + bq = 0$.

Assume $\mathrm{SD}[X] = 0.5$, *i.e.*, $\sqrt{pq}(b - a) = 0.5$.

Compute $\lim_{n \to \infty} [f(t/\sqrt{n})]^n$.