Financial Mathematics

Regular review session C, Final

Discussion:

n-jet at a of f

nth order Maclaurin approximation of f

Maclaurin expansion of f

the renormalized nth power of f

$e^{it} =$?????

Discussion:

for one-variable functionals:

- Maclaurin approximation
- Maclaurin expansion
- k-jet
- increasing, decreasing on I
- increasing vs. positive derivative

for multi-variable functionals:

- Maclaurin approximation
- Maclaurin expansion
- k-jet
- gradient
- Hessian
- Laplacian = trace of the Hessian

$f'(a)$ $\qquad (\nabla f)(a)$ $\qquad f''(a)$ $\qquad (\Delta f)(a)$

$$f''(a) = (Hf)(a) = (\text{Hess } f)(a) = (\nabla\nabla f)(a) = (\nabla^2 f)(a)$$

Discussion:

for multi-variable functions $\mathbb{R}^n \to \mathbb{R}^m$:

gradient

Hessian

$f'(a)$ ~~$f''(a)$~~

for one-variable functions $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$:

chain rule:

$$(g \circ f)'(p) = [\, g'(f(p)) \,][\, f'(p) \,]$$

for multi-variable functions $\mathbb{R}^l \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^n$,

chain rule:

$$(g \circ f)'(p) = [\, g'(f(p)) \,][\, f'(p) \,]$$

$n \times l$ $\quad$ $n \times m$ $\quad$ $m \times l$

Discussion: vector field

flowline = integral curve

"footed at" existence? uniqueness?

compute: flowline footed at p of L_M

inhomogeneous linear?

integration of a vector field through a point

integration of a vector field

contraction $X \to X$ (w.r.t. dist)

contraction factor

Fact: Contractions have fixpts,

provided dist is complete.

Discussion:

derivative of e^x w.r.t. x

derivative of $e^x + 1$ w.r.t. x

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gradient of $x^2 + y^2 + z^3$ w.r.t. (x, y, z)

gradient of $x^2 + y^2 + z^3 + 1$ w.r.t. (x, y, z)

reverse gradient   reverse gradient flow

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integrate $V(x, y) = (x, y)$ through $(2, 3)$

Cauchy-Riemann equations

$U + iV$ is complex analytic iff

$\partial_x U = \partial_y V$ and $\partial_y U = -\partial_x V$

Cauchy-Schwarz inequality $|v \cdot w| \leq |v| \cdot |w|$

$$|B(v, w)| \leq \sqrt{[Q(v)][Q(w)]}$$

Discussion:

critical point of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

second derivative test for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
at a critical point

positive definite test

$f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$
critical point of f on $g^{-1}(k)$

e.g., $f(x, y) = 3x^3 + 2xy$,
$g(x, y) = 2(\sin x) + (\cos(xy))$, $k = 1$

Set up, don't solve.

Discussion:

$$\int_L \omega := \cdots, \qquad \omega = p(x,y)\,dx + q(x,y)\,dy$$

$$\int_L \omega := \cdots, \qquad d\omega = p(z)\,dz$$

$$\int_C \omega := \cdots, \qquad \omega = p(x,y)\,dx + q(x,y)\,dy$$

$$\int_C \omega := \cdots, \qquad d\omega = p(z)\,dz$$

$$\int_R \omega := \cdots, \qquad \omega = p(x,y)\,dx \wedge dy$$

$$\partial R = \cdots, \qquad dF = \cdots$$

$$d\omega = \cdots, \quad \omega = P\,dx + Q\,dy$$

Green's Theorem, Cauchy's Theorem

Discussion: Two variables: x and y

C a 1-chain, ω a 1-form $\quad \int_C \omega :=$

R a 2-rectangle $\quad \int\int_R f(x,y)\,dx\,dy :=$

R a 2-rectangle, ω a 2-form $\quad \int_R \omega :=$

ω, η both 1-forms $\quad \omega \wedge \eta$

ω a 0-form $\quad d\omega$

ω a 1-form $\quad d\omega$

$$d[(f(x)\,dx) + (f(y)\,dy)] =$$

Counterclockwise boundary of R

Stokes' Theorem

Discussion:

PCRV

distribution

partition of a PCRV

finer vs. coarser

expected value — E linear?

variance — Var linear?

standard deviation — SD linear?

standard PCRV

covariance

correlation — uncorrelated

independence

$\Pr[E|F]$ — $\mathrm{Odds}[E|F]$

Bayes, odds-Bayes

2nd odds-Bayes

Discussion:

Fourier transform of distribution of X

Fourier transform of distribution of $X + Y$
assuming X and Y are independent.

Fourier transform of distribution of $X/8$

Discussion:

Partition of a PCRV
coarse vs. fine

measurability of a PRCV w.r.t. a partition

Conditional expectation
of a PCRV, given an event
Conditional expectation
of a PCRV, given a partition
Conditional expectation
of a PCRV, given another PCRV
Taking out what you know (a.k.a. linearity)

The Tower Law

The "Power" Tower Law

Discussion:

Example of two different PCRVs with the same distribution.

What implies what?

$\mathsf{E}[X|Y]$ is deterministic

X is independent of Y

Y is independent of X

$\mathrm{Cov}[X,Y] = 0$

$\mathrm{Corr}[X,Y] = 0$

$\mathsf{E}[X|Y] = \mathsf{E}[X]$

$\mathsf{E}[X|X] = \cdots$

$\mathrm{Corr}[X,X] = \cdots$

$\mathrm{Corr}[X,2X] = \cdots$

Condition that ensures:

$$\mathsf{E}[\ \mathsf{E}[X|\mathcal{S}]\ \mid\ \mathcal{R}] = \mathsf{E}[\ X\ \mid\ \mathcal{R}\ ]$$

Discussion:

$$\mathsf{E}[X] = a(\mathsf{E}[X])$$

$$\mathsf{Var}[aX] =$$

$$\mathsf{SD}[aX] =$$

$$\mathsf{E}[X + Y] = (\mathsf{E}[X]) + (\mathsf{E}[Y])$$

$$\mathsf{Var}[X + Y] =$$

$$\mathsf{SD}[X + Y] =$$

Discussion:

$\forall$PCRVs X, Y, $\quad X * Y := \text{Cov}[X, Y]$

v, w both $n \times 1$ column vectors of PCRVs

$\quad \text{Cov}[v, w] := v * w^t \in \mathbb{R}^{n \times n}$

A, B constant $n \times n$ matrices

$$\begin{aligned} \text{Cov}[Av, Bw] &= (Av) * (Bw)^t \\ &= (Av) * (w^t B^t) \\ &= A(v * w^t) B^t \\ &= A(\text{Cov}[v, w]) B^t \end{aligned}$$

Say PCRVs in v are variance 1, uncorrelated

$$\begin{aligned} \text{Cov}[Av, Av] &= A(\text{Cov}[v, v]) A^t \\ &= AA^t \end{aligned}$$

Cholesky decomposition

conditions: symmetric, pos semidef

Discussion:

asymptotics of $n!$

asymptotics of $\binom{2n}{n}$.

$\lim_{n\to\infty} \mathsf{E}[f(Z_n)] =$

$X_n \to aZ + b$ in distribution

$X_n \to aZ + b$ in distr. against contin.,exp-bdd

$X_n \to 2Z + 3 \quad \Rightarrow \quad \mathsf{E}[X_n^6] \to ?$

$(C_1 + \cdots + C_n)/? \to ??$

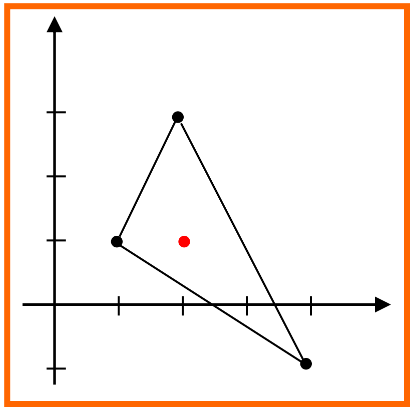
$\triangle$CLT: $X_n \in \sum^n \mathcal{B}_{q_n}^{p_n}, \ \ldots \quad \Rightarrow \quad X_n \to \sigma Z + \mu$

$\mathsf{E}[e^{\sigma Z + \mu}] = e^{??}$

Homework#(03/05)-1

(03/05)-1-1c. Compute $\int_{1+i}^{4-i} e^x \, dx$.

(03/05)-1-1d. Show that the complex limit



$$\lim_{h \to 0} \frac{e^{2+i+h} - e^{2+i}}{h} \quad \text{exists,}$$

and is equal to e^{2+i}.

Homework#(03/05)-1

(03/05)-1-2c. Compute $\int_i^{3i+2} \overline{x}\, dx$.

(03/05)-1-1d. Show that the complex limit

$$\lim_{h \to 0} \frac{\overline{1+2i+h} - \overline{1+2i}}{h}$$

does not exist.

1. Compute $\int_{2+2i}^{4+6i} e^{x/2}\,dx$.

a. $e^{(2+3i)-(1+i)}$

c. $2(e^{2+3i} - e^{1+i})$

b. $e^{2+3i} - e^{1+i}$

d. $(1/2)(e^{2+3i} - e^{1+i})$

e. None of the above

Antiderivative for $e^{x/2}$: $\dfrac{e^{x/2}}{1/2} = 2(e^{x/2})$

$$\int_{2+2i}^{4+6i} e^{x/2}\,dx = [2(e^{x/2})]_{x=2+2i}^{x=4+6i}$$

$$= [2(e^{2+3i} - e^{1+i})]$$

$f(x) = [\cos x]^{-3}$ | $g(x) = [e^{-x^2/2}]^{-3} = e^{3x^2/2}$

$$[g(x/\sqrt{N})]^N = [e^{3x^2/(2\not{N})}]^{\not{N}} = e^{3x^2/2} = g(x)$$

g is a fixpoint

$[g(\underset{7}{x}/\sqrt{\underset{100}{N}})]^{\overset{100}{N}} = g(\underset{7}{x})$ | $[g(7/\sqrt{100})]^{100} = g(7)$

2. Let $f(x) = \sec^3 x$.
Let $g(x) = \lim_{n\to\infty} [f(x/\sqrt{n})]^n$.

Compute $[g(7/\sqrt{100})]^{100} - [g(7)]$.

a. 0
b. -1
c. $e^{7\cdot 3^2/2}$
d. $e^{-3\cdot 7^2/2}$
e. None of the above

Let W, X and Y

all have the same partition.

Then $\mathrm{E}[W|Y] = W$

and $\mathrm{E}[X|Y] = X$,

but it's not necessarily true that $W = X$.

2. True or False: If W, X, Y are PCRVs,
and W and X have the same partition,
then $\mathrm{E}[W|Y] = \mathrm{E}[X|Y]$.

t. True

f. False

Homework#(03/05)-1

(03/05)-1-3.

Assume $f(0) = 3$, $f'(0) = 4$ and $f''(0) = 5$.

Assume $f'''(t) \leq 1$, for all $t \in [0, 4]$.

What is the maximum possible value of $f(4)$?

(03/05)-1-4.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x[\cos(x)]$.

Let p be the second order Macl. approx. of f.

a. Compute $p(x)$.

b. Compute $p(0.1)$.

c. Compute $f(0.1)$.

Homework#(02/27)-1

(02/27)-1-4. Let X be a binary PCRV
s.t. $\Pr[X = a] = p$
and $\Pr[X = b] = q$. ($p + q = 1$)

Let $f(t)$ be the Fourier transform of the distribution of X.

Assume $E[X] = 0$.

Compute $\lim_{N \to \infty} [f(x/\sqrt{N})]^N$.

(Answers should be expressions of a, b, p, q.)

1-1. Find the first order Macl. expansion of $\tan x$.

a. $1 + x$

b. x

c. $1 + (x/6)$

d. $x^2/3$

e. None of the above

$f(x) = \tan x$ $\quad f(0) = 0$

$f'(x) = \sec^2 x$ $\quad f'(0) = 1$

$p(x) = x$ $\quad p(0) = 0$

$p'(x) = 1$ $\quad p'(0) = 1$

$$f(t) = \mathrm{E}[e^{-itX}]$$

$$f'(t) = \mathrm{E}[e^{-itX}(-iX)]$$

$$f''(t) = \mathrm{E}[e^{-itX}(-iX)^2]$$

$$f''(0) = \mathrm{E}[(-iX)^2] = \mathrm{E}[-X^2] = -\mathrm{E}[X^2]$$

1-2. Let X be a PCRV. Let $f(t) = \mathrm{E}[e^{-itX}]$. Find a formula for $f''(0)$.

a. $\mathrm{E}[X]$

b. $\mathrm{Var}[X]$

c. $-\mathrm{E}[X^2]$

d. $-\mathrm{Var}[X]$

e. None of the above

2-1. T or F: If two PCRVs have the same distribution, then they are equal.

T. True

F. False

Counterexample: C_1 and C_2.

T or F: If two PCRVs have the same distribution, then they have equal expectation and variance?

Counterexample: C_1 and C_2.

2-2. T or F: If two PCRVs are independent then they are equal.

T. True

F. False

1. Suppose $\Pr[B|A] = 6\%$,

$\Pr[B] = 4\%$ and $\Pr[A] = 2\%$.

Find $\Pr[A|B]$.

a. 6%

b. 12%

c. 1%

d. 3%

e. None of the above

$$\Pr[A|B] = \Pr[B|A] \cdot \frac{\Pr[A]}{\Pr[B]}$$

$$= (6\%) \cdot \frac{2\%}{4\%} = 3\%$$

Let W, X and Y
all have the same partition.

Then $\mathrm{E}[W|Y] = W$
and $\mathrm{E}[X|Y] = X$,
but it's not necessarily true that $W = X$.

2. True or False: If W, X, Y are PCRVs,
and W and X have the same partition,
then $\mathrm{E}[W|Y] = \mathrm{E}[X|Y]$.

t. True

f. False

1. True or False:
There exists a 2×2 real matrix A s.t.

$$AA^t = \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}$$

t. True

f. False

$$\forall A, v, \quad AA^t v \cdot v = A^t v \cdot A^t v \geq 0$$

Then $\forall A$, AA^t is pos. semidef.

so $\det(AA^t) \geq 0$.

$$\det \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix} = 21 - 25 < 0$$

Let $W := 2X + 3Y$.

$$\begin{aligned}\mathrm{Corr}[2X + 3Y, 4X + 6Y] &= \mathrm{Corr}[W, 2W] \\ &= \mathrm{Corr}[W, W] \\ &= 1\end{aligned}$$

2. Say X, Y independent, std PCRVs. Compute $\mathrm{Corr}[2X + 3Y, 4X + 6Y]$.

a. -1

c. 1

b. 0

d. 0.6

e. None of the above

1. Compute $\boxed{\frac{1}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} x^8 e^{-x^2/2}\,dx$.

a. 8!

c. 0

b. $(3)(5)(7)\sqrt{2\pi}$

d. $\boxed{(3)(5)(7)}$

e. None of the above

$$\int_{-\infty}^{\infty} x^8 e^{-x^2/2}\,dx = 7\int_{-\infty}^{\infty} x^6 e^{-x^2/2}\,dx$$

$$= 7\cdot 5\int_{-\infty}^{\infty} x^4 e^{-x^2/2}\,dx = 7\cdot 5\cdot 3\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$$

$$= 7\cdot 5\cdot 3\cdot 1\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 7\cdot 5\cdot 3\cdot \cancel{1}\cdot \sqrt{2\pi}$$

Now divide by $\sqrt{2\pi}$.

Eigenvalues: 3, 2 and 0

Symmetric? Yes.

2. T or F: The matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is positive semi-definite.

T. True

F. False

Homework#(05/07)-1 ungraded

(05/07)-1-5. Maximize $2x - 5y$ subject to the constraint $x^4 + y^4 = 1$.

(05/07)-1-6. Minimize $7x - y$ subject to the constraint $x^6 + y^6 = 1$.

(05/07)-1-7a. For every integer $n \geq 1$, maximize $x + y$ subject to the constraint $x^{2n} + y^{2n} = 1$.

(05/07)-1-7b. Let (a_n, b_n) denote the answer to (05/07)-1-7a. Compute $\lim_{n\to\infty} (a_n, b_n)$.

Homework#(04/30)-1

(04/30)-1-3.

Compute $d(ze^{xy})$, the exterior derivative of

$$ze^{xy},$$

with respect to x, y, z.

(04/30)-1-4.

Compute $d(xy\,dx + ye^z\,dy - \sin(xz)\,dz)$, the exterior derivative of

$$xy\,dx + ye^z\,dy - \sin(xz)\,dz,$$

with respect to x, y, z.

Homework#(04/30)-1

(04/30)-1-5.

$$\text{Compute } \int_{(1,2,3)}^{(4,5,6)} x\,dx + y\,dy + xe^z\,dz,$$

$$\textit{i.e.}, \text{ compute } \int_L x\,dx + y\,dy + xe^z\,dz,$$

where L is the directed line segment from $(1,2,3)$ to $(4,5,6)$.

(04/30)-1-6. Let $R := (1,2) \times (3,4)$.

$$\text{Compute } \int_R [e^{x+y}]\,dy \wedge dx.$$

Homework#(02/20)-1

(02/20)-1-2.

a. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} xe^{-x^2/2}\,dx$.

b. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-5}^{7} xe^{-x^2/2}\,dx$.

c. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} x^4e^{-x^2/2}\,dx$.

d. Compute $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} x^5e^{-x^2/2}\,dx$.

Homework#(02/20)-1

(02/20)-1-3.

a. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{3x+4}e^{-x^2/2}\,dx$.

b. Compute $\frac{1}{\sqrt{2\pi}}\int_{-5}^{7} e^{3x+4}e^{-x^2/2}\,dx$.

c. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (e^{3x+4}-5)e^{-x^2/2}\,dx$.

d. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (e^{3x+4}-5)_{+}e^{-x^2/2}\,dx$.

e. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (e^{3ix+4}-5)e^{-x^2/2}\,dx$.

$i=\sqrt{-1}$

STOP

IRREGULARITIES: