

# Financial Mathematics

## The binomial formula

$$(x + y)^0 = \underline{1} \quad 2^0 = 1 \text{ terms}$$

$$(x + y)^1 = \underline{x + y} \quad 2^1 = 2 \text{ terms}$$

$$\begin{aligned}(x + y)^2 &= \underline{x(x + y) + y(x + y)} \\ &= \underline{xx + xy + yx + yy} \quad \text{d duplication} \\ &\equiv \underline{x^2 + 2xy + y^2} \quad 2^2 = 4 \text{ terms}\end{aligned}$$

$$\begin{aligned}(x + y)^3 &= x(xx + xy + yx + yy) \\ &+ y(xx + xy + yx + yy)\end{aligned}$$

$$\begin{aligned}&\stackrel{\text{duplications}}{=} \underline{xxx + xxy + xyx + xyy} \\ &+ \underline{yxx + yxy + yyx + yyy} \quad 2^3 = 8 \text{ terms}\end{aligned}$$

$$\begin{aligned}(x + y)^4 &= x(xxx + xxy + xyx + xyy) \\ &+ y(xxx + xxy + xyx + xyy)\end{aligned}$$

$$\begin{aligned}&+ y(yxx + yxy + yyx + yyy) \\ &+ y(yxx + yxy + yyx + yyy) = \text{etc.} \quad \text{d duplications} \\ &2^4 = 16 \text{ terms}\end{aligned}$$

Lots of duplications... e.g.

$$(x + y)^5 = \begin{aligned} &xxxxx + xxxx y + xxxy x + xxxy y \\ &+ xxy x x + xxy xy + xx y y x + xx y yy \\ &+ xy x x x + xy x x y + xy x y x + xy x y y \\ &+ xy y x x + xy y x y + xy y y x + xy y y y \\ &+ y x x x x + y x x x y + y x x y x + y x x y y \\ &+ y x y x x + y x y x y + y x y y x + y x y y y \\ &+ y y x x x + y y x x y + y y x y x + y y x y y \\ &+ y y y x x + y y y x y + y y y y x + y y y y y \end{aligned}$$

*x<sup>4</sup>y*  



Start over, avoiding duplications...

$$2^5 = 32 \text{ terms}$$

$$(x+y)^0 \stackrel{x+y \neq 0}{=} 1 = 1$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$\begin{aligned}
 (x+y)^3 &= (x+y)(x+y)^2 \\
 &= (x+y)(1x^2 + 2xy + 1y^2) \\
 &= x(1x^2 + 2xy + 1y^2) \\
 &\quad + y(1x^2 + 2xy + 1y^2) \\
 &= 1x^3 + 2x^2y + 1xy^2 \\
 &\quad + 1x^2y + 2xy^2 + 1y^3 \\
 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3
 \end{aligned}$$

$$(x+y)^0 \stackrel{x+y \neq 0}{=} 1 = 1$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^3 = 1x^3 + 2x^2y + 1xy^2$$

$$+ 1x^2y + 2xy^2 + 1y^3$$

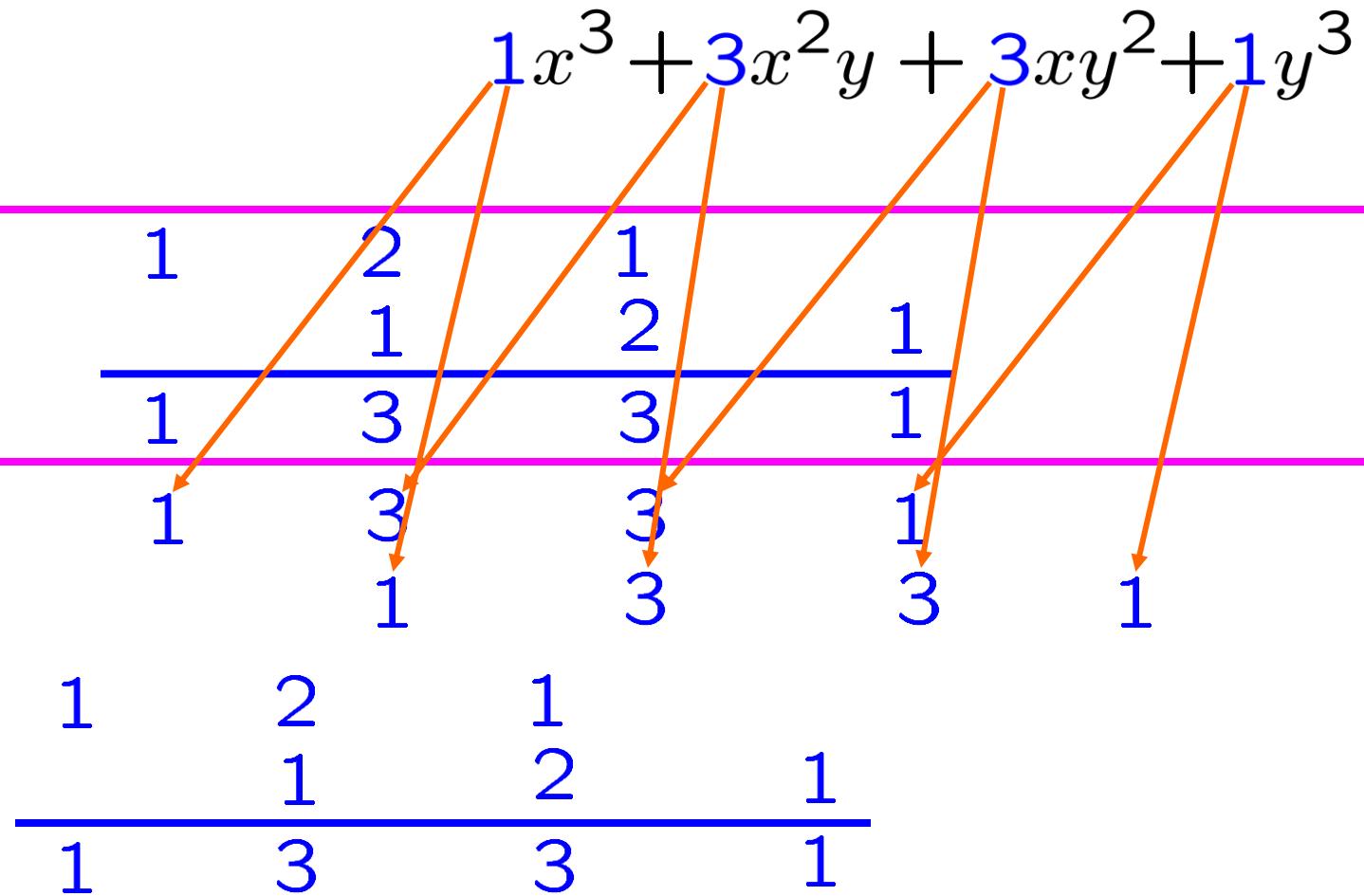
$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$= 1x^3 + \frac{2}{1}x^2y + \frac{1}{2}xy^2$$

$$+ \cancel{1x^2y + 3xy^2 + 1y^3}$$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$\begin{aligned}
 (x+y)^0 &= 1 & 1 &= 1 \\
 (x+y)^1 &= x + y & = 1x + 1y \\
 (x+y)^2 &= x^2 + 2xy + y^2 & = 1x^2 + 2xy + 1y^2 \\
 (x+y)^3 &= & 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
 (x+y)^4 &
 \end{aligned}$$



$$(x+y)^0 \stackrel{x+y \neq 0}{=} 1 = 1$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4$$

1	2	1		
	1	2	1	
<hr/>	<hr/>	<hr/>	<hr/>	
1	3	3	1	
<hr/>	<hr/>	<hr/>	<hr/>	
1	3	3	1	
	1	3	3	1
<hr/>	<hr/>	<hr/>	<hr/>	
1	4	6	4	1

$$(x+y)^0 = \text{ } x + y \neq 0$$

$$(x+y)^1 = 1x + 1y \quad 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy \quad 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y \quad 1x^3 + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The diagram illustrates the expansion of  $(x+y)^4$  using Pascal's triangle. The top row shows the expansion terms:  $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$ . Below this, four rows of Pascal's triangle are shown, each with blue numbers. Orange arrows point from each term in the expansion to its corresponding row in the triangle. The first row has 1. The second row has 1, 2, 1. The third row has 1, 3, 3, 1. The fourth row has 1, 4, 6, 4, 1. The fifth row has 1, 5, 10, 10, 5, 1.

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

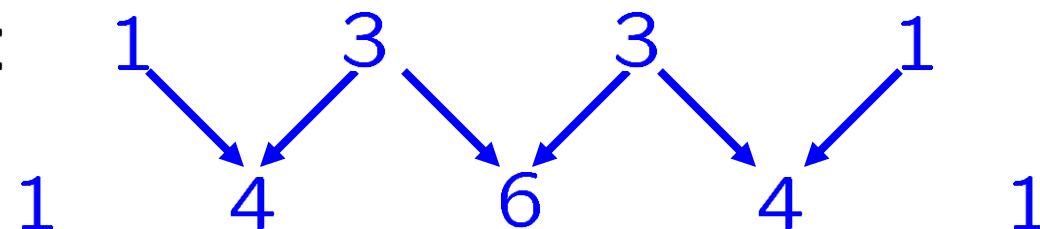
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1\boxed{x^4} + 4\boxed{x^3y} + 6\boxed{x^2y^2} + 4\boxed{xy^3} + 1\boxed{y^4}$$

Start with four xs.  
Change an x to a y.  
Continue...  
until...  
four ys.

1	3	3	1
	1	3	3
1	4	6	4
1	4	6	4

Easier:



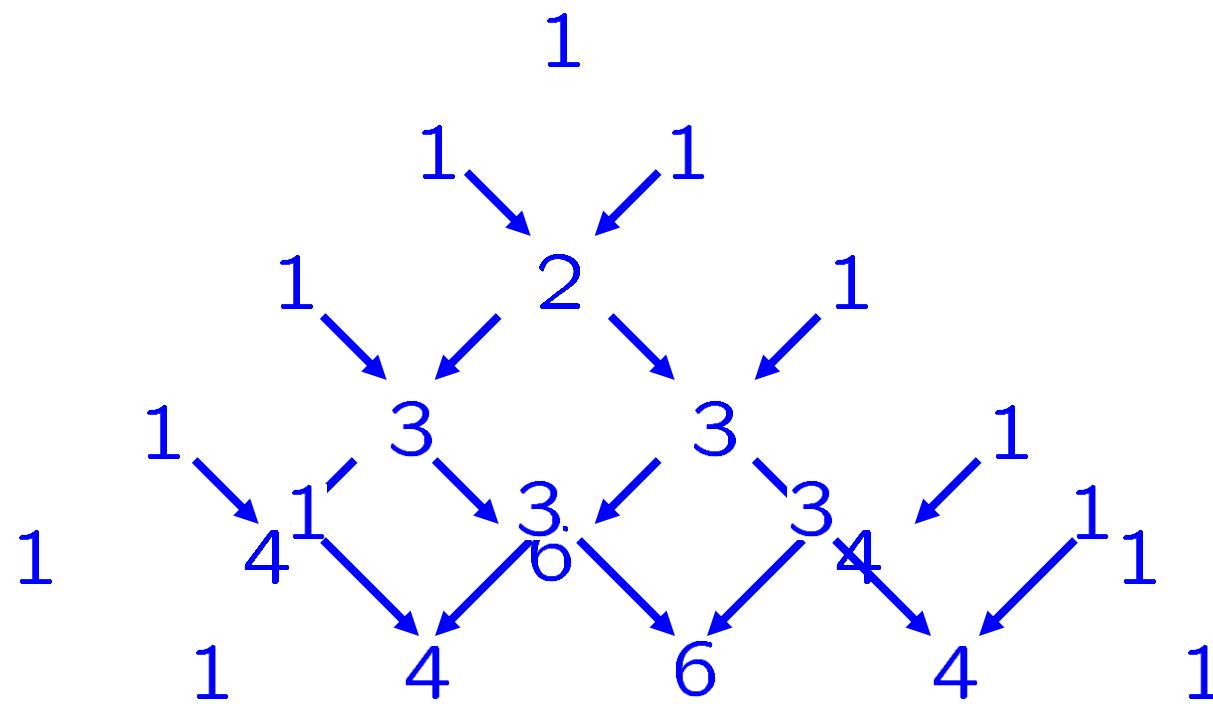
$$(x + y)^0 = \textcolor{red}{1}$$

$$(x + y)^1 = \textcolor{blue}{1}x + \textcolor{blue}{1}y$$

$$(x + y)^2 = \textcolor{blue}{1}x^2 + \textcolor{blue}{2}xy + \textcolor{blue}{1}y^2$$

$$(x + y)^3 = \textcolor{blue}{1}x^3 + \textcolor{blue}{3}x^2y + \textcolor{blue}{3}xy^2 + \textcolor{blue}{1}y^3$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$



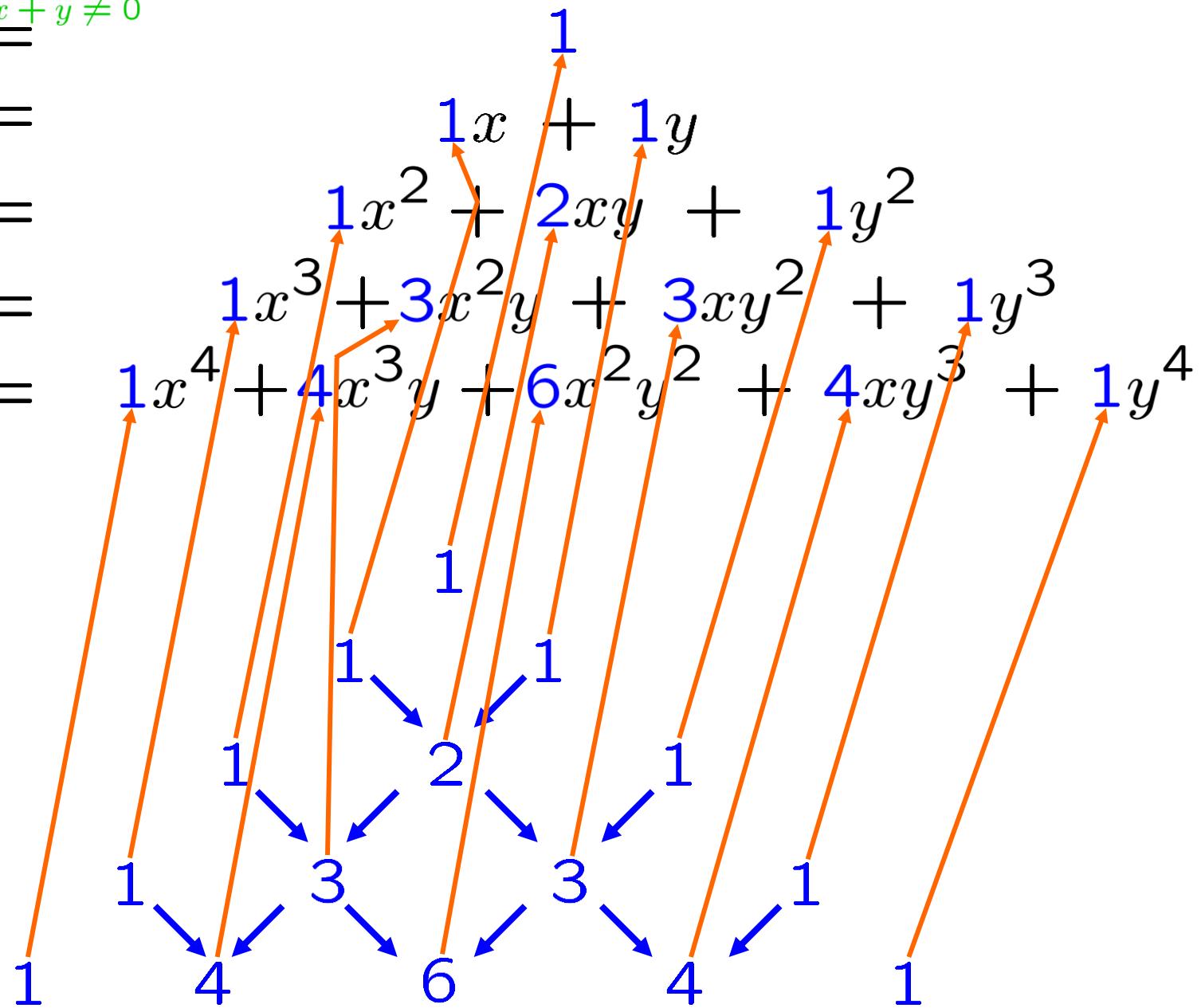
$$(x + y)^0 = \text{ } x + y \neq 0$$

$$(x + y)^1 =$$

$$(x + y)^2 =$$

$$(x + y)^3 =$$

$$(x + y)^4 =$$



$$(x + y)^0 = \text{$x+y \neq 0$}$$

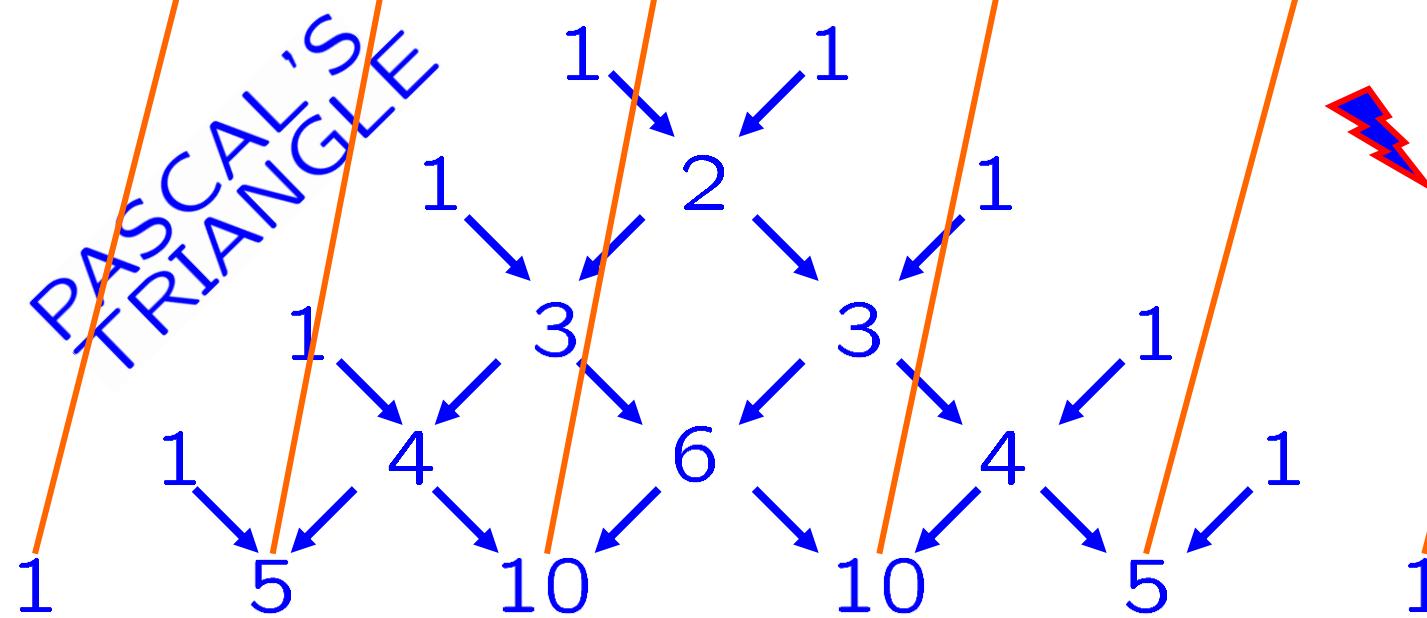
$$(x + y)^1 = 1x + 1y$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^2y^3 + 10x^3y^2 + 5xy^4 + 1y^5$$



Question: How many  $x^2y^3$  in  $(x+y)^5$ ?

$$(x+y)^5 = \begin{aligned} &xxxxx + xxxx y + xxx y x + xxx y y \\ &+ xxy x x + xxy x y + xxy y x + \underline{\underline{xxy y y}} \\ &+ xy x x x + xy x x y + xy x y x + \underline{\underline{xy x y y}} \\ &+ xy y x x + \underline{\underline{xy y x y}} + \underline{\underline{xy y y x}} + xy y y y \\ &+ y x x x x + y x x x y + y x x y x + \underline{\underline{y x x y y}} \\ &+ y x y x x + \underline{\underline{y x y x y}} + \underline{\underline{y x y y x}} + y x y y y \\ &+ yy x x x + \underline{\underline{yy x x y}} + \underline{\underline{yy x y x}} + yy x y y \\ &+ \underline{\underline{yy y x x}} + yy y x y + yy y y x + yy y y y \end{aligned}$$

$xxyy$

$xyxy$

$xyyx$

$xyyy$

$yxxx$

$yxyx$

$yxyy$

$yyxx$

$yyxy$

$yyyx$

position  
12345

---

$xxyy$

$xyxy$

$yyxy$

$yyyx$

$yxx$

$yxyx$

$yxxy$

$yxuyx$

$yyxx$

$yyxy$

$yyyxx$

$xxyy$

$xyxy$

$yyyx$

$yxx$

$xyyy$

$yyxy$

position 12345	$x$ positions	$y$ positions
$xxyyy$	12, 345	
$xyxyy$	13, 245	
$xyyxu$	14, 235	
$xyyyx$	15, 234	
$yxxuy$	23, 145	
$yxyxy$	24, 135	
$yxyyx$	25, 134	
$yyxxu$	34, 125	
$yyxyx$	35, 124	
$yyyxx$	45, 123	

For  $x$ ,

choose two from  
 $\{1,2,3,4,5\}$

For  $y$ ,

choose three from  
 $\{1,2,3,4,5\}$

Question: How many  $x^2y^3$  in  $(x + y)^5$ ?

For  $x$ ,

choose two from  
 $\{1,2,3,4,5\}$

For  $y$ ,

choose three from  
For  $x$ ,  
 $\{1,2,3,4,5\}$  two from  
 $\{1,2,3,4,5\}$

For  $y$ ,

choose three from  
 $\{1,2,3,4,5\}$

Question: How many  $x^2y^3$  in  $(x+y)^5$ ?

For  $x$ ,

choose two from  
 $\{1,2,3,4,5\}$

For  $y$ ,

choose three from  
 $\{1,2,3,4,5\}$

---

Question: How many ways of choosing  
two objects from among five?

three

Answer: “5 choose 2”, written  $\binom{5}{2}$

$$\binom{5}{2} = 10$$

Question: How many  $x^2y^3$  in  $(x+y)^5$ ?

For  $x$ ,

choose two from  
 $\{1,2,3,4,5\}$

For  $y$ ,

choose three from  
 $\{1,2,3,4,5\}$

---

Question: How many ways of choosing three objects from among five?

Answer: “5 choose 3”, written  $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many  $x^2y^3$  in  $(x + y)^5$ ?

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$$(x + y)^5 =$$

$$\dots + \binom{5}{3} x^2 y^3 + \dots$$

---

Answer: “5 choose 3”, written  $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many  $x^2y^3$  in  $(x + y)^5$ ?

$$(x + y)^5 = \binom{5}{5}x^0y^5 + \binom{5}{4}x^1y^4 + \binom{5}{3}x^2y^3 + \binom{5}{2}x^3y^2 + \binom{5}{1}x^4y^1 + \binom{5}{0}x^5y^0$$

Can reverse  
the coefficients,  
by symmetry. . .

The binomial formula

Answer: “5 choose 3”, written

$$\binom{5}{3}$$

$$\binom{5}{3} = 10$$

Question: How many  $x^2y^3$  in  $(x + y)^5$ ?

Binomial  
coefficients



$$(x + y)^5 =$$

$$\binom{5}{0}x^0y^5 +$$

$$\binom{5}{1}x^1y^4 +$$

$$\binom{5}{2}x^2y^3 +$$

$$\binom{5}{3}x^3y^2 +$$

$$\binom{5}{4}x^4y^1 +$$

$$\binom{5}{5}x^5y^0$$

The binomial formula

Answer: “5 choose 2”, written

$$\binom{5}{2}$$

$$\binom{5}{2} = 10$$