Financial Mathematics

The binomial formula

$$(x+y)^0 \overset{x+y\neq 0}{=} 1 \qquad 2^0 = 1 \text{ terms}$$

$$(x+y)^1 = x + y \qquad 2^1 = 2 \text{ terms}$$

$$\begin{aligned}(x+y)^2 &= x(x+y) \\ &\quad + y(x+y) \\ &= xx + xy + yx + yy = x^2 + 2xy + y^2\end{aligned} \qquad 2^2 = 4 \text{ terms}$$

duplication

$$\begin{aligned}(x+y)^3 &= x(xx+xy+yx+yy) \\ &\quad + y(xx+xy+yx+yy) \\ &= xxx+xxy+xyx+xyy \\ &\quad +yxx+yxy+yyx+yyy\end{aligned} \qquad 2^3 = 8 \text{ terms}$$

duplications

$$\begin{aligned}(x+y)^4 &= x(xxx+xxy+xyx+xyy \\ &\quad +yxx+yxy+yyx+yyy) \\ &\quad + y(xxx+xxy+xyx+xyy \\ &\quad +yxx+yxy+yyx+yyy) = \text{etc.}\end{aligned}$$

duplications

$2^4 = 16$ terms

Spp

Lots of duplications. . . *e.g.*

$$
\begin{aligned}
(x+y)^5 = {} & xxxxx + xxxxy + xxxyx + xxxyy \\
& + xxyxx + xxyxy + xxyyx + xxyyy \\
& + xyxxx + xyxxy + xyxyx + xyxyy \\
& + xyyxx + xyyxy + xyyyx + xyyyy \\
& + yxxxx + yxxxy + yxxyx + yxxyy \\
& + yxyxx + yxyxy + yxyyx + yxyyy \\
& + yyxxx + yyxxy + yyxyx + yyxyy \\
& + yyyxx + yyyxy + yyyyx + yyyyy
\end{aligned}
$$

x^4y

Start over, avoiding duplications. . .

$2^5 = 32$ terms

$$(x+y)^0 = 1 = 1 \quad (x+y \neq 0)$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$\begin{aligned}
(x+y)^3 &= (x+y)\,(x+y)^2 \\
&= (x+y)(1x^2 + 2xy + 1y^2) \\
&= x\,(1x^2 + 2xy + 1y^2) \\
&\quad + y\,(1x^2 + 2xy + 1y^2) \\
&= 1x^3 + 2x^2y + 1xy^2 \\
&\quad + 1x^2y + 2xy^2 + 1y^3 \\
&= 1x^3 + 3x^2y + 3xy^2 + 1y^3
\end{aligned}$$

Spp

$$(x+y)^0 = 1 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^3 = 1x^3 + 2x^2y + 1xy^2$$

$$+ \quad 1x^2y + 2xy^2 + 1y^3$$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$= 1x^3 + 2x^2y + 1xy^2$$

$$\quad 1 \quad 2 \quad 1$$

$$+ \quad 1 \quad 3x^2y + 3xy^2 + 1y^3$$

$$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$(x+y)^0 = 1 = 1$ ($x+y \neq 0$)

$(x+y)^1 = x + y = 1x + 1y$

$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4$

1	2	1		
	1	2	1	
1	3	3	1	
1	3	3	1	
	1	3	3	1

1	2	1	
	1	2	1
1	3	3	1

$(x+y)^0 = 1 = 1$ ($x+y \neq 0$)

$(x+y)^1 = x + y = 1x + 1y$

$(x+y)^2 = x^2 + 2xy + y^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4$

1	2	1		
	1	2	1	
1	3	3	1	

1	3	3	1	
	1	3	3	1
1	4	6	4	1

Spp

$(x+y)^0 =$ 1 ($x+y\neq 0$)

$(x+y)^1 = 1x + 1y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

1	2	1		
	1	2	1	
1	3	3	1	

1	3	3	1	
	1	3	3	1
1	4	6	4	1

$$(x+y)^0 \underset{x+y\neq 0}{=} 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1\boxed{x^4} + 4\boxed{x^3y} + 6\boxed{x^2y^2} + 4\boxed{xy^3} + 1\boxed{y^4}$$

Start with four xs.

Change an x to a y.

Continue. . .

until. . .

four ys.

1	3	3	1	
	1	3	3	1
1	4	6	4	1

Easier: 1 3 3 1

1 4 6 4 1

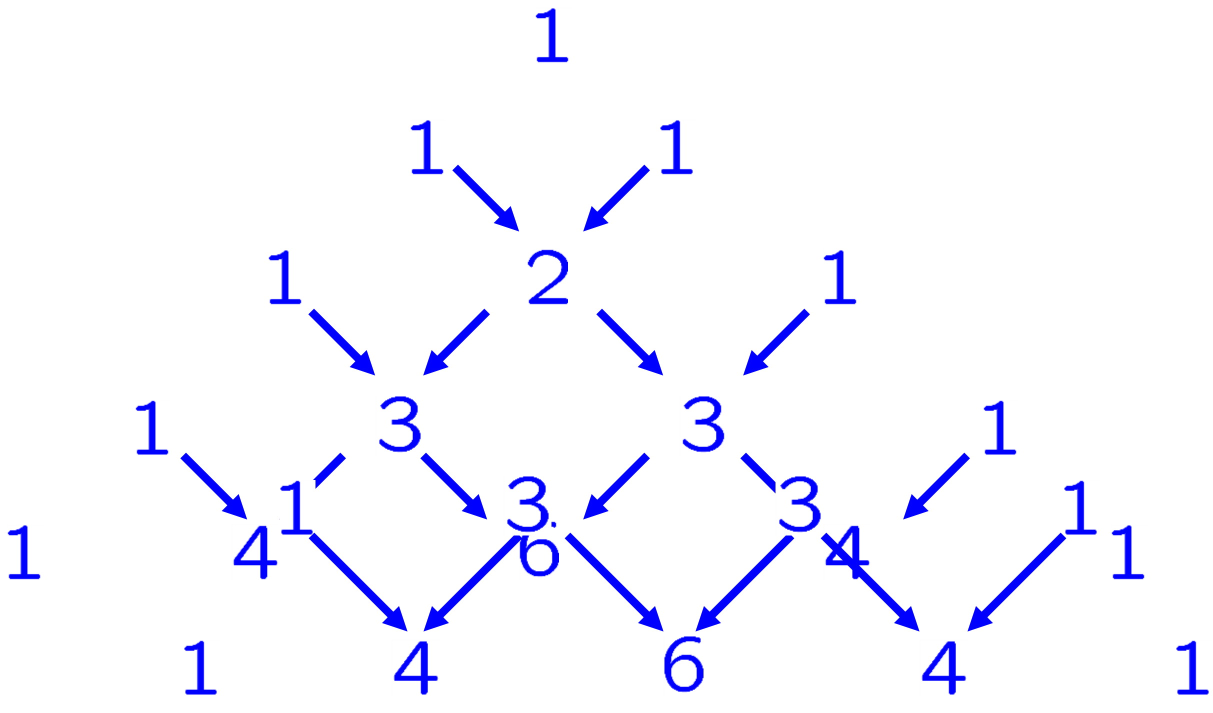
$$(x+y)^0 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$



$$(x+y)^0 = 1 \quad x+y \neq 0$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

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$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$(x+y)^0 = 1$ $\quad x+y \neq 0$

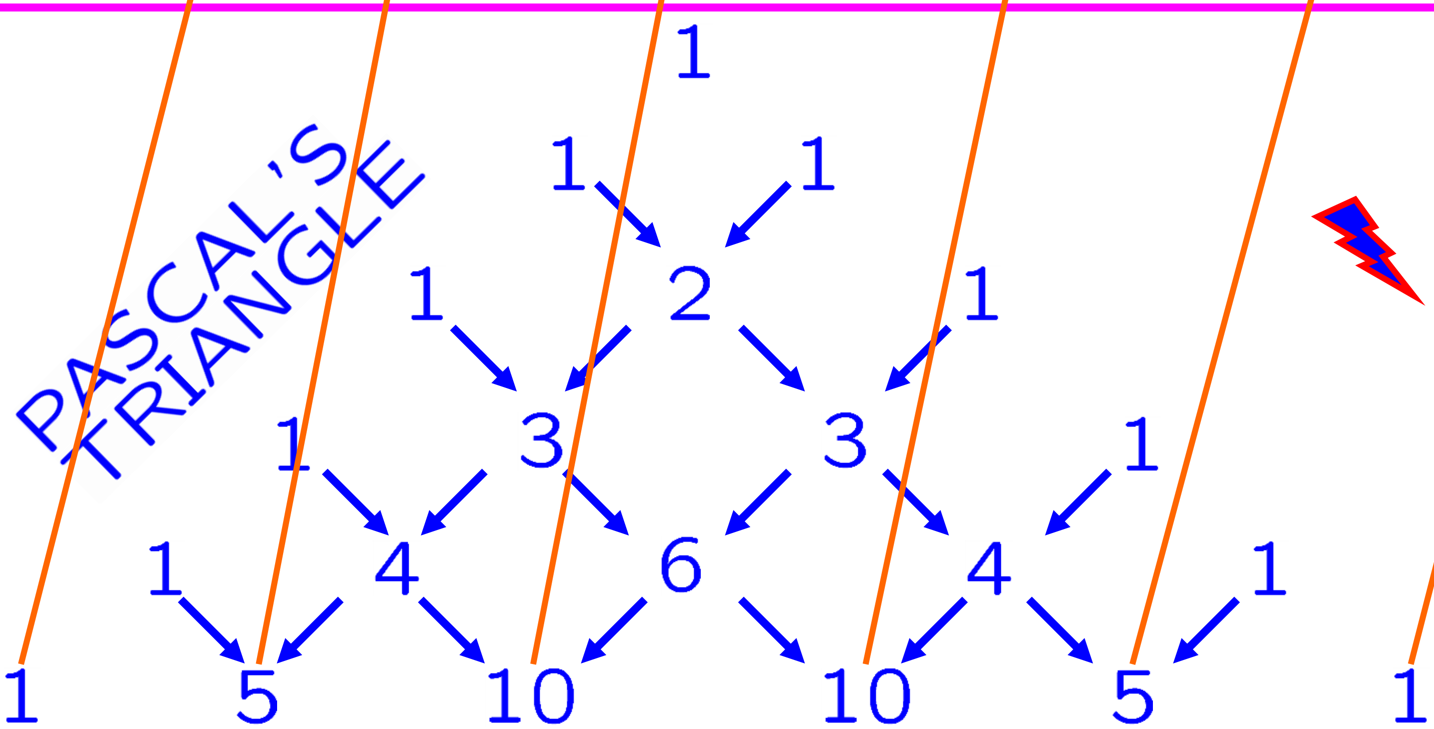
$(x+y)^1 = 1x + 1y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

$(x+y)^5 = 1x^5 + 5x^4y + 10x^2y^3 + 10x^3y^2 + 5xy^4 + 1y^5$



Question: How many x^2y^3 in $(x+y)^5$?

$$
\begin{aligned}
(x+y)^5 = & \ xxxxx + xxxxy + xxxyx + xxxyy \\
& + xxyxx + xxyxy + xxyyx + \underline{xxyyy} \\
& + xyxxx + xyxxy + xyxyx + \underline{xyxyy} \\
& + xyyxx + \underline{xyyxy} + \underline{xyyyx} + xyyyy \\
& + yxxxx + yxxxy + yxxyx + \underline{yxxyy} \\
& + yxyxx + \underline{yxyxy} + \underline{yxyyx} + yxyyy \\
& + yyxxx + \underline{yyxxy} + \underline{yyxyx} + yyxyy \\
& + \underline{yyyxx} + yyyxy + yyyyx + yyyyy
\end{aligned}
$$

$$xxyyy$$

$$xyxyy$$

$$xyyxy \qquad xyyyx$$

$$yxxyy$$

$$yxyxy \qquad yxyyx$$

$$yyxxy \qquad yyxyx$$

$$yyyxx$$

position

12345

$xxyyy$

$xyxyy$

$xyyxy$

$xyyyx$

$yxxyy$

$yxyxy$

$yxyyx$

$yyyxx$

$yyxxy$

$yyxyx$

$yyyxx$

$xyyxy$ $xyyyx$

$yxyxy$ $yxyyx$

$yyxxy$ $yyxyx$

$xxyyy$

$xyxyy$

$yxxyy$

position 12345	x positions, y positions
$xxyyy$	$12, 345$
$xyxyy$	$13, 245$
$xyyxy$	$14, 235$
$xyyyx$	$15, 234$
$yxxyy$	$23, 145$
$yxyxy$	$24, 135$
$yxyyx$	$25, 134$
$yyxxy$	$34, 125$
$yyxyx$	$35, 124$
$yyyxx$	$45, 123$

For x,
choose two from
{1,2,3,4,5}

For y,
choose three from
{1,2,3,4,5}

Question: How many x^2y^3 in $(x+y)^5$?

For x,
choose two from
$\{1,2,3,4,5\}$

For y,
choose three from
$\{1,2,3,4,5\}$

For x,
choose two from
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For y,
choose three from
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Question: How many x^2y^3 in $(x+y)^5$?

For x,
choose two from
{1,2,3,4,5}

For y,
choose three from
{1,2,3,4,5}

Question: How many ways of choosing two objects from among five?

three

Answer: "5 choose 2", written $\binom{5}{2}$

$$\binom{5}{2} = 10$$

Question: How many x^2y^3 in $(x+y)^5$?

For x,
choose two from
$\{1,2,3,4,5\}$

For y,
choose three from
$\{1,2,3,4,5\}$

Question: How many ways of choosing three objects from among five?

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many x^2y^3 in $(x+y)^5$?

$$(x+y)^5 =$$

$$\cdots \quad + \binom{5}{3} x^2y^3 + \cdots$$

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many x^2y^3 in $(x+y)^5$?

Can reverse the coefficients, by symmetry...

$$(x+y)^5 =$$

$$\binom{5}{5}x^0y^5 + \binom{5}{4}x^1y^4 + \binom{5}{3}x^2y^3 + \binom{5}{2}x^3y^2 + \binom{5}{1}x^4y^1 + \binom{5}{0}x^5y^0$$

The binomial formula

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many x^2y^3 in $(x+y)^5$?

$$(x+y)^5 =$$

Binomial coefficients

STOP

$$\binom{5}{0}x^0y^5 + \binom{5}{1}x^1y^4 + \binom{5}{2}x^2y^3 + \binom{5}{3}x^3y^2 + \binom{5}{4}x^4y^1 + \binom{5}{5}x^5y^0$$

The binomial formula

Answer: "5 choose 2", written $\binom{5}{2}$

$$\binom{5}{2} = 10$$