# Financial Mathematics

## The binomial formula

$x+y \neq 0$

$$(x+y)^0 = 1 \qquad 2^0 = 1 \text{ terms}$$

$$(x+y)^1 = x + y \qquad 2^1 = 2 \text{ terms}$$

$$(x+y)^2 = x(x+y) + y(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^2 \qquad 2^2 = 4 \text{ terms}$$

duplication

$$(x+y)^3 = x(xx+xy+yx+yy) + y(xx+xy+yx+yy)$$

$$= xxx+xxy+xyx+xyy + yxx+yxy+yyx+yyy \qquad 2^3 = 8 \text{ terms}$$

duplications

$$(x+y)^4 = x(xxx+xxy+xyx+xyy + yxx+yxy+yyx+yyy) + y(xxx+xxy+xyx+xyy + yxx+yxy+yyx+yyy) = \text{etc.}$$

duplications

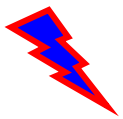
$2^4 = 16$ terms

Spp

# Lots of duplications. . . *e.g.*

$$
\begin{aligned}
(x+y)^5 = {} & xxxxx + xxxxy + xxxyx + xxxyy \\
& + xxyxx + xxyxy + xxyyx + xxyyy \\
& + xyxxx + xyxxy + xyxyx + xyxyy \\
& + xyyxx + xyyxy + xyyyx + xyyyy \\
& + yxxxx + yxxxy + yxxyx + yxxyy \\
& + yxyxx + yxyxy + yxyyx + yxyyy \\
& + yyxxx + yyxxy + yyxyx + yyxyy \\
& + yyyxx + yyyxy + yyyyx + yyyyy
\end{aligned}
$$

$x^4y$



Start over, avoiding duplications. . .

$2^5 = 32$ terms



Spp

$$(x+y)^0 \overset{x+y\neq 0}{=} 1 = 1$$

$$(x+y)^1 = x + y = 1x + 1y$$

$$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy+1y^2$$

$$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$$

---

$$\begin{aligned}
(x+y)^3 &= (x+y)\quad (x+y)^2 \\
&= (x+y)(\,1x^2+2xy+1y^2) \\
&= \quad x\quad (\,1x^2+2xy+1y^2) \\
&\quad + \;y\quad (\,1x^2+2xy+1y^2) \\
&= \quad 1x^3+2x^2y+1xy^2 \\
&\quad + \qquad 1x^2y+2xy^2+1y^3 \\
&= \quad 1x^3+3x^2y+3xy^2+1y^3
\end{aligned}$$



Spp

$(x+y)^0 = 1 = 1$ $x+y \neq 0$

$(x+y)^1 = x + y = 1x + 1y$

$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy + 1y^2$

$(x+y)^3 = 1x^3+3x^2y + 3xy^2+1y^3$

---

$$(x+y)^3 = 1x^3+2x^2y+1xy^2$$
$$+ \quad 1x^2y+2xy^2+1y^3$$
$$= 1x^3+3x^2y + 3xy^2+1y^3$$

---

$$= 1x^3+2x^2y+1xy^2$$
$$+ \quad 1x^2y+2xy^2+1y^3$$
$$= 1x^3+3x^2y + 3xy^2+1y^3$$



Spp

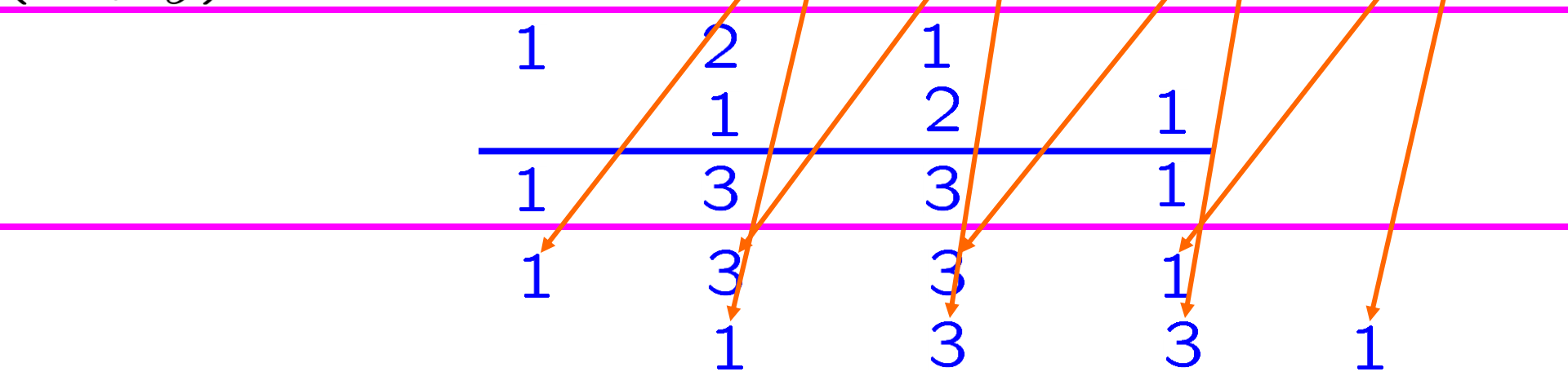
$(x+y)^0 = 1 = 1$ $x+y \neq 0$

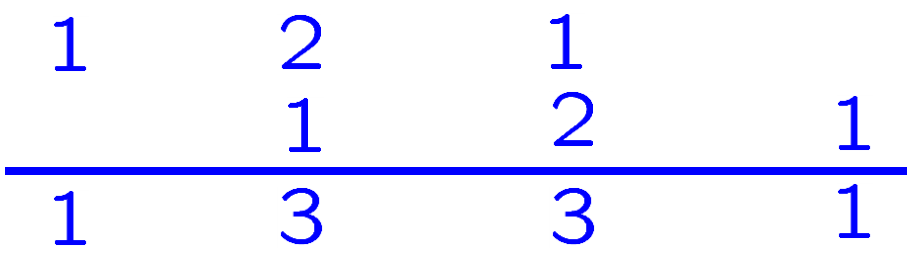
$(x+y)^1 = x + y = 1x + 1y$

$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy+1y^2$

$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$

$(x+y)^4$






Spp

$(x+y)^0 = 1 = 1$ $x+y \neq 0$

$(x+y)^1 = x + y = 1x + 1y$

$(x+y)^2 = x^2+2xy+y^2 = 1x^2+2xy+1y^2$

$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$

$(x+y)^4$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 1 | | |
| | 1 | 2 | 1 | |
| 1 | 3 | 3 | 1 | |
| 1 | 3 | 3 | 1 | |
| | 1 | 3 | 3 | 1 |
| 1 | 4 | 6 | 4 | 1 |



Spp

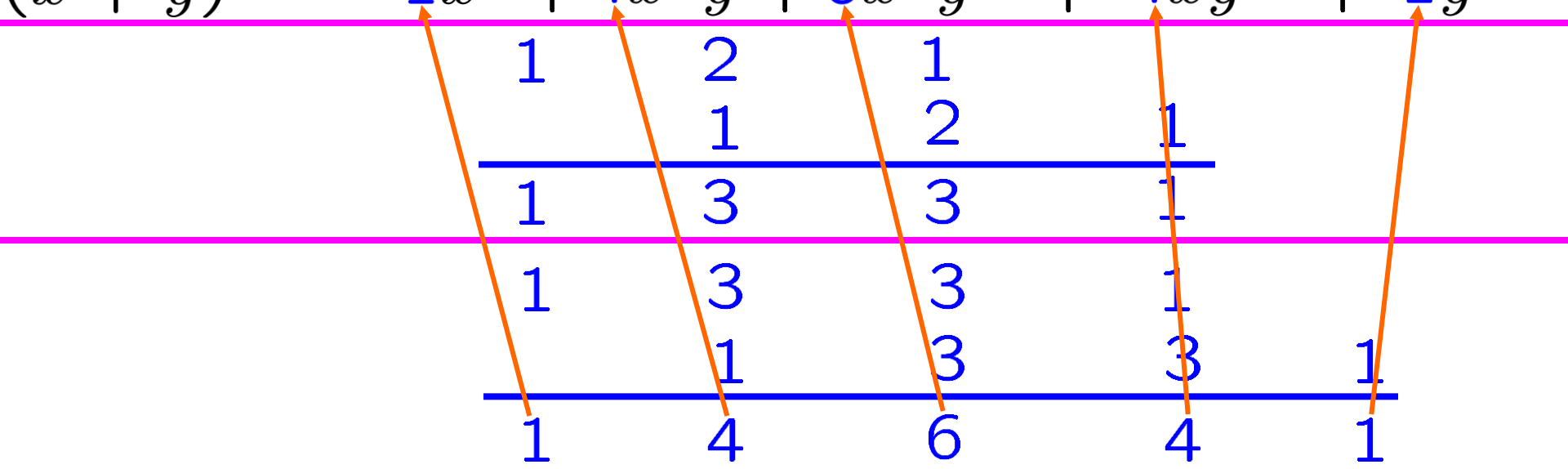
$(x+y)^0 =$ $1$ $1$ $x+y \neq 0$

$(x+y)^1 =$ $1x + 1y$ $1x + 1y$

$(x+y)^2 =$ $1x^2 + 2xy$ $1x^2 + 2xy + 1y^2$

$(x+y)^3 =$ $1x^3 + 3x^2y$ $1x^2y + 3xy^2$ $y + 1y^3$ $1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 =$ $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$



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$(x+y)^0 =$ $1$ $\quad x+y \neq 0$

$(x+y)^1 =$ $1x + 1y$

$(x+y)^2 =$ $1x^2 + 2xy + 1y^2$

$(x+y)^3 =$ $1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 =$ $1\boxed{x^4} + 4\boxed{x^3y} + 6\boxed{x^2y^2} + 4\boxed{xy^3} + 1\boxed{y^4}$

Start with four $x$s.

Change an $x$ to a $y$.

Continue...

until...

four $y$s.

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| | | | | |
|---|---|---|---|---|
| 1 | 3 | 3 | 1 | |
| | 1 | 3 | 3 | 1 |
| 1 | 4 | 6 | 4 | 1 |

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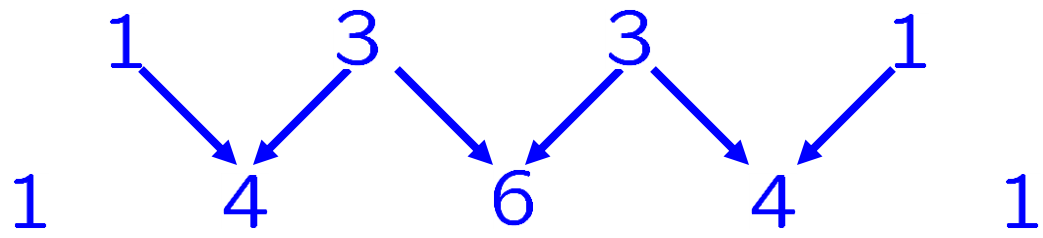
Easier:




Spp

$x + y \neq 0$

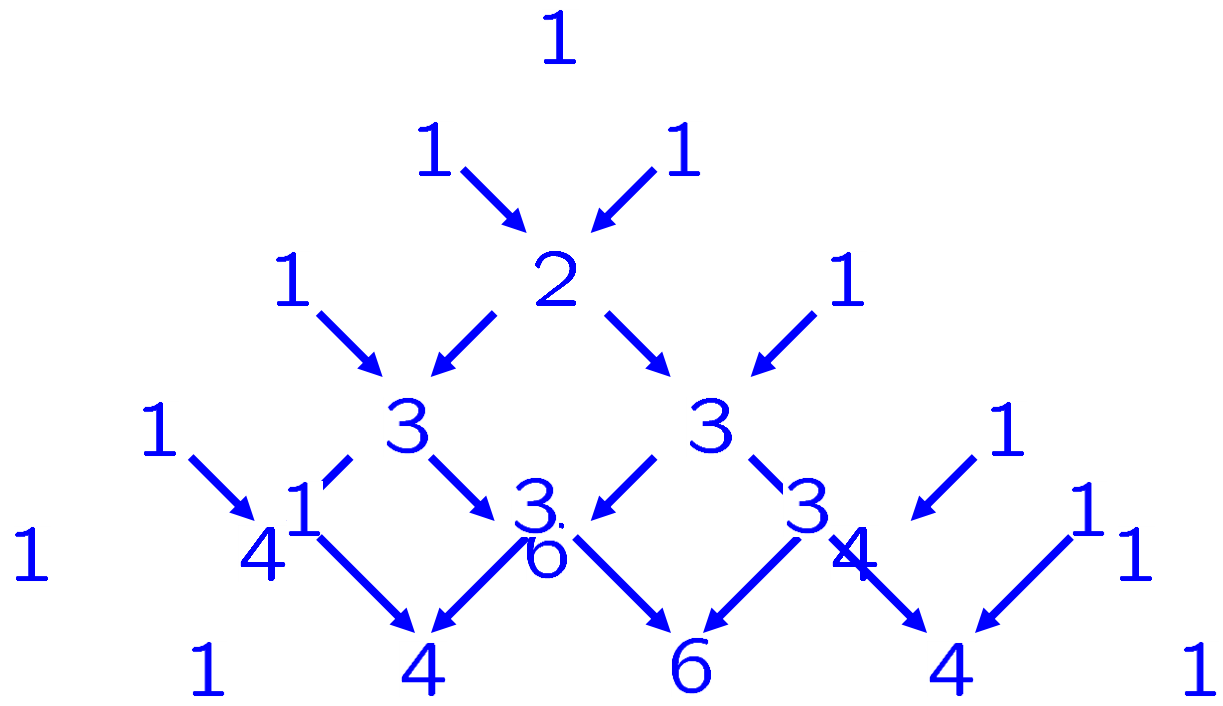
$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$




Spp

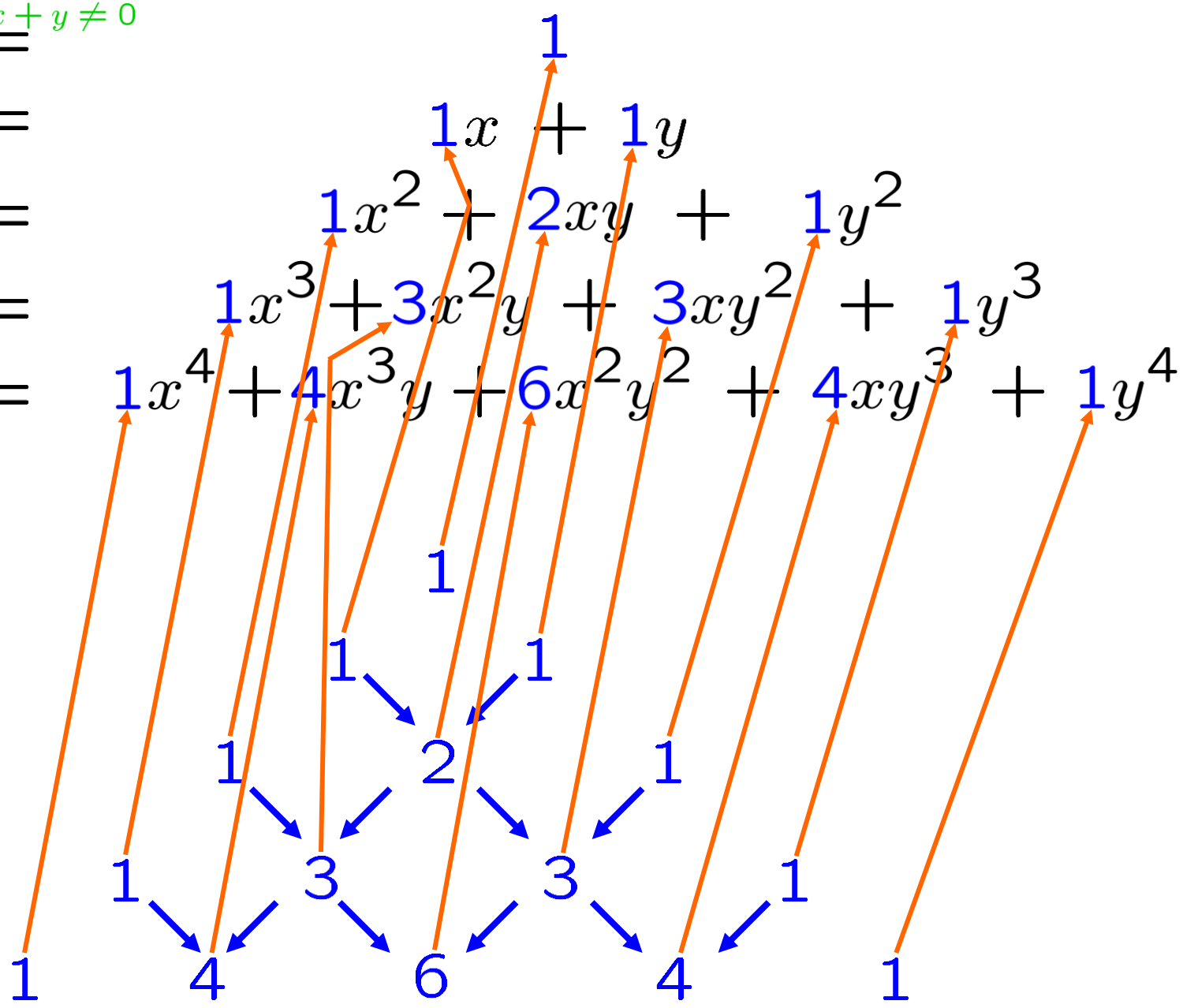
$(x+y)^0 =$ $x+y \neq 0$

$(x+y)^1 =$

$(x+y)^2 =$

$(x+y)^3 =$

$(x+y)^4 =$






Spp

$x + y \neq 0$

$$(x+y)^0 = 1$$

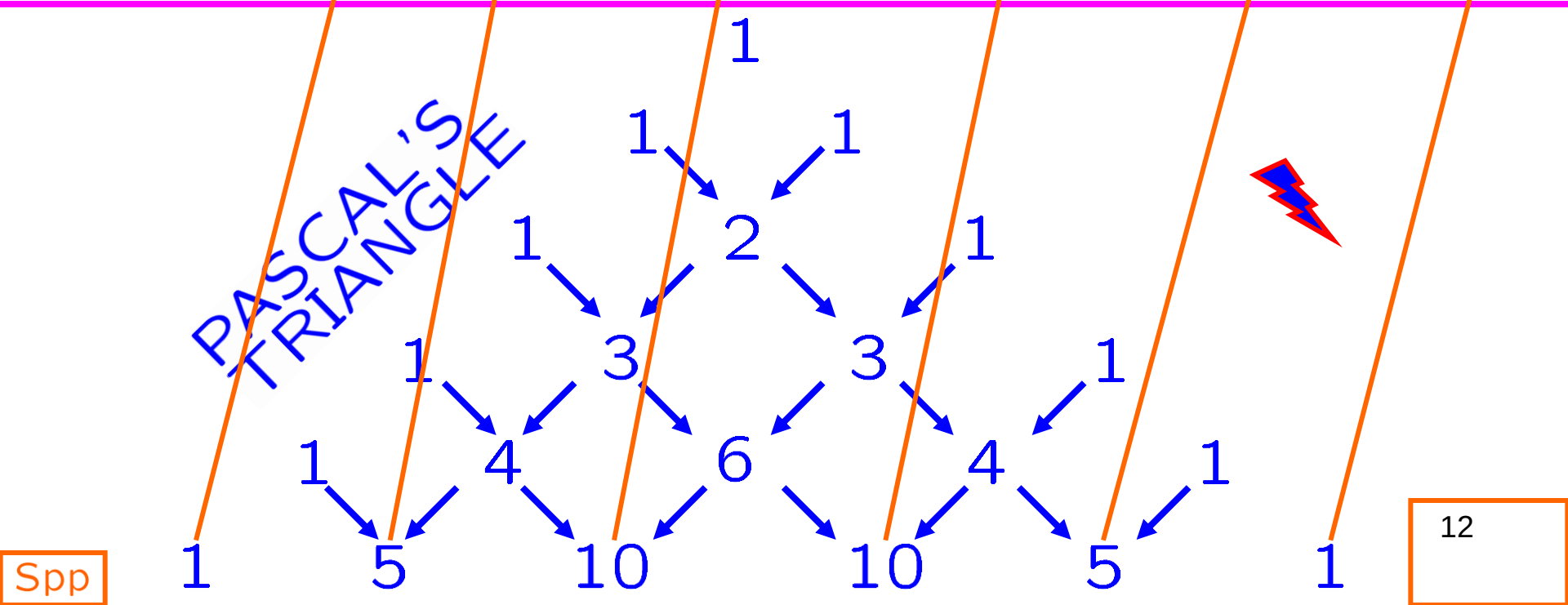
$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^2y^3 + 10x^3y^2 + 5xy^4 + 1y^5$$



Question: How many $x^2y^3$ in $(x+y)^5$?

$$
\begin{aligned}
(x+y)^5 = {} & xxxxx + xxxxy + xxxyx + xxxyy \\
& + xxyxx + xxyxy + xxyyx + \underline{xxyyy} \\
& + xyxxx + xyxxy + xyxyx + \underline{xyxyy} \\
& + xyyxx + \underline{xyyxy} + \underline{xyyyx} + xyyyy \\
& + yxxxx + yxxxy + yxxyx + \underline{yxxyy} \\
& + yxyxx + \underline{yxyxy} + \underline{yxyyx} + yxyyy \\
& + yyxxx + \underline{yyxxy} + \underline{yyxyx} + yyxyy \\
& + \underline{yyyxx} + yyyxy + yyyyx + yyyyy
\end{aligned}
$$

$xxyyy$

$xyxyy$

$xyyxy$ $xyyyx$

$yxxyy$

$yxyxy$ $yxyyx$

$yyxxy$ $yyxyx$

$yyyxx$

position

12345

$xxyyy$

$xyxyy$

$xyyxy$

$xyyyx$

$yxxyy$

$yxyxy$

$yxyyx$

$yyxxy$

$yyxyx$

$yyyxx$

$xxyyy$

$xyxyy$

$yxxyy$

$xyyxy$ $xyyyx$

$yxyxy$ $yxyyx$

$yyxxy$ $yyxyx$

| position 12345 | $x$ positions | $y$ positions |
|---|---|---|
| $xxyyy$ | 12, 345 | |
| $xyxyy$ | 13, 245 | |
| $xyyxy$ | 14, 235 | |
| $xyyyx$ | 15, 234 | |
| $yxxyy$ | 23, 145 | |
| $yxyxy$ | 24, 135 | |
| $yxyyx$ | 25, 134 | |
| $yyxxy$ | 34, 125 | |
| $yyxyx$ | 35, 124 | |
| $yyyxx$ | 45, 123 | |

For $x$,
choose two from
{1,2,3,4,5}

For $y$,
choose three from
{1,2,3,4,5}

# Question: How many $x^2y^3$ in $(x+y)^5$?

For $x$,
choose two from
$\{1,2,3,4,5\}$

For $y$,
choose three from
$\{1,2,3,4,5\}$

For $x$,
$\{1,2,3,4,5\}$two from
$\{1,2,3,4,5\}$

For $y$,
choose three from
$\{1,2,3,4,5\}$

Question: How many $x^2y^3$ in $(x+y)^5$?

For $x$,
choose two from
{1,2,3,4,5}

For $y$,
choose three from
{1,2,3,4,5}

---

Question: How many ways of choosing two objects from among five?

three

Answer: "5 choose 2", written $\binom{5}{2}$

$$\binom{5}{2} = 10$$

Question: How many $x^2y^3$ in $(x+y)^5$?

For $x$,
choose two from
$\{1,2,3,4,5\}$

For $y$,
choose three from
$\{1,2,3,4,5\}$

---

Question: How many ways of choosing three objects from among five?

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many $x^2y^3$ in $(x+y)^5$?

$$(x+y)^5 =$$

$$\cdots + \binom{5}{3} x^2y^3 + \cdots$$

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many $x^2y^3$ in $(x+y)^5$?

Can reverse the coefficients, by symmetry...

$$(x+y)^5 = \binom{5}{5}x^0y^5 + \binom{5}{4}x^1y^4 + \binom{5}{3}x^2y^3 + \binom{5}{2}x^3y^2 + \binom{5}{1}x^4y^1 + \binom{5}{0}x^5y^0$$

The binomial formula

Answer: "5 choose 3", written $\binom{5}{3}$

$$\binom{5}{3} = 10$$

Question: How many $x^2y^3$ in $(x+y)^5$?


$$(x+y)^5 = \binom{5}{0}x^0y^5 + \binom{5}{1}x^1y^4 + \binom{5}{2}x^2y^3 + \binom{5}{3}x^3y^2 + \binom{5}{4}x^4y^1 + \binom{5}{5}x^5y^0$$

Binomial coefficients

The binomial formula

Answer: "5 choose 2", written $\binom{5}{2}$

$$\binom{5}{2} = 10$$