Financial Mathematics

The Intermediate Value Theorem and the Mean Value Theorem

Definition:

An output of a function is called a **value** of the function.

Intermediate Value Theorem:

Let f be continuous on $[a, c]$.

Suppose $f(a) = s$ and $f(c) = u$.

Let t be a number strictly between s and u.

Then, for some $b \in (a, c)$,

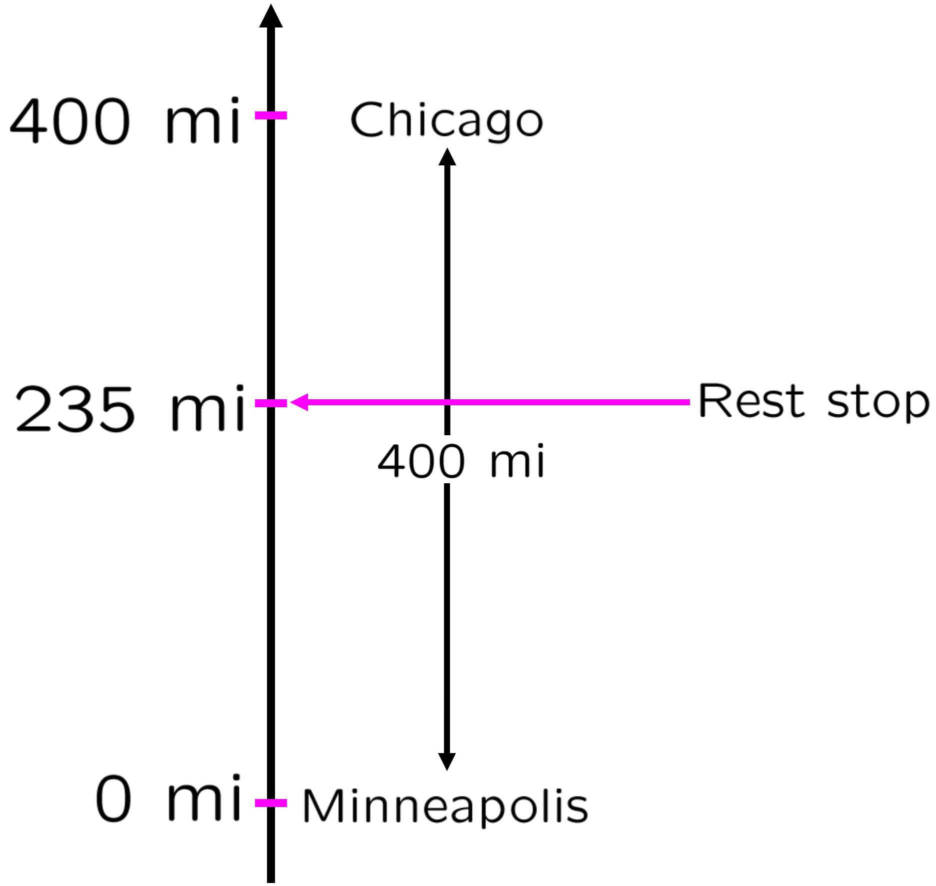
$$f(b) = t.$$

(*I.e.*, let $m := \min\{s, u\}$, let $M := \max\{s, u\}$, and let $t \in (m, M)$.)

Buzz phrase:

Continuous functions attain intermediate values.

MEAN VALUE THEOREM:



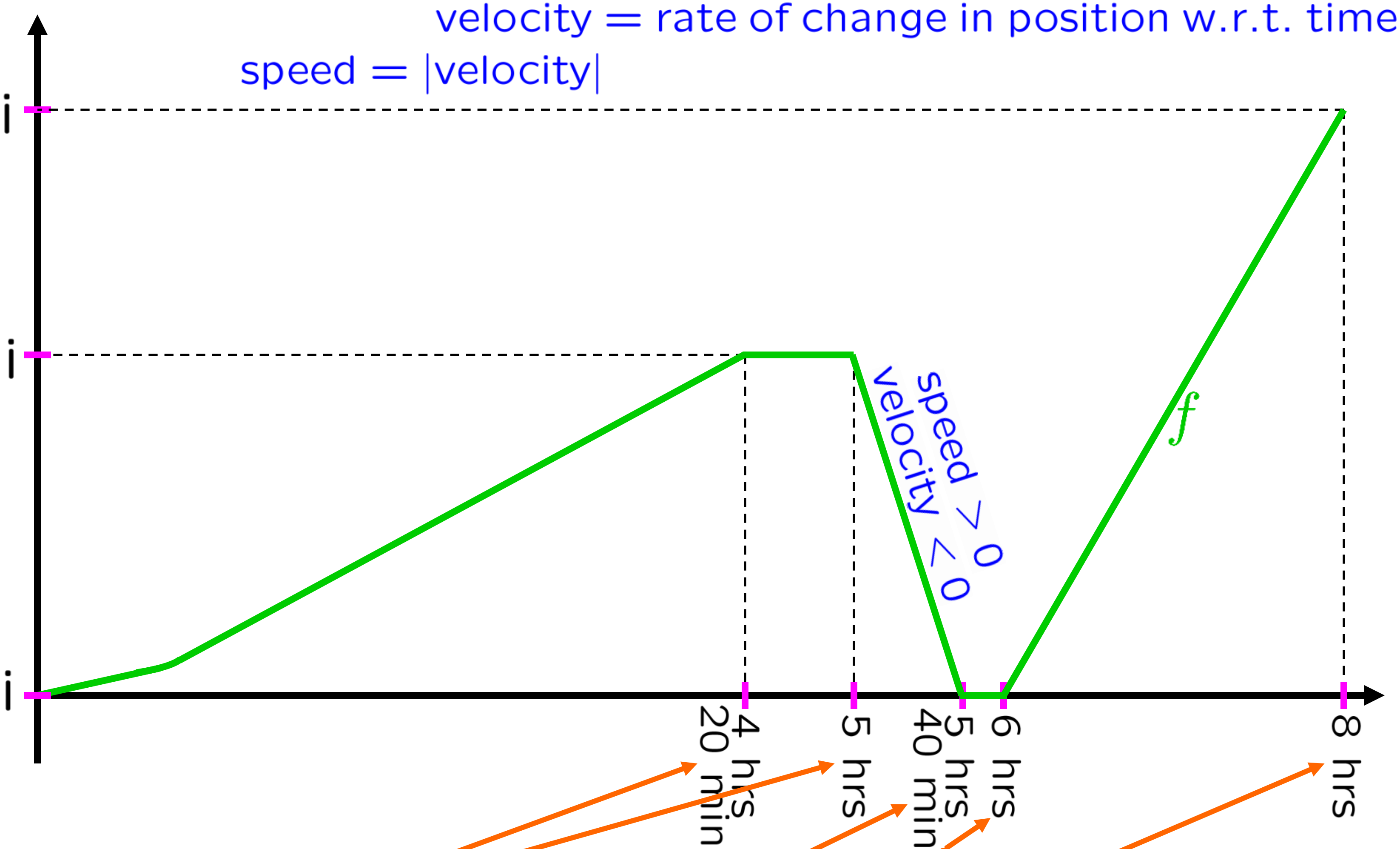
0 hrs, leave Minneapolis
4 hrs 20 mins, start break
5 hrs, notice problem,
head back
5 hrs 40 mins, arrive Minneapolis

6 hrs, leave Minneapolis
8 hrs, arrive Chicago

MEAN VALUE THEOREM:

velocity = rate of change in position w.r.t. time

speed = |velocity|



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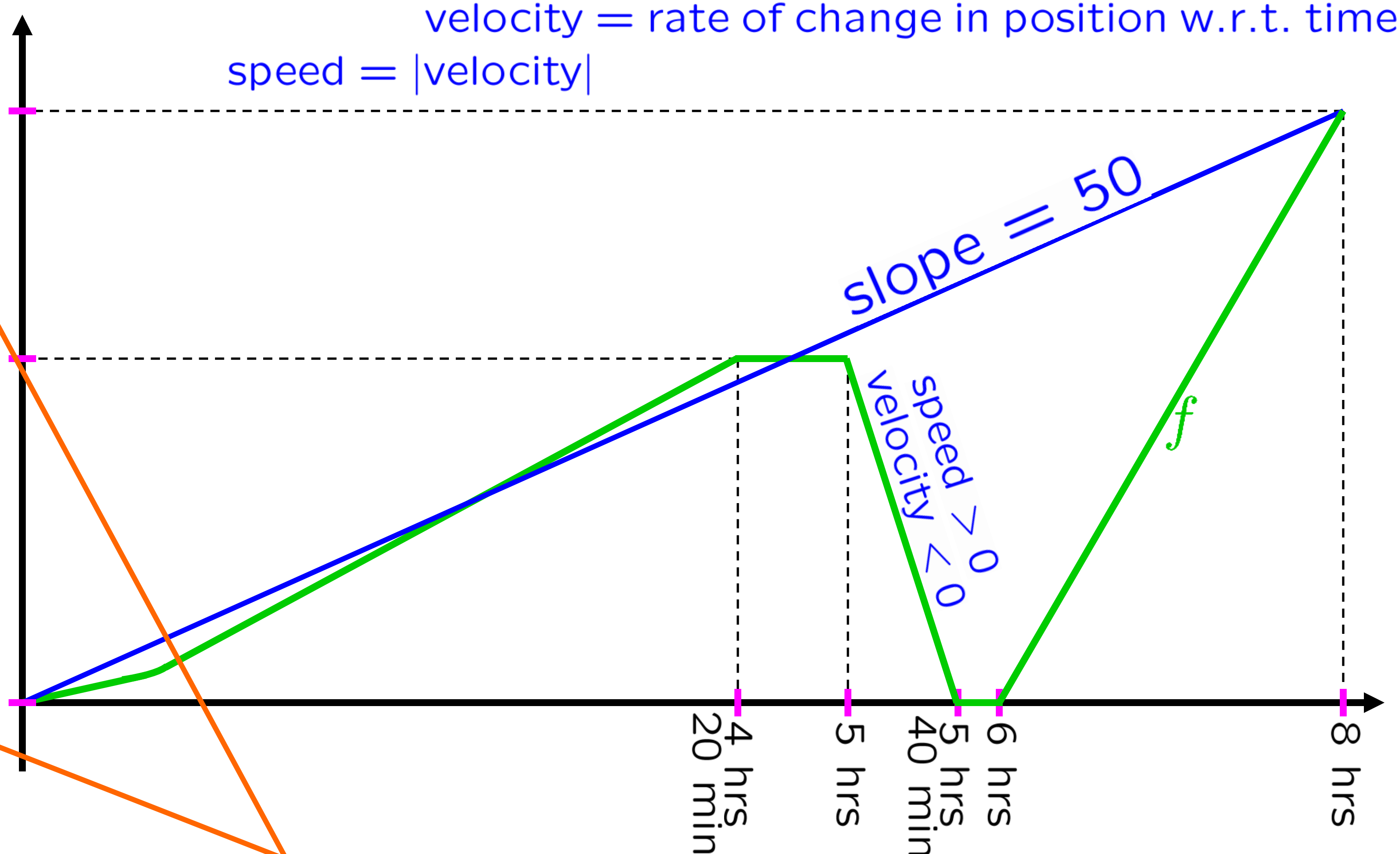


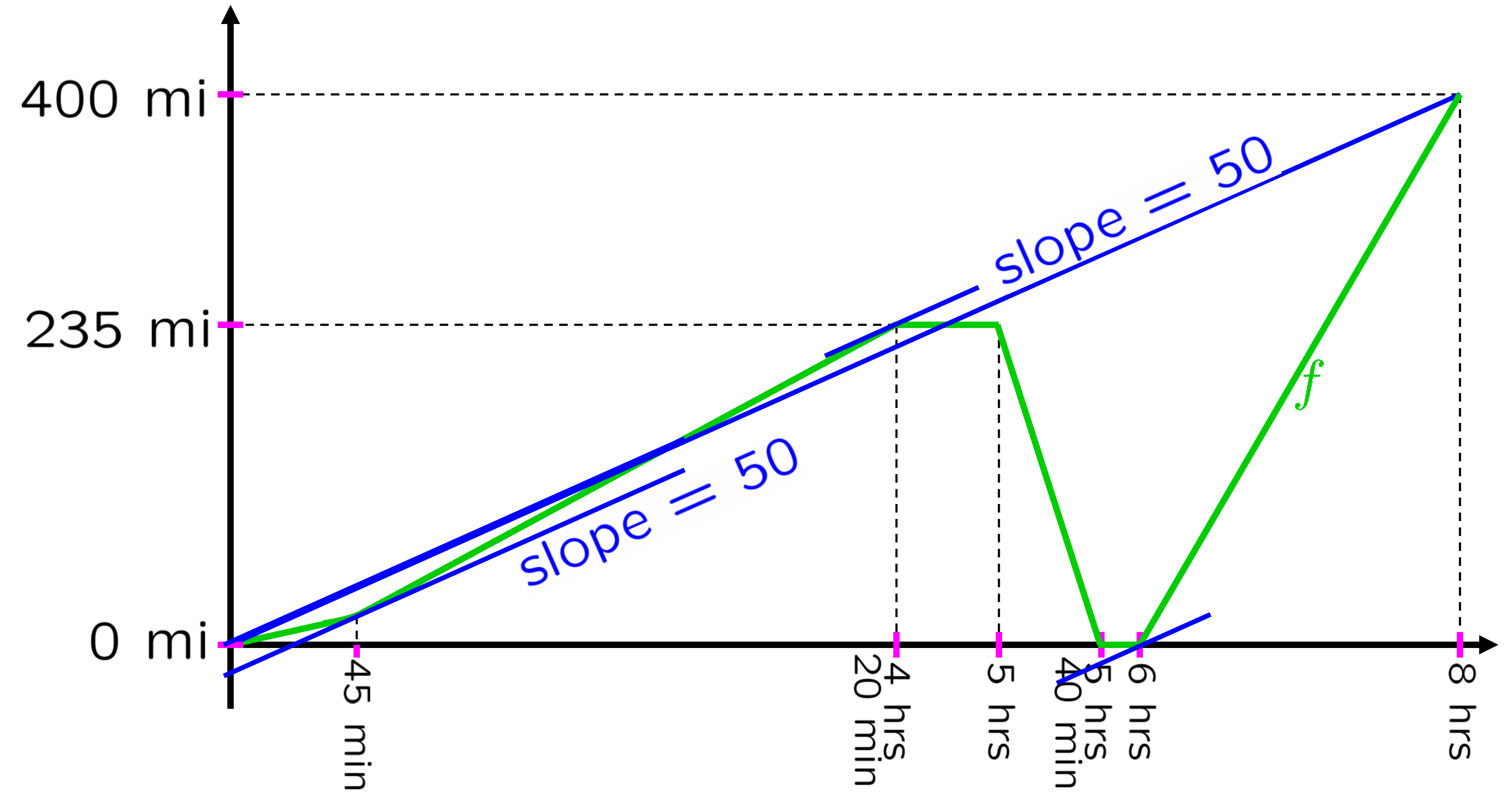
Average velocity over the eight hrs:

$$\frac{[f(8)] - [f(0)]}{8-0} = \frac{400-0}{8} = 50$$

Average velocity is 50 mph from 0 hrs to 8 hrs.

instantaneous velocity is 50 mph at some time

MEAN VALUE THEOREM:

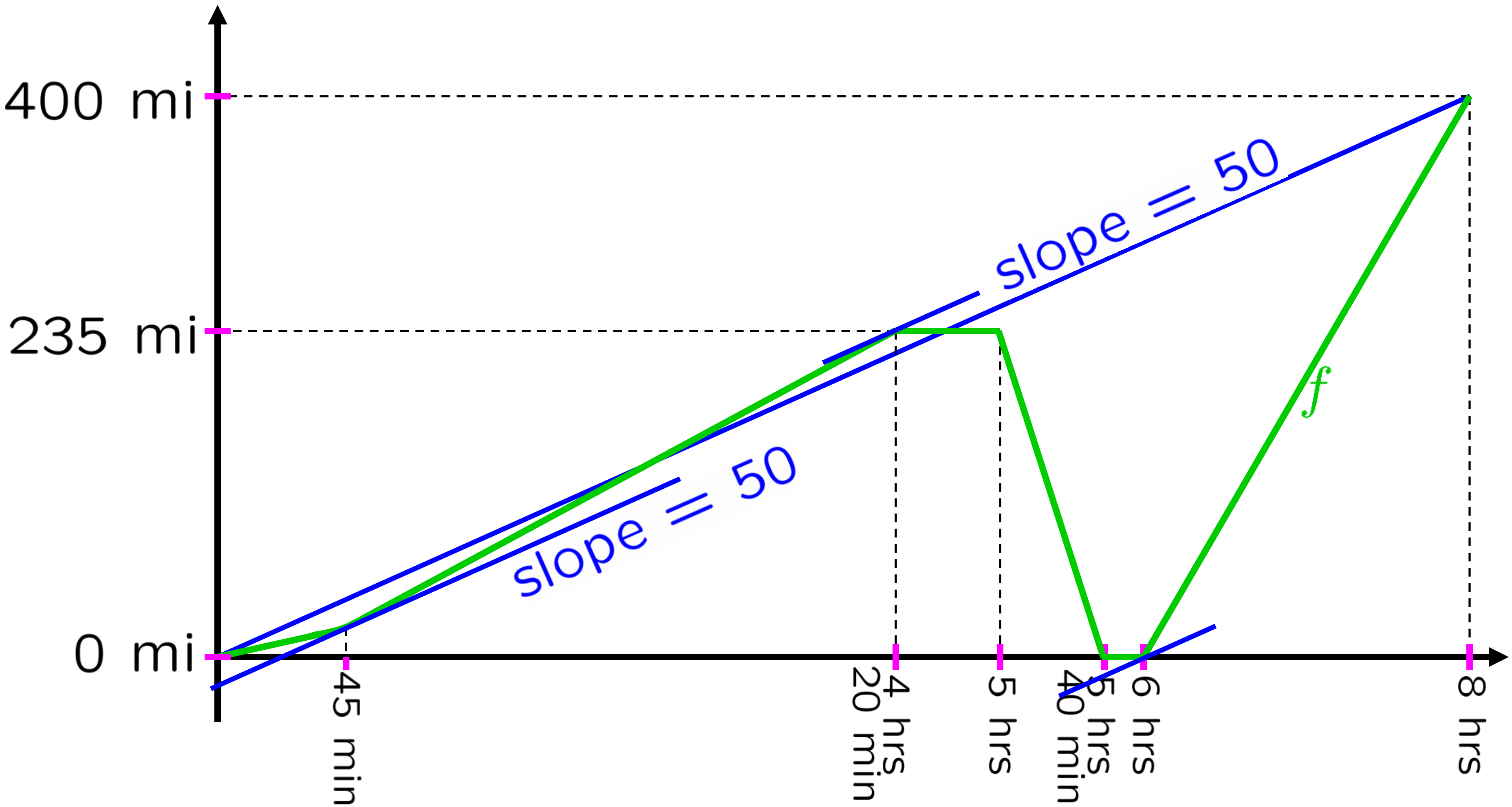


Pf: Always $>$ 50 mph $\Rightarrow$ overshoot
Sometimes $\leq$ 50 mph
Always $<$ 50 mph $\Rightarrow$ undershoot
Sometimes $\geq$ 50 mph
IVT: Sometimes $=$ 50 mph QED

instantaneous velocity is 50 mph at some time (45 min)

Sometimes more than one time.

MEAN VALUE THEOREM:



Expect: Every avg. velocity is an instantaneous velocity.
Expect: Every sec. slope is a tangent slope.

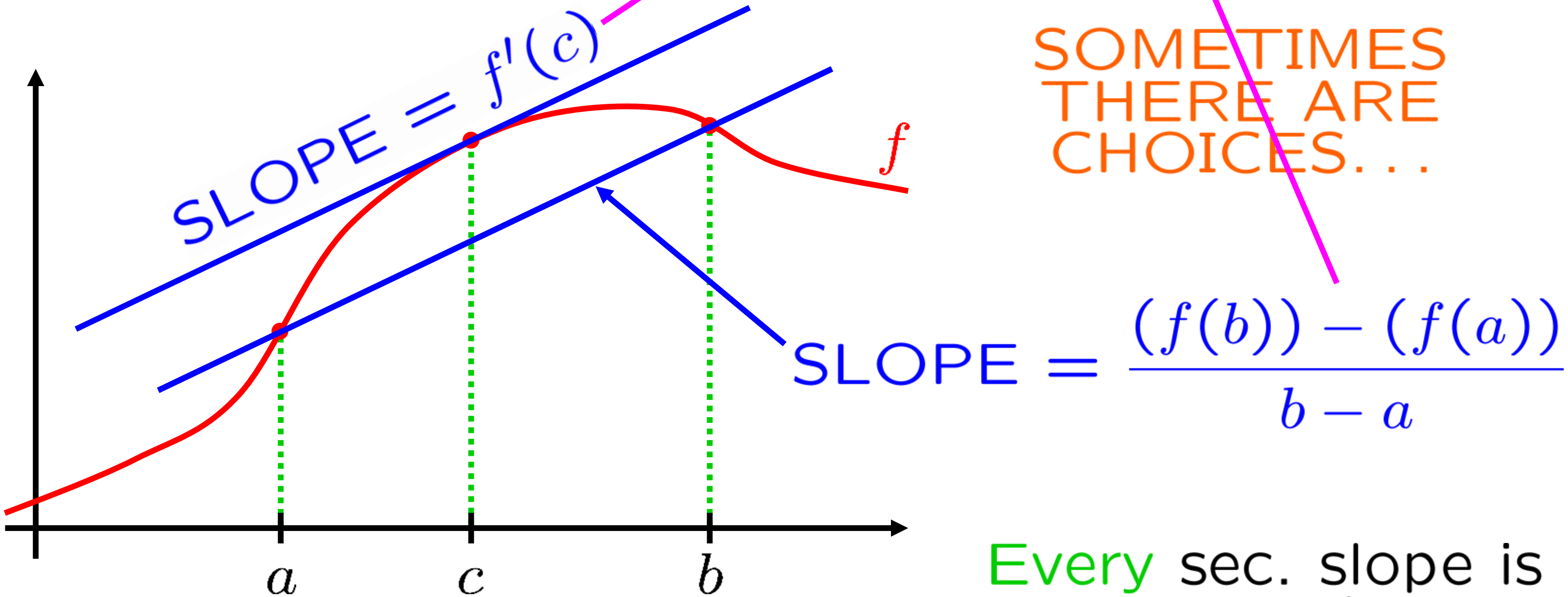
instantaneous velocity is 50 mph at some time (45 min)

Sometimes more than one time.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.



SOMETIMES THERE ARE CHOICES. . .

Every sec. slope is a tangent slope.

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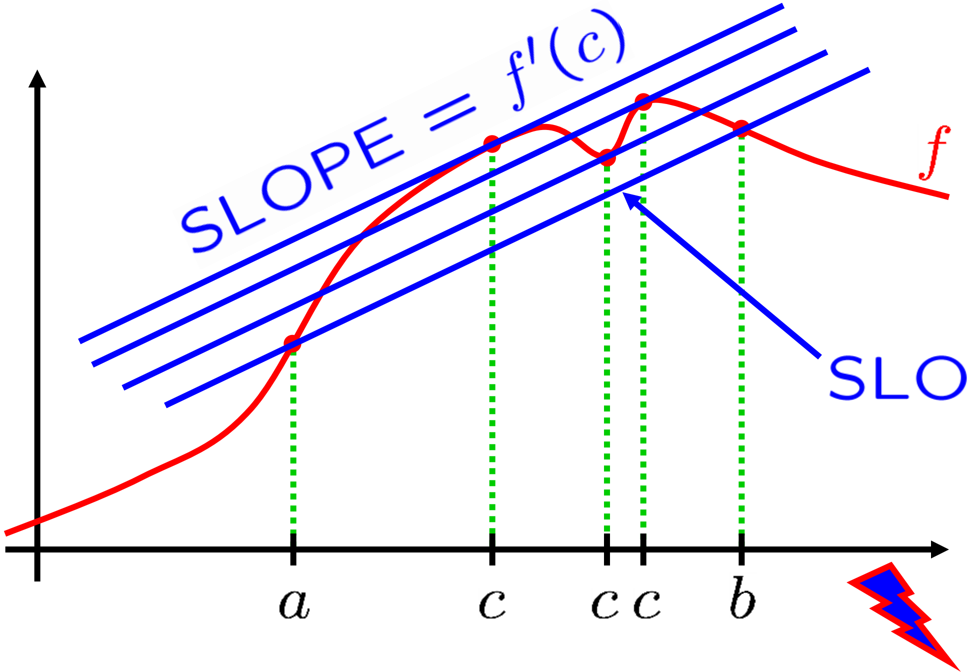

SOMETIMES THERE ARE CHOICES...

Every sec. slope is a tangent slope.

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Let $a, b \in \mathbb{R}$ and assume that $a < b$.
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Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

0

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I,
then f is increasing on I.

CONSEQUENSES . . .

ROLLE'S THEOREM:

Assume that f is continuous on $[a, b]$,
that f is differentiable on (a, b)
and that $f(a) = f(b)$.
Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Every sec. slope is a tangent slope.

Idea: If some secant line is horizontal, then some tangent line is horizontal.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I,
then f is increasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) < f(b)$.
Assume $a < b$. Assume $f(a) \geq f(b)$.
Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$(a, b) \subseteq I$

$\dfrac{(f(b)) - (f(a))}{b - a} \leq 0$

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line runs uphill,
then every secant line runs uphill.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

DECREASING TEST:

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

Proof: Let $a, b \in I$. Want: $f(a) > f(b)$.
Assume $a < b$. Assume $f(a) \le f(b)$.
Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$(a, b) \subseteq I$

$\dfrac{(f(b)) - (f(a))}{b - a} \geq 0$

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line runs downhill,
then every secant line runs downhill.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (ONE-TO-ONE TEST):

If $f'(x) \neq 0$, for all x in an interval I,
then f is one-to-one on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) \neq f(b)$.

Assume $a \neq b$. Assume $f(a) = f(b)$.

Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$(a, b) \subseteq I$

$\dfrac{(f(b)) - (f(a))}{b - a} = 0$

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line is not horizontal,
then every secant line is not horizontal.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (CONSTANT TEST):

If $f'(x) = 0$, for all x in an interval I,
then f is constant on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $a, b \in I$. Want: $f(a) = f(b)$.

Assume $f(a) \neq f(b)$. $a \neq b$

Want: Contradiction.

Choose $c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

$(a,b) \subseteq I$

$$f'(c) = 0 \neq \frac{(f(b)) - (f(a))}{b - a}$$

Contradiction. QED

Every sec. slope is a tangent slope.

Idea: If every tangent line is horizontal,
then every secant line is horizontal.

MEAN VALUE THEOREM:

Let $a, b \in \mathbb{R}$ and assume that $a < b$.
Assume that f is continuous on $[a, b]$,
and that f is differentiable on (a, b).

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{(f(b)) - (f(a))}{b - a}$.

THEOREM (CONSTANT TEST):

If $f'(x) = 0$, for all x in an interval I,
then f is constant on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

EQUALITY OF DERIVATIVES:

If $g'(x) = h'(x)$, for all x in an interval I,
then $g - h$ is constant on I;
that is, $\exists c \in \mathbb{R}$ s.t., $\forall x \in I$,
$g(x) = (h(x)) + c$.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

Proof: Let $f := g - h$.
Then $\forall x \in I$, $f'(x) = (g'(x)) - (h'(x)) = 0$.
So f is constant on I. Choose $c \in \mathbb{R}$ s.t. $f = c$ on I.
That is, $\forall x \in I$, $f(x) = c$.
That is, $\forall x \in I$, $(g(x)) - (h(x)) = c$. QED

f is **decreasing on** I if: $\forall u, v \in I, \quad u < v \Rightarrow f(v) < f(u)$

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I, then f is decreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

f is **nonincreasing on** I if: $\forall u, v \in I, \quad u \leq v \Rightarrow f(v) \leq f(u)$

NONINCREASING TEST:

If $f'(x) \leq 0$, for all x in an interval I, then f is nonincreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

f is **increasing on** I if: $\forall u, v \in I, \quad u < v \Rightarrow f(u) < f(v)$

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I, then f is increasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

f is **nondecreasing on** I if: $\forall u, v \in I, \quad u \leq v \Rightarrow f(u) \leq f(v)$

NONDECREASING TEST:

If $f'(x) \geq 0$, for all x in an interval I, then f is nondecreasing on I.

works for any kind of interval (open, closed, half-open) (bdd, unbdd)

f is **decreasing on** I if: $\forall u, v \in I, \quad u < v \Rightarrow f(v) < f(u)$

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I, then f is decreasing on I.

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