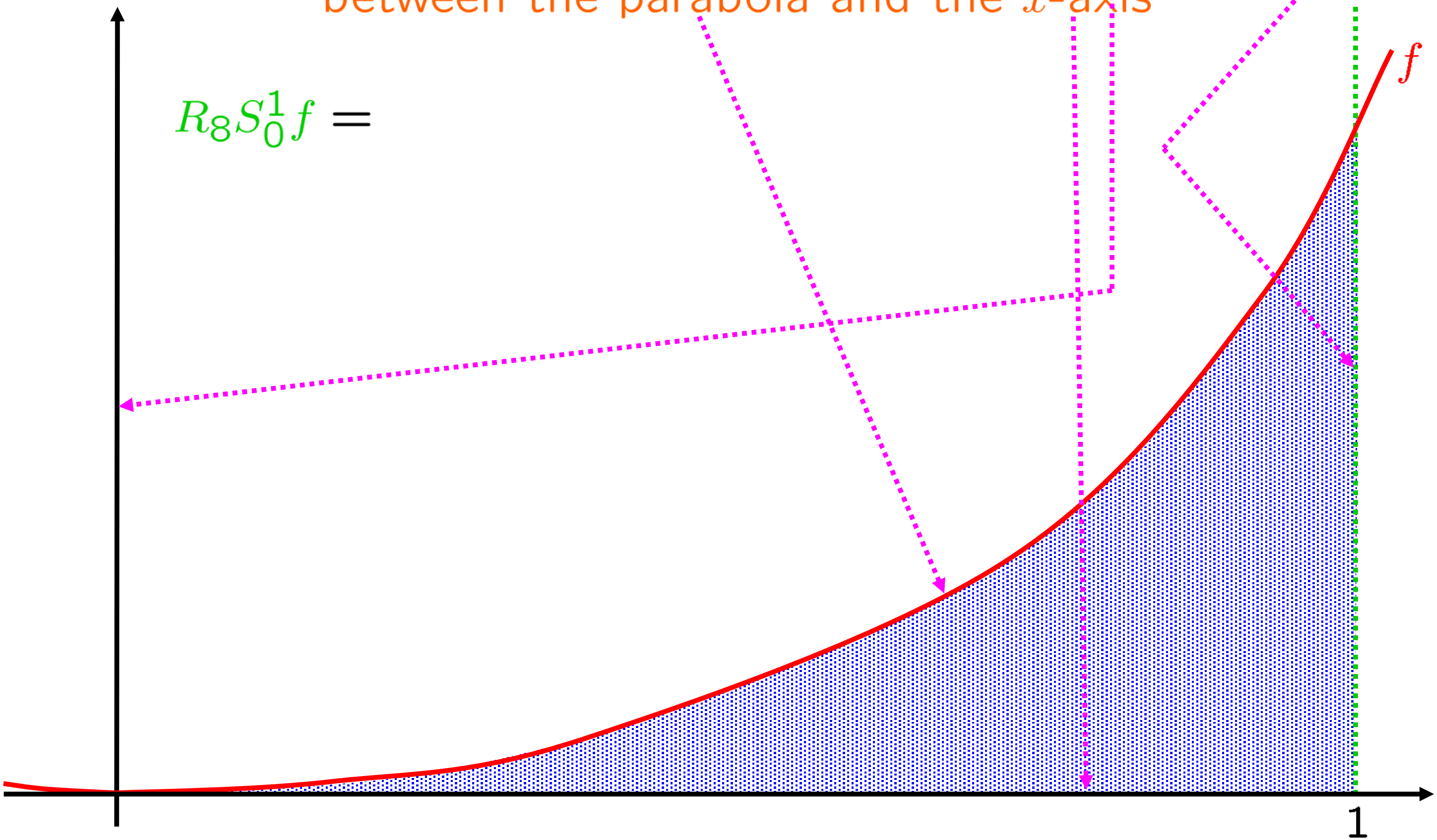
Financial Mathematics

Example of a definite integral

EXAMPLE: Use rectangles to estimate the area "under" the parabola $y = x^2$ from $x = 0$ to $x = 1$.

between the parabola and the x-axis



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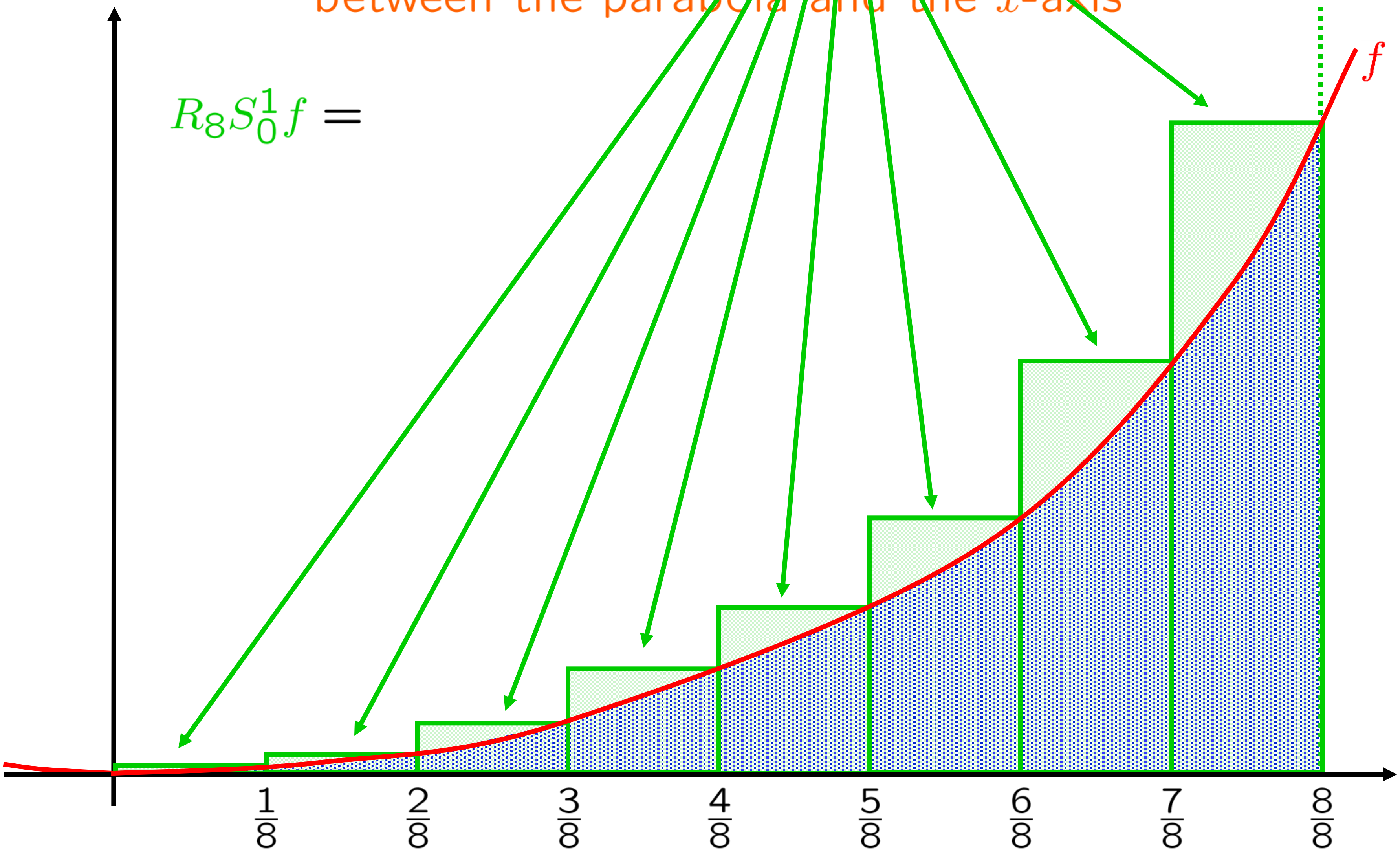

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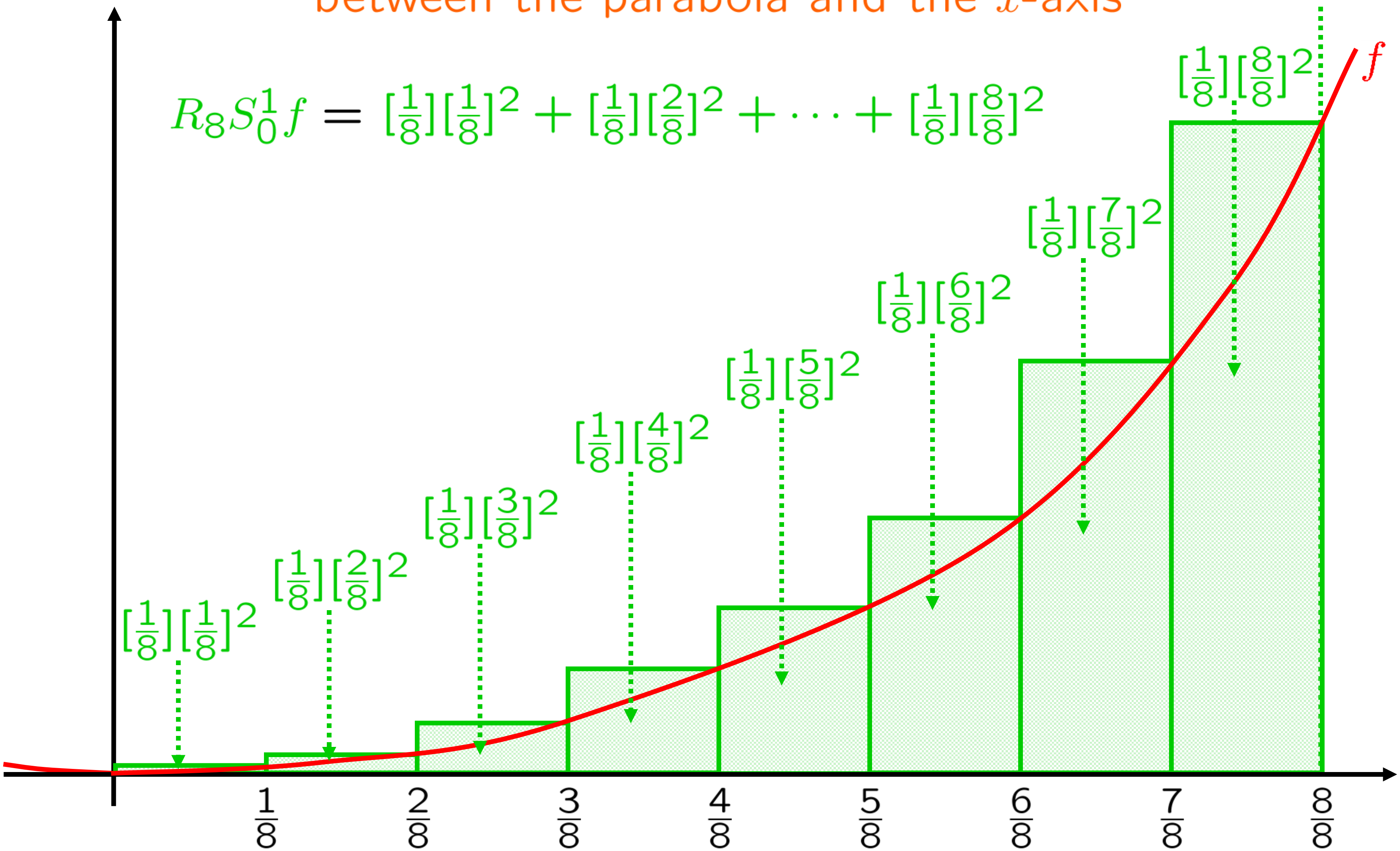

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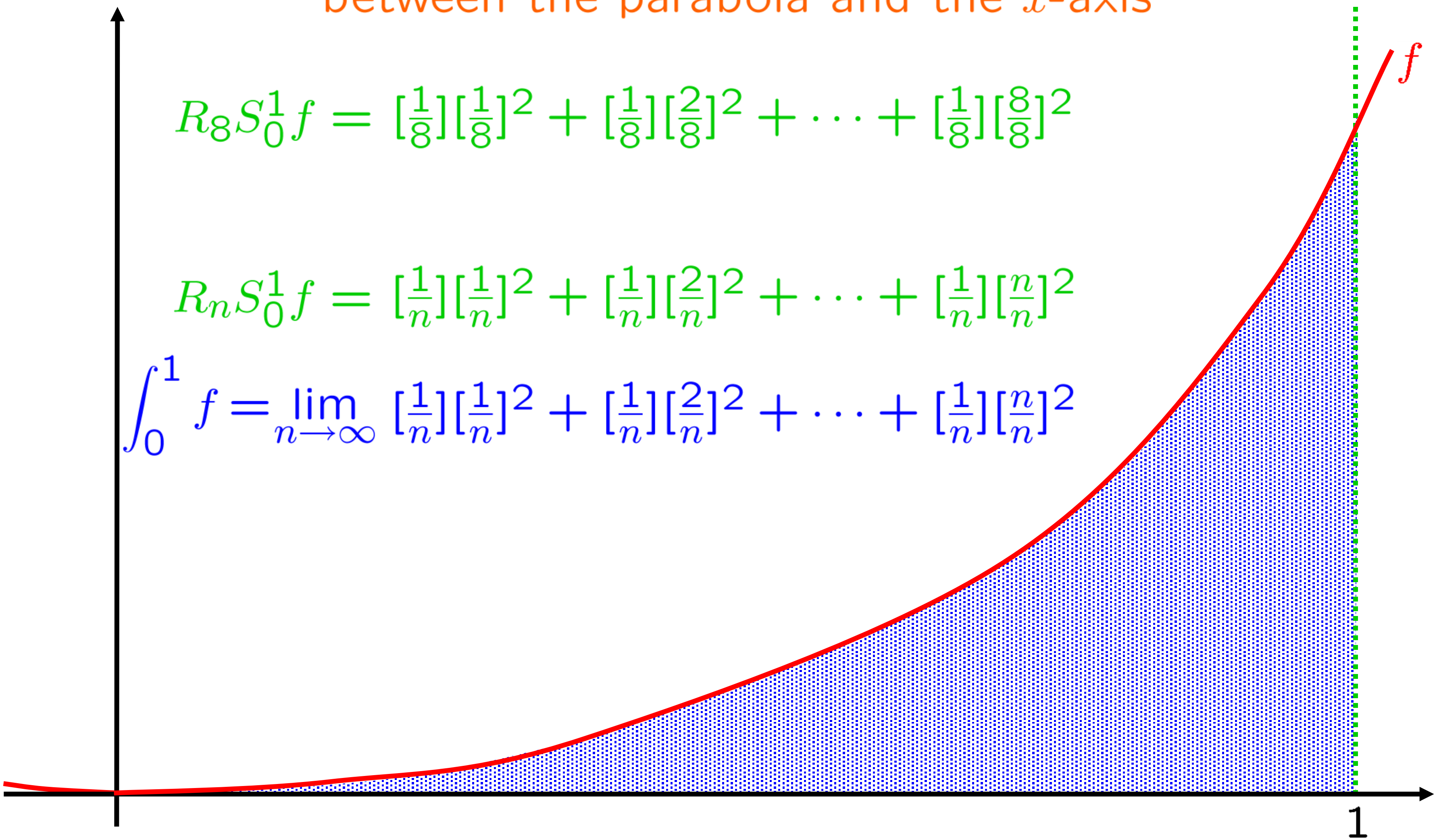
$$R_8 S_0^1 f = [\tfrac{1}{8}][\tfrac{1}{8}]^2 + [\tfrac{1}{8}][\tfrac{2}{8}]^2 + \cdots + [\tfrac{1}{8}][\tfrac{8}{8}]^2$$

$$R_n S_0^1 f = [\tfrac{1}{n}][\tfrac{1}{n}]^2 + [\tfrac{1}{n}][\tfrac{2}{n}]^2 + \cdots + [\tfrac{1}{n}][\tfrac{n}{n}]^2$$

$$\int_0^1 f = \lim_{n\to\infty} [\tfrac{1}{n}][\tfrac{1}{n}]^2 + [\tfrac{1}{n}][\tfrac{2}{n}]^2 + \cdots + [\tfrac{1}{n}][\tfrac{n}{n}]^2$$



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$$= \lim_{n\to\infty} \tfrac{1^2}{n^3} + \tfrac{2^2}{n^3} + \cdots + \tfrac{n^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{1^2 + \cdots + n^2}{n^3} = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}$$

But first... prove the IOU.

IOU: $1^2 + \cdots + n^2 = \dfrac{2n^3 + 3n^2 + n}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 f(v)\,dv = \int_0^1 f(t)\,dt = \int_0^1 f(s)\,ds = \int_0^1 f = \frac{1}{3}$$

Next topic: An easier way to show $\int_0^1 x^2\,dx = \frac{1}{3}$

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Definition:
A **(real) sequence** is
an infinite ordered list of real numbers,
separated by commas.

IOU: $1^2 + \cdots + n^2 \quad = \quad \frac{2n^3 + 3n^2 + n}{6}$

Pf of IOU: For any sequence $a_0, a_1, a_2, \ldots,$

$\boxed{\triangle a_0, \triangle a_1, \triangle a_2, \ldots}$ is the sequence

$a_1 - a_0, \quad a_2 - a_1, \quad a_3 - a_2, \quad \ldots$

E.g.: Let $x_n := 2n$, so $x_0, x_1, x_2, \ldots$ is

$0, \quad 2, \quad 4, \quad 6, \quad \ldots$

Then $\triangle x_0, \triangle x_1, \triangle x_2, \ldots$ is

$2-0, \quad 4-2, \quad 6-4, \quad 8-6, \quad \ldots$

More simply, $\triangle(2n) = (2(n+1)) - (2n) = 2.$

Generally, $\boxed{\triangle a_n} = ([a_n]_{n:\to n+1}) - (a_n).$

$$\triangle n^4 = ((n+1)^4) - (n^4)$$

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More simply, $\triangle(2n) = (2(n+1)) - (2n) = 2.$

Generally, $\triangle a_n = ([a_n]_{n:\to n+1}) - (a_n).$

$$\triangle n^4 = ((n+1)^4) - (n^4)$$
$$= (\cancel{n^4} + 4n^3 + 6n^2 + 4n + 1) - (\cancel{n^4})$$
$$= 4n^3 + 6n^2 + 4n + 1$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

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n^4 is quartic (degree $= 4$)

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cubic (degree $= 3$)

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1 2 1
1 3 3 1
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Let $s_0, s_1, s_2, \ldots$ be the sequence

$0, \quad 1^2, \quad 1^2 + 2^2, \quad 1^2 + 2^2 + 3^2, \quad 1^2 + 2^2 + 3^2 + 4^2, \quad \ldots$

$\boxed{\triangle a_n} = ([a_n]_{n:\to n+1}) - (a_n)$

n^4 is quartic (degree = 4)

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

$$\begin{aligned} \triangle n^4 &= ((n+1)^4) - (n^4) \\ &= (\cancel{n^4} + 4n^3 + 6n^2 + 4n + 1) - (\cancel{n^4}) \\ &= 4n^3 + 6n^2 + 4n + 1 \end{aligned}$$

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i.e., $s_n = 1^2 + 2^2 + \cdots + n^2$ (understood that $s_0 = 0$).

Then $\triangle s_n = (n+1)^2$

$$s_{n+1} = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2$$
$$s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

---

n^4 is quartic (degree $= 4$)

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$$= 4n^3 + 6n^2 + 4n + 1$$

cubic (degree $= 3$)

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IOU: $1^2 + \cdots + n^2 \quad = \quad \dfrac{2n^3 + 3n^2 + n}{6}$ cubic (degree = 3)

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$\frac{1}{3} \times \{\triangle n^3 = 3n^2 + 3n + 1\}$

$\frac{1}{2} \times \{\triangle n^2 = 2n + 1\}$

$\triangle n = 1$

ADD

$$\triangle\left(\tfrac{1}{3}n^3\right) = n^2 + n + \tfrac{1}{3}$$

$$\triangle\left(\tfrac{1}{2}n^2\right) = n + \tfrac{1}{2}$$

$$\tfrac{1}{3} + \tfrac{1}{2} = \tfrac{2}{6} + \tfrac{3}{6} = \tfrac{5}{6}$$

$$\triangle\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2\right) = n^2 + 2n + \tfrac{5}{6}$$

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$$\triangle n^3 = 3n^2 + 3n + 1$$

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$$\frac{1}{6} \times \left[\triangle n = 1 \right]$$

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$$\triangle\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n - s_n\right) = 0$$

$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n - s_n$ is a constant sequence.

$$\left[\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n - s_n\right]_{n:\to 0} = 0 + 0 + 0 - 0 = 0$$

$$\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n - s_n = 0$$

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$\dfrac{2n^3 + 3n^2 + n}{6}$ QED

STOP

$\|$

$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = s_n = 1^2 + 2^2 + \cdots + n^2$

$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n - s_n = 0$