Financial Mathematics

Sequences and series

Definition:

A **(real) sequence** is an infinite ordered list of real numbers, separated by commas.

e.g.: $1, 2, 3, 4, 5, 6, \ldots \neq 2, 1, 3, 4, 5, 6, \ldots$

reverse same

$-1, -1, -1, \ldots$

$1, (1/2), (1/3), \ldots$

$n = 0 \quad n = 1 \quad n = 2 \quad n = 3$

$$\underbrace{1, (1/2), (1/4), (1/8), \ldots}$$

$$\|$$

nth term $\quad (2^{-n})_{n=0}^{\infty}$

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e.g.: $1, 2, 3, 4, 5, 6, \ldots$

Tails of this sequence:

0th: $1, 2, 3, 4, 5, 6, \ldots$

1st: $2, 3, 4, 5, 6, \ldots$

2nd: $3, 4, 5, 6, \ldots$

3rd: $4, 5, 6, \ldots$

4th: $5, 6, \ldots$

etc.

e.g.: $\liminf 0, 1, 0, 1, 0, 1, \ldots = 0$

0th tatail infima: $0, 0, 0, 0, \ldots \to 0$

Next: $\limsup$

Def'n: $\liminf_{m\to\infty} a_m$ is the limit of

tail infima

$$\begin{cases} \inf\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \inf\{a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \inf\{a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \inf\{a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \inf\{a_5, a_6, a_7, a_8, \ldots\}, \\ \inf\{a_6, a_7, a_8, \ldots\}, \\ \vdots \end{cases}$$

$$\liminf_{m\to\infty} a_m := \lim_{m\to\infty} \left[\inf_{n\geq m} a_n \right]$$

e.g.: $\limsup 0, 1, 0, 1, 0, 1, \ldots = 1$

tail suprema: 1,1,1,1,... $\to 1$

Def'n: $\limsup_{m\to\infty} a_m$ is the limit of

tail suprema

$$\begin{cases} \sup\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \sup\{a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \sup\{a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \sup\{a_4, a_5, a_6, a_7, a_8, \ldots\}, \\ \sup\{a_5, a_6, a_7, a_8, \ldots\}, \\ \sup\{a_6, a_7, a_8, \ldots\}, \\ \vdots \end{cases}$$

$$\limsup_{m\to\infty} a_m := \lim_{m\to\infty} \left[\sup_{n\geq m} a_n \right]$$

e.g.: $\limsup 0, 1, 0, 1, 0, 1, \ldots = 1$

Fact: lim exists iff

lim inf and lim sup are equal

Def'n: $\limsup\limits_{m\to\infty} a_m$ is the limit of

tail suprema

$$\begin{aligned}
&\sup\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},\\
&\sup\{a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},\\
&\sup\{a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},\\
&\sup\{a_4, a_5, a_6, a_7, a_8, \ldots\},\\
&\sup\{a_5, a_6, a_7, a_8, \ldots\},\\
&\sup\{a_6, a_7, a_8, \ldots\},\\
&\vdots
\end{aligned}$$

$$\limsup_{m\to\infty} a_m := \lim_{m\to\infty}\left[\sup_{n\geq m} a_n\right]$$

Problem: Find lim inf and lim sup of

$$9, 3, 9-\frac{1}{2}, 3+\frac{1}{2}, 9-\frac{2}{3}, 3+\frac{2}{3}, 9-\frac{3}{4}, 3+\frac{3}{4}, 9-\frac{4}{5}, 3+\frac{4}{5}, \ldots$$

Note: blue sequence $\longrightarrow 8$, red sequence $\longrightarrow 4$

Def'n: $\limsup\limits_{m\to\infty} a_m$ is the limit of

$$\sup\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},$$
$$\sup\{a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},$$
$$\sup\{a_3, a_4, a_5, a_6, a_7, a_8, \ldots\},$$
$$\sup\{a_4, a_5, a_6, a_7, a_8, \ldots\},$$
$$\sup\{a_5, a_6, a_7, a_8, \ldots\},$$
$$\sup\{a_6, a_7, a_8, \ldots\},$$
$$\vdots$$

$$\limsup_{m\to\infty} a_m := \lim_{m\to\infty}\left[\sup_{n\geq m} a_n\right]$$

Problem: Find $\liminf$ and $\limsup$ of

$$9, 3, 9-\frac{1}{2}, 3+\frac{1}{2}, 9-\frac{2}{3}, 3+\frac{2}{3}, 9-\frac{3}{4}, 3+\frac{3}{4}, 9-\frac{4}{5}, 3+\frac{4}{5}, \ldots$$

Note: blue sequence $\to 8$, red sequence $\to 4$

Tail suprema:

$$9, 9-\frac{1}{2}, 9-\frac{1}{2}, 9-\frac{2}{3}, 9-\frac{2}{3}, 9-\frac{3}{4}, 9-\frac{3}{4}, 9-\frac{4}{5}, \ldots \to 9-1=8$$

$$\text{limsup} = 8$$

Tail infima:

$$3, 3, 3+\frac{1}{2}, 3+\frac{1}{2}, 3+\frac{2}{3}, 3+\frac{2}{3}, 3+\frac{3}{4}, 3+\frac{3}{4}, \ldots \to 3+1=4$$

$$\text{liminf} = 4$$

Fact: $\limsup = \sup\{\text{limits of subsequences}\}$

Fact: $\liminf = \inf\{\text{limits of subsequences}\}$

Definition:

A **(real) series** is an infinite ordered list of real numbers, separated by plus signs.

e.g.: $1 + 2 + 3 + 4 + 5 + 6 + \cdots$

$(-1) + (-1) + (-1) + \cdots$ ← $-1 - 1 - 1 - \cdots$

$1 + (1/2) + (1/3) + \cdots$

$$\underbrace{1 + (1/2) + (1/4) + (1/8) + \cdots}_{} = \sum_{n=0}^{\infty} 2^{-n}$$

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sequence of partial sums:

$1,$
$1 + 2,$ 3
$1 + 2 + 3,$ 6
$1 + 2 + 3 + 4,$ 10
$1 + 2 + 3 + 4 + 5,$ 15
$1 + 2 + 3 + 4 + 5 + 6,$ 21
$1 + 2 + 3 + 4 + 5 + 6 + 7, \ldots$ 28

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sequence of partial sums:

$$1, 3, 6, 10, 15, 21, 28, \ldots$$

3

6

10

15

21

28

Definition:
A **(real) series** is
an infinite ordered list of real numbers,
separated by plus signs.

e.g.: $1+2+3+4+5+6+\cdots=\infty$

sequence of partial sums:

$$1,3,6,10,15,21,28,\ldots\to\infty$$

Definition: $\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,\infty\}$

An **extended real number** is
a real number or $\infty=+\infty$ or $-\infty$.

Def: We say $s_1+s_2+s_3+\cdots$ **converges to** L
and write $s_1+s_2+s_3+\cdots=L$ if

$$\lim_{n\to\infty}(s_1+\cdots+s_n)=L.$$

A series may converge (to a real number), or diverge.

$$1 + (1/2) + (1/4) + \cdots + (1/2^{n-1}) = 2 - (1/2^n)$$

$n \to \infty$

e.g.: $1 + (1/2) + (1/4) + (1/8) + \cdots = 2$

e.g.: $1 + 2 + 3 + 4 + 5 + 6 + \cdots = \infty$

sequence of partial sums:

$$1, 3, 6, 10, 15, 21, 28, \ldots \to \infty$$

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$$\lim_{n\to\infty} (s_1 + \cdots + s_n) = L.$$

A series may converge (to a real number), or diverge. It may diverge to ∞ or to $-\infty$.

e.g.: $-1-1-1-\cdots = -\infty$

e.g.: $1+(1/2)+(1/4)+(1/8)+\cdots = 2$

e.g.: $1+2+3+4+5+6+\cdots = \infty$

sequence of partial sums:

$$1, 3, 6, 10, 15, 21, 28, \ldots \to \infty$$

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$$\lim_{n\to\infty}(s_1+\cdots+s_n) = L.$$

A series may converge (to a real number), or diverge. It may diverge to ∞ or to $-\infty$.
It may have no sum at all.

e.g.: $-1 - 1 - 1 - \cdots = -\infty$

e.g.: $1 + (1/2) + (1/4) + (1/8) + \cdots = 2$

e.g.: $1 + 2 + 3 + 4 + 5 + 6 + \cdots = \infty$

$1 + 2 + 3 + 4 + 5 + 6 + \cdots$ has a sum.

e.g.: $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ has no sum.

Definition: $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

An **extended real number** is a real number or $\infty = +\infty$ or $-\infty$.

Partial sums: $1, 0, 1, 0, 1, 0, 1, 0, \ldots$ does not converge

Definition:

A **(real) series** is
an infinite ordered list of real numbers,
separated by plus signs.

e.g.: $1+2+3+4+5+6+\cdots = \infty$

$-1-1-1-\cdots = -\infty$

more on this later...

$1+(1/2)+(1/3)+(1/4)+\cdots = ??$

$$\underbrace{1+(1/2)+(1/4)+(1/8)+\cdots}_{\displaystyle \| \atop \displaystyle \sum_{n=0}^{\infty} 2^{-n}} = 2$$

Infinite linear combinations of functions and expressions

Def'n: The **linear operations** are scalar multiplication and addition.

Def'n: $\forall$integers $j \geq 1$,

let $f_j : A_j \to \mathbb{R}$ be a function.

Let $c_1, c_2, c_3, \ldots \in \mathbb{R}$ be a sequence.

The **(infinite) linear combination**

of $f_1, f_2, f_3, \ldots$

with coefficients $c_1, c_2, c_3, \ldots$

is $c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots$.

expressions of q...

Domain:

$\{s \in \mathbb{R} \,|\, c_1[f_1(s)] + c_2[f_2(s)] + c_3[f_3(s)] + \cdots$ converges$\}$

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The **(infinite) linear combination**

of $f_1(q), f_2(q), f_3(q), \ldots$ expressions of s...

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Let $c_1, c_2, c_3, \ldots \in \mathbb{R}$ be a sequence.
The **(infinite)** **linear combination**
of $f_1(s), f_2(s), f_3(s), \ldots$
with coefficients $c_1, c_2, c_3, \ldots$
is $c_1[f_1(s)] + c_2[f_2(s)] + c_3[f_3(s)] + \cdots$.

Power series

Def'n: A **polynomial** **in** x is a

finite linear combination of $1, x, x^2, x^3, x^4, \ldots$.

e.g.: $4 + 7x + 8x^2$

$2 - 6x + 3x^2 + \pi x^3 - ex^4$

8

$x^{1000000}$

$2 - 7x^{\text{googol plex}}$

Def'n: A **power series** **in** x is a (possibly)

infinite linear combination of $1, x, x^2, x^3, x^4, \ldots$.

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$1 + x + x^2 + x^3 + \cdots$

$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$1 + x + (2!)x^2 + (3!)x^3 + \cdots$

$1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$

expressions of u...

Power series

Def'n: A **power series in** u is a (possibly) infinite linear combination of $1, u, u^2, u^3, u^4, \ldots$.

e.g.: $4 + 7u + 8u^2$

$2 - 6u + 3u^2 + \pi u^3 - eu^4$

8

$u^{1000000}$

$2 - 7u^{\text{googol plex}}$

$1 + u + u^2 + u^3 + \cdots$

$1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots$

$1 + u + (2!)u^2 + (3!)u^3 + \cdots$

$1 - u + \frac{1}{2}u^2 - \frac{1}{3}u^3 + \cdots$