Financial Mathematics

Complex numbers

Convention on scalars

We're "real people", meaning that,
unless otherwise specified,
number means *real* number, as does **scalar**.

However, sometimes we use...

Complex numbers

Def'n: A **complex number** is an expression of the form $a + bi$, where $a, b \in \mathbb{R}$.

$\mathrm{Re}(3+4i) = 3$ $\quad$ $\mathrm{Re}(\sqrt{2}-8i) = \sqrt{2}$ $\quad$ $\mathrm{Re}(e-\pi i) = e$

e.g.: $3 + 4i$, $\quad$ $\sqrt{2} - 8i$, $\quad$ $e - \pi i$

$\mathrm{Im}(3+4i) = 4$ $\quad$ $\mathrm{Im}(\sqrt{2}-8i) = -8$ $\quad$ $\mathrm{Im}(e-\pi i) = -\pi$

Def'n: The **real part** of $z = a + bi$ is

$$\boxed{\mathrm{Re}(z)} = a.$$

Def'n: The **imaginary part** of $z = a + bi$ is

$$\boxed{\mathrm{Im}(z)} = b.$$

SKILL
Finding real & imaginary parts of cx numbers

Addition of complex numbers

Definition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

i.e.: addition of complex numbers is accomplished by adding together real and imaginary parts.

e.g.: $(3 + 4i) + (7 - 8i) = 10 - 4i$

SKILL
addition of cx numbers

The geometry of complex addition

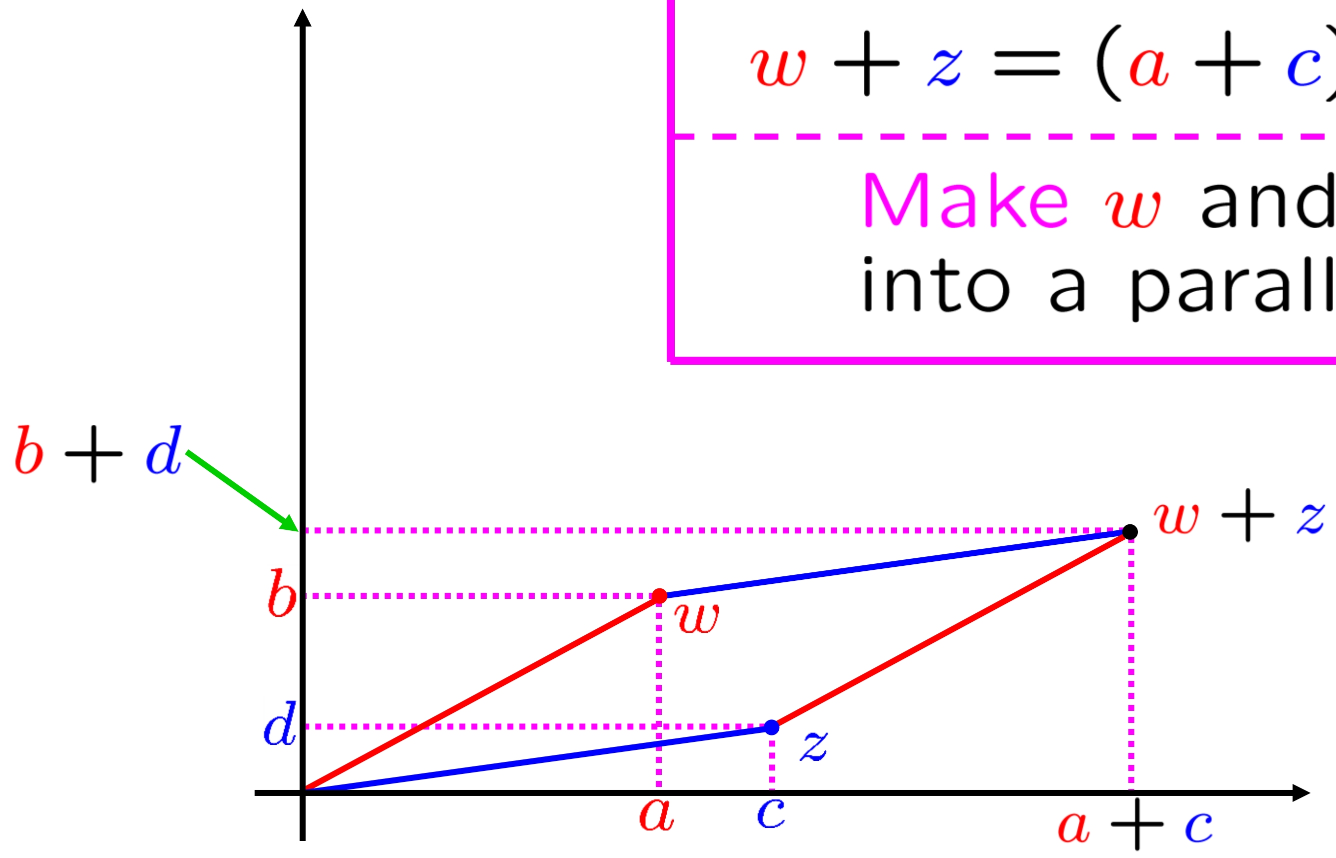
Question: Where is $w + z$?

Answer:

$w = a + bi \qquad z = c + di$

$$w + z = (a + c) + (b + d)i$$

Make w and z into a parallelogram.



The geometry of complex addition

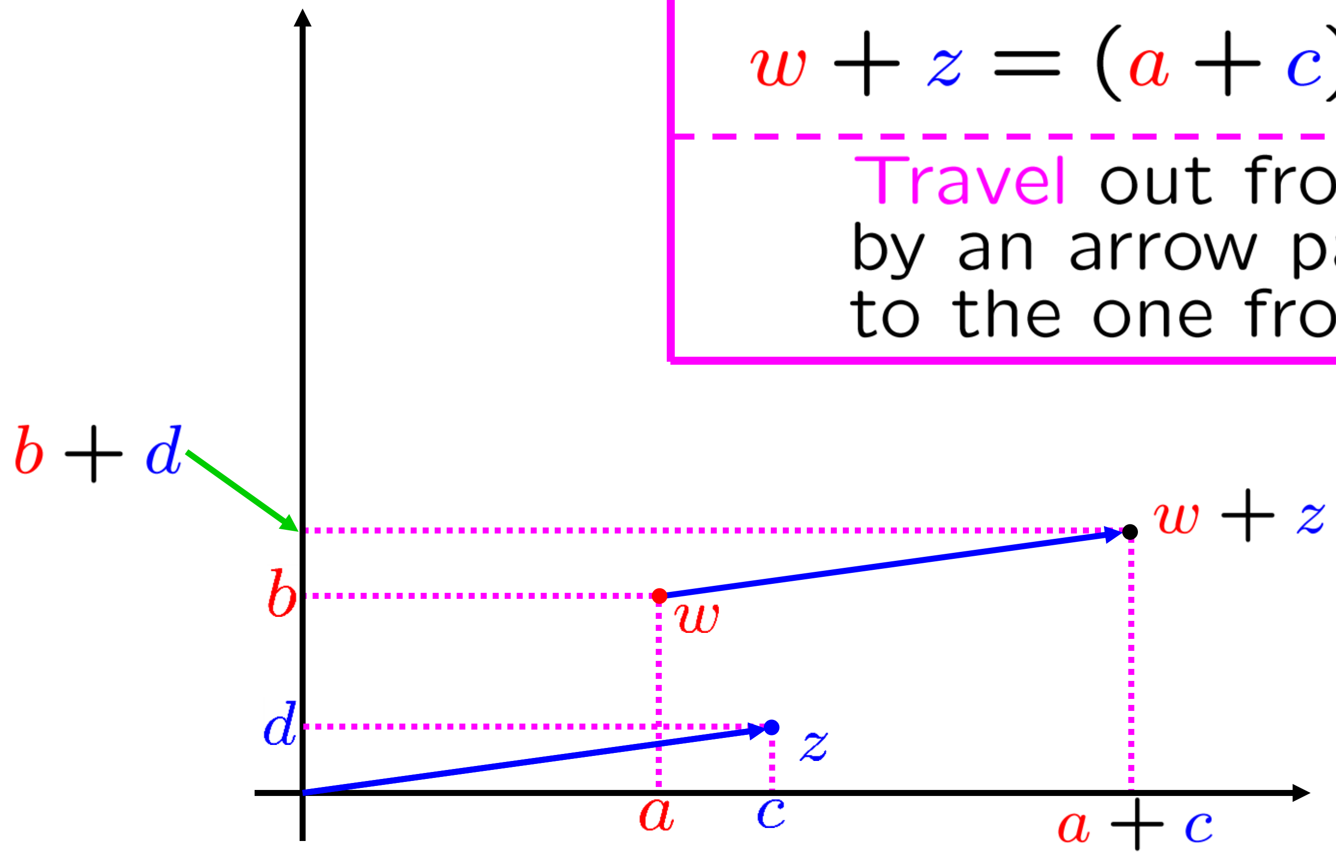
Question: Where is $w + z$?

Answer:

$w = a + bi \qquad z = c + di$

$w + z = (a + c) + (b + d)i$

Travel out from w
by an arrow parallel
to the one from 0 to z.



The geometry of complex addition

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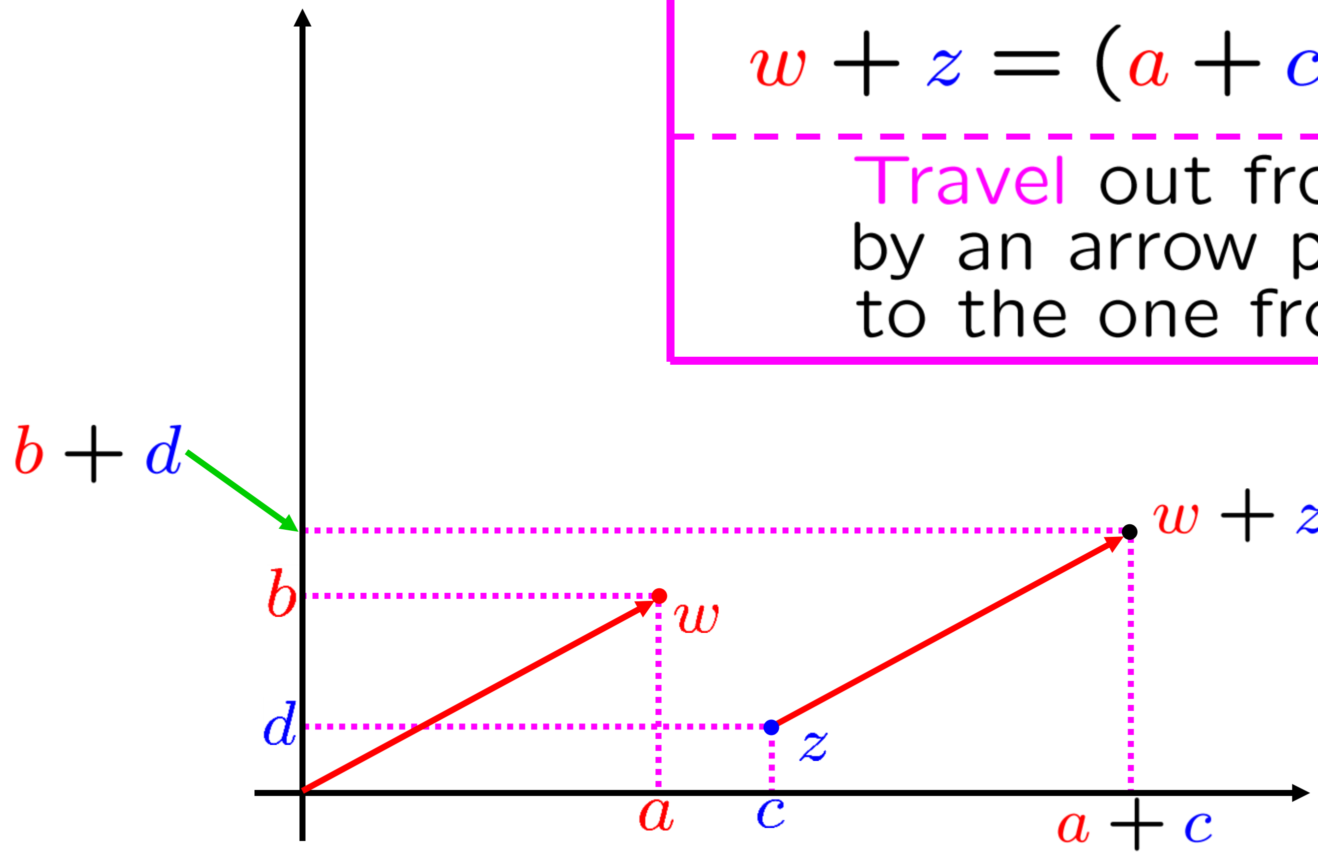

Absolute value of complex numbers

Definition:

The $\underbrace{\textbf{absolute value}}^{\text{or } \textbf{modulus}}$ of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

e.g.: $|4 - 3i| = \sqrt{16 + 9} = 5$

SKILL
Finding absolute values

Absolute value of complex numbers

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The $\underbrace{\textbf{absolute value}}^{\text{or } \textbf{modulus}}$ of $z = a + bi$

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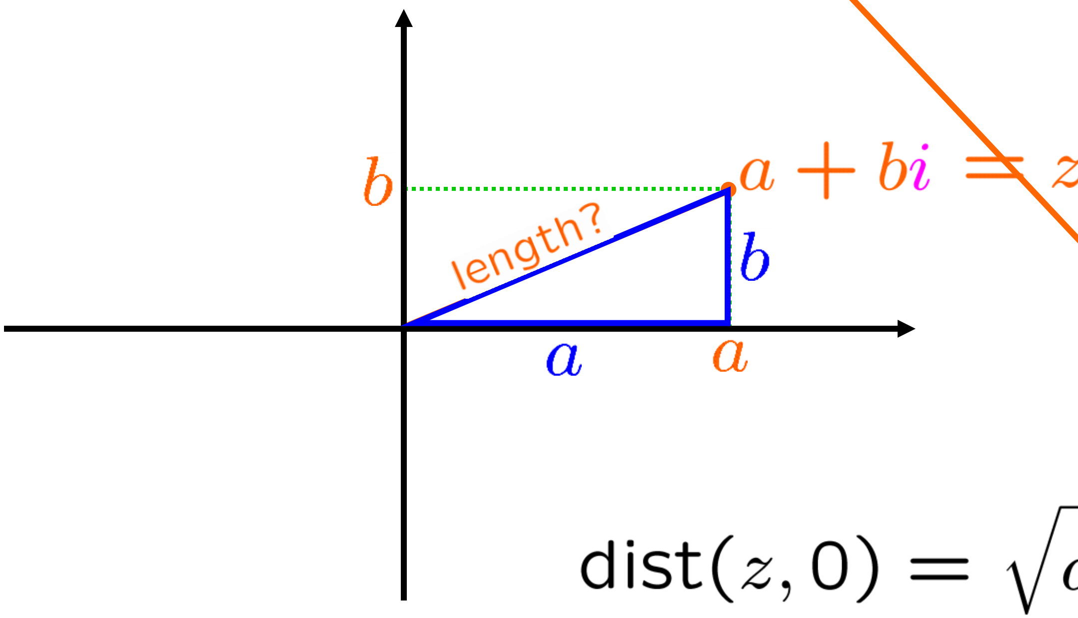


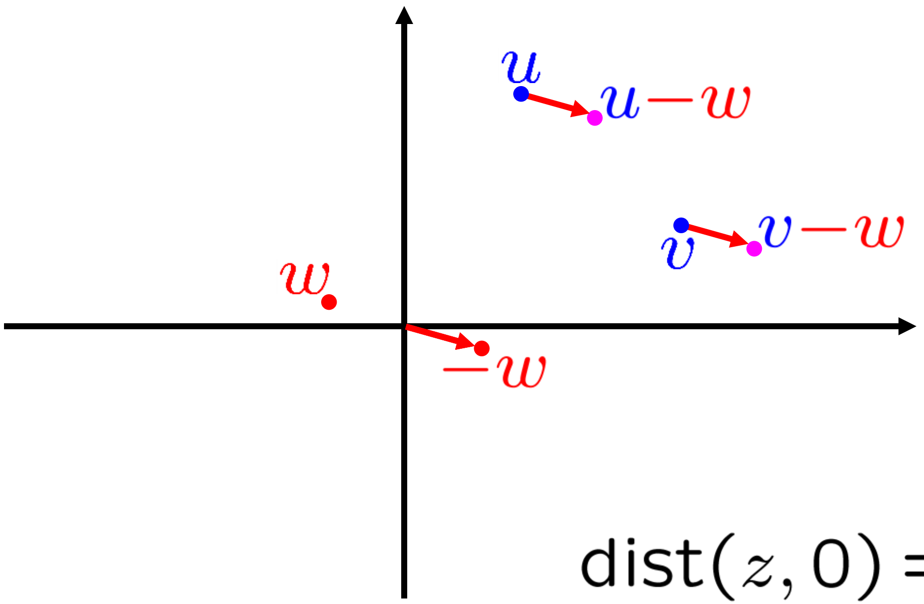
$\text{dist}(z, 0) = \sqrt{a^2 + b^2} = |z|$

Absolute value of complex numbers

Definition:

The **absolute value** (or **modulus**) of $z = a + bi$

is $|z| = \sqrt{a^2 + b^2}$.



$\text{dist}(z, 0) = |z|$ $= |z|$

Absolute value of complex numbers

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The **absolute value** $\overbrace{}^{\text{or } \textbf{modulus}}$ of $z = a + bi$

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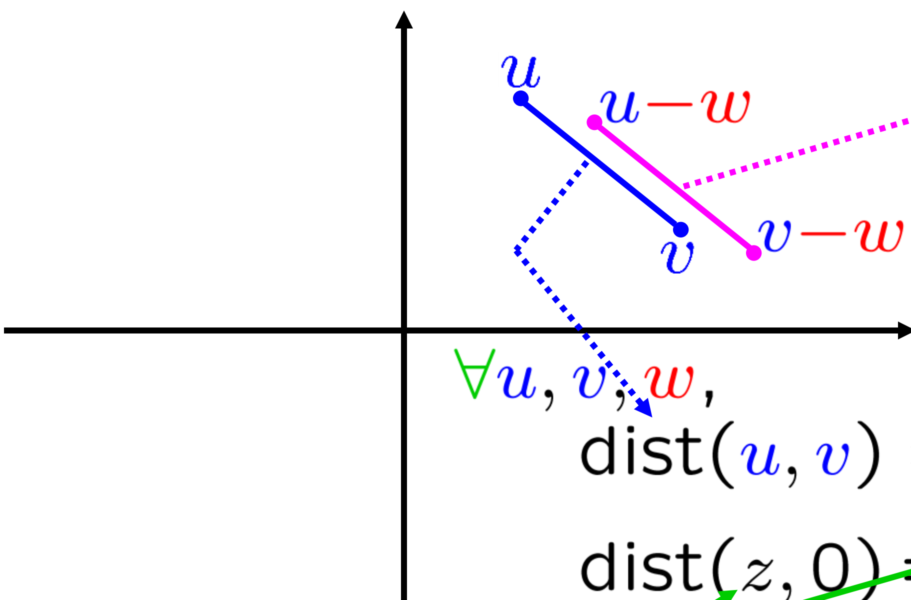


$\forall u, v, w,$ $\quad w :\to v$

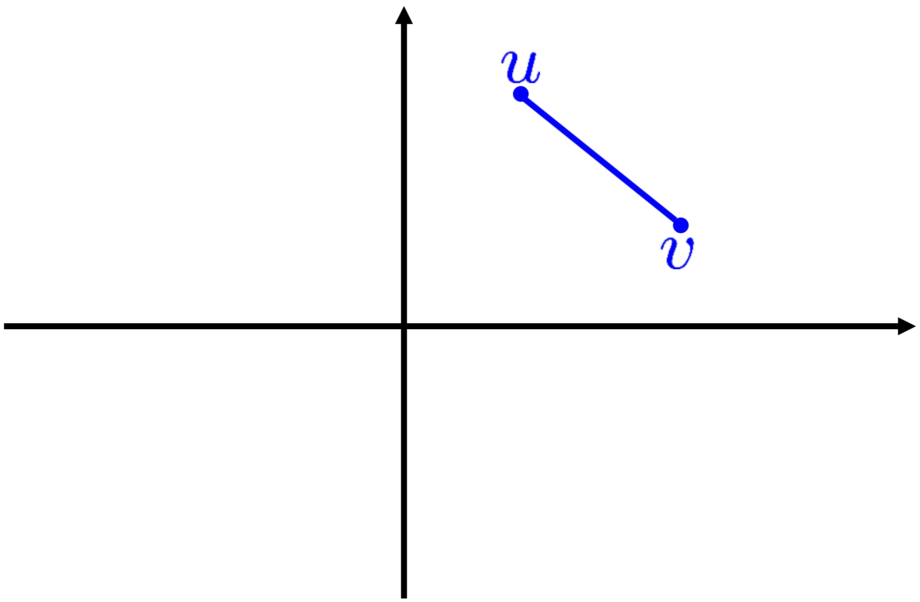
$$\mathrm{dist}(u, v) = \mathrm{dist}(u-w, v-w)$$

$$\mathrm{dist}(z, 0) = |z|$$

$$\mathrm{dist}(u, v) = \mathrm{dist}(u-v, \cancel{v}-\cancel{v})$$
$$= \mathrm{dist}(u-v, 0) = |u-v|$$

Absolute value of complex numbers

e.g.: $\text{dist}(2+8i, 5+4i)$



$\text{dist}(u, v)$

$$\text{dist}(u, v) = |u - v|$$
$$= |u - v|$$

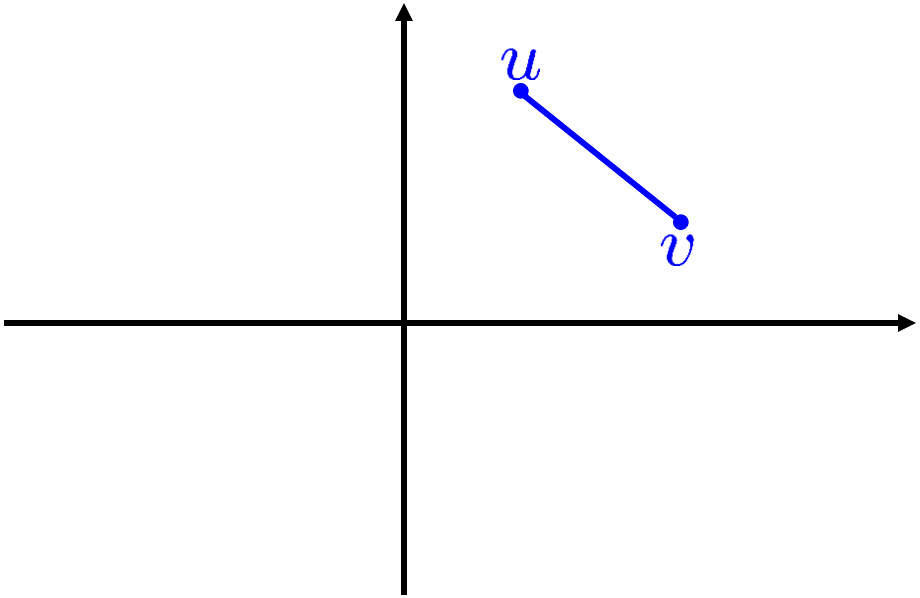
Absolute value of complex numbers

e.g.: $\mathrm{dist}(2+8i, 5+4i) = |(2+8i) - (5+4i)|$

$$= |(2-5) + (8-4)i|$$

$$= |-3+4i| = \sqrt{(-3)^2 + 4^2}$$

$$= 5 \quad ■$$



$\mathrm{dist}(u, v) = |u - v|$

Absolute value of complex numbers

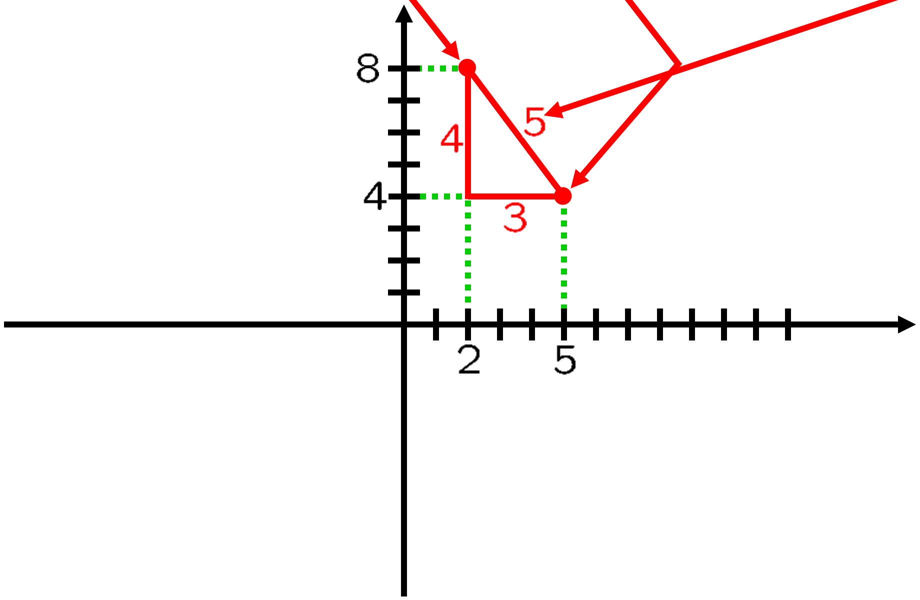
e.g.: $\mathrm{dist}(2+8i, 5+4i) = |(2+8i)-(5+4i)|$

$$= |(2-5)+(8-4)i|$$

$$= |-3+4i| = \sqrt{(-3)^2+4^2}$$

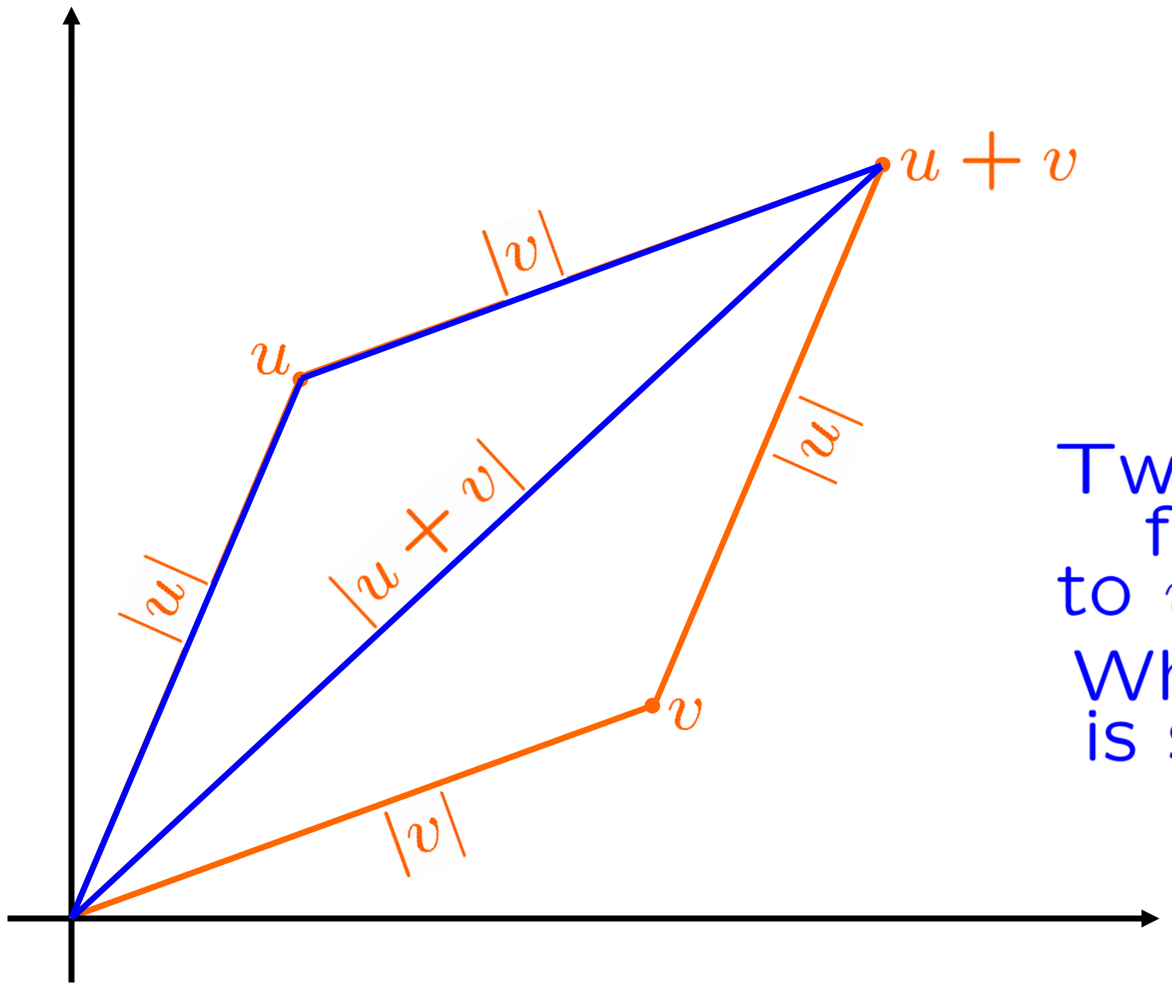
$$= 5 \blacksquare$$

SKILL
compute distance between two complex numbers



$$\mathrm{dist}(u, v) = |u - v|$$

Fact: $\boxed{|u+v| \leq |u| + |v|}$ **Triangle Inequality**



Two routes from 0 to $u+v$ . . .

Which one is shorter?

Multiplication of complex numbers

Definition:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

i.e.: multiplication of complex numbers is accomplished by expanding,

replacing i^2 by -1

and collecting real and imaginary parts.

e.g.: $(3+4i)(7-8i) =$

expl. . .

$$(3)(7) + (4)(-8)i^2 + (3)(-8)i + (4)(7)i =$$

expl. . .

$$((3)(7) - (4)(-8)) + ((3)(-8) + (4)(7))i =$$

$$53 + 4i$$

SKILL

multiplication of cx numbers

Properties of addition and multiplication

Facts:

$\forall u, v \in \mathbb{C}$,

$$u + v = v + u$$

addition is commutative

$$uv = vu$$

multiplication is commutative

$\forall u, v, w \in \mathbb{C}$,

$$(u + v) + w = u + (v + w)$$

addition is associative

$$(uv)w = u(vw)$$

multiplication is associative

$$u(v + w) = uv + uw$$

multiplication distributes over addition

Complex conjugation of complex numbers

Definition:

The **complex conjugate** of $z = a + bi$ is $\boxed{\bar{z}} := a - bi$.

e.g.: $\overline{4 - 3i} = 4 + 3i$

SKILL
Finding complex conjugates

Fact: $\overline{w + z} = \overline{w} + \bar{z}$ Pf: Exercise.

Fact: $\overline{wz} = \overline{w}\,\bar{z}$

Complex conjugation of complex numbers

Proof:

$$w = a + b\textcolor{magenta}{i} \qquad \overline{w} = a - b\textcolor{magenta}{i}$$

$$z = c + d\textcolor{magenta}{i} \qquad \overline{z} = c - d\textcolor{magenta}{i}$$

$$wz = (ac - bd) + (ad + bc)\textcolor{magenta}{i} \qquad \overline{w}\,\overline{z} = (ac - bd) + (-ad - bc)\textcolor{magenta}{i}$$

expl. . .

equal

$$\boxed{\overline{}}\overline{wz} = \boxed{(ac - bd) - (ad + bc)\textcolor{magenta}{i}} = \boxed{(ac - bd) - (ad + bc)\textcolor{magenta}{i}}$$

equal

QED

Fact: $\overline{wz} = \overline{w}\;\overline{z}$

Absolute value of complex numbers

Fact: $|z| = \sqrt{z\overline{z}}$

Proof: $z = a + bi$

$p :\to a \qquad q :\to bi$

$(p+q)(p-q) = p^2 - q^2$

$\overline{z} = a - bi$

$$\begin{aligned} z\overline{z} &= (a+bi)(a-bi) \\ &= a^2 \boxed{-} (b\boxed{i})^{\boxed{2}} \\ &= a^2 \boxed{+} b^2 \end{aligned}$$

$\sqrt{z\overline{z}} = \sqrt{a^2 + b^2} = |z|$ QED

Absolute value of complex numbers

Fact: $|uv| = |u||v|$

Fact: $|z| = \sqrt{z\bar{z}}$

Absolute value of complex numbers

Fact: $|uv| = |u||v|$

Proof:

$$|uv| = \sqrt{uv\boxed{\overline{uv}}}$$

$$= \sqrt{u\boxed{v}\boxed{\overline{u}}\boxed{\overline{v}}}$$

$$= \sqrt{u\boxed{\overline{u}v}\overline{v}}$$

$$= \sqrt{u\overline{u}}\sqrt{v\overline{v}}$$

$$= |u||v| \qquad \text{QED}$$

Fact: $|z| = \sqrt{z\overline{z}}$

Fact: $\overline{wz} = \overline{w}\,\overline{z}$

Exponentiation of complex numbers

Definition:

$z :\longrightarrow w$

$$\boxed{e^z} := \boxed{\exp(z)} := \underline{\lim_{n\to\infty}(1+(z/n))^n}$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Fact: $e^z e^w = e^{z+w}$

$n \to \infty$ $\quad$ $n \to \infty$

Pf: $\boxed{\left[1+\frac{z}{n}\right]^n}\boxed{\left[1+\frac{w}{n}\right]^n} = \left[1+\frac{z}{n}+\frac{w}{n}+\boxed{\frac{zw}{n^2}}\right]^n$

$$\delta_n := \frac{zw}{n} \qquad = \boxed{\left[1+\frac{z+w}{n}+\boxed{\frac{\delta_n}{n}}\right]^n}$$ QED

$x :\longrightarrow z + w$

Fact: $\forall x \in \overset{\mathbb{C}}{\cancel{\mathbb{R}}}$, $\forall \delta_n \to 0$, $\left[1+\frac{x}{n}+\frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$x :\longrightarrow ix$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$i \quad i^2 \quad i^3 \quad i^4 \quad i^5$

$$= 1 \boxed{+\ i} x \boxed{-} \frac{x^2}{2!} \boxed{-\ i} \frac{x^3}{3!} \boxed{+} \frac{x^4}{4!} \boxed{+\ i} \frac{x^5}{5!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$x :\longrightarrow ix$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

COLLECT REAL PARTS

COLLECT IMAGINARY PARTS

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$x :\longrightarrow ix$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

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$$= \cos x + i \sin x$$

$x :\rightarrow \theta$

$$e^{ix} = \cos x + i \sin x$$

e^{ix}

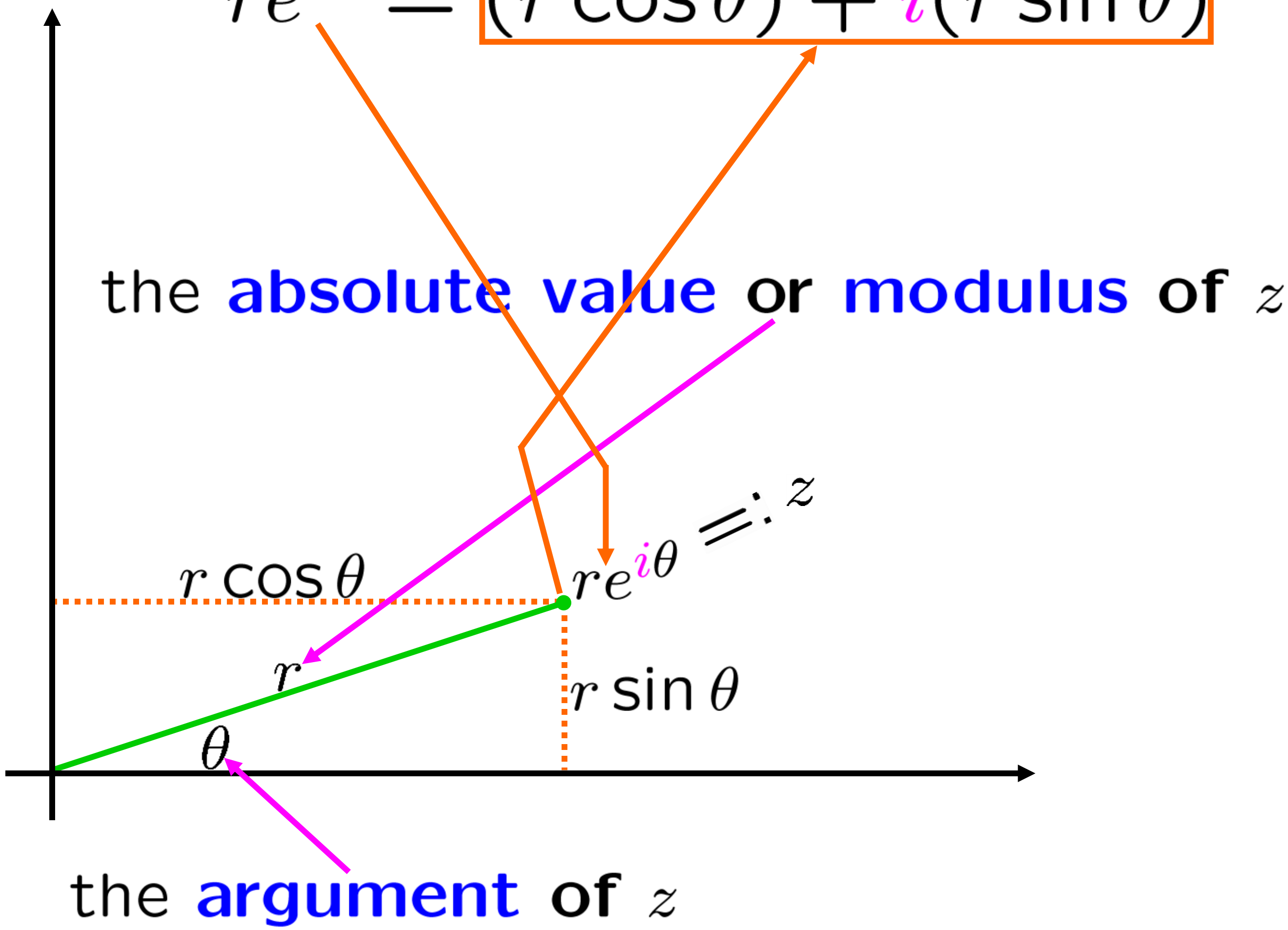
$= \qquad \cos x \qquad + \, i \qquad \sin x$

$x :\longrightarrow \theta$

$$e^{ix} = \cos x + i \sin x$$

$$r \times \left[\; e^{i\theta} = \cos\theta + i\sin\theta \; \right]$$

$$re^{i\theta} = \boxed{(r\cos\theta) + i(r\sin\theta)}$$



The geometry of complex multiplication

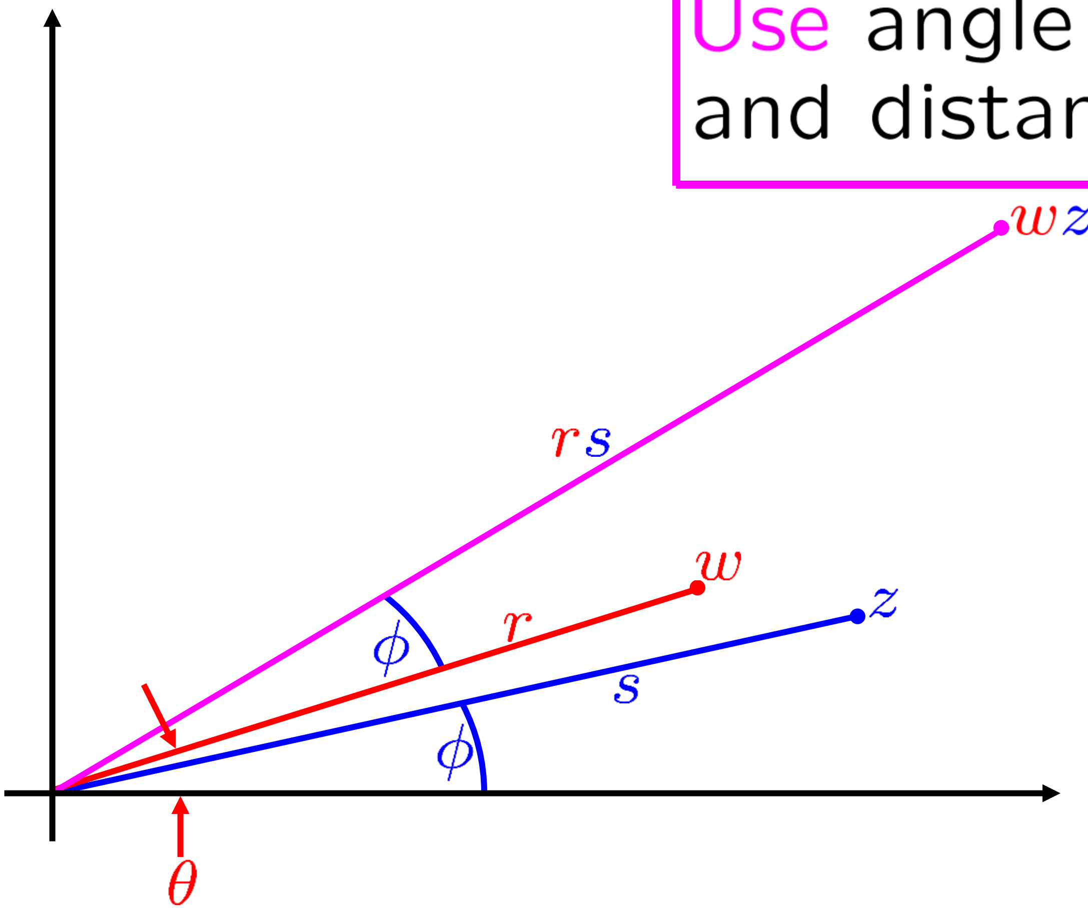
Question: Where is wz?

Answer:

$$w = re^{i\theta} \qquad z = se^{i\phi}$$

$$wz = (rs)e^{i(\theta+\phi)}$$

Use angle $\theta + \phi$ arguments add
and distance rs. moduli multiply



Exponentiation of complex numbers

$$e^{ix} = \cos x + i \sin x$$

Fact: $e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b)$

e.g.: $e^{6+\pi i} = e^6(\cos \pi + i \sin \pi) = -e^6$

SKILL
Exponentiating complex numbers

Exponentiation of complex numbers

Definition:

$$\boxed{e^z} = \boxed{\exp(z)} = \lim_{n \to \infty} (1 + (z/n))^n$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Fact: $\overline{w + z} = \overline{w} + \overline{z}$

Fact: $\overline{wz} = \overline{w}\,\overline{z}$

Corollary: $\overline{e^z} = e^{\overline{z}}$

Exponentiation of complex numbers

e^z

Proof:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

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Exponentiation of complex numbers

Proof:

$$\overline{e^z} = \overline{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

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Exponentiation of complex numbers

Proof:

$$\overline{e^z} = \bar{1} + \bar{z} + \overline{\frac{z^2}{2!}} + \overline{\frac{z^3}{3!}} + \cdots$$

Fact: $\boxed{\overline{w + z} = \overline{w} + \bar{z}}$

Fact: $\boxed{\overline{wz} = \overline{w}\,\bar{z}}$

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Exponentiation of complex numbers

Proof:

$$\overline{e^{z}} = \bar{1} + \bar{z} + \frac{\bar{z}^2}{2!} + \frac{\bar{z}^3}{3!} + \cdots = e^{\bar{z}}$$

STOP

QED

Fact: $\overline{w+z} = \overline{w} + \bar{z}$

Fact: $\boxed{\overline{wz} = \overline{w}\,\bar{z}}$

Corollary: $\overline{e^{z}} = e^{\bar{z}}$