

# Financial Mathematics

## Cauchy-Schwarz

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Can  $v \cdot w$  be big, even if  $|v|$  and  $|w|$  are small?

Say we know  $|v| \leq 3$  and  $|w| \leq 5$ .

How big can  $v \cdot w$  be?

Say we know  $|v|^2 \leq 9$  and  $|w|^2 \leq 25$ .

How big can  $v \cdot w$  be?

## Cauchy-Schwarz inequality

$$|v|^2 \leq 9$$

$$|w|^2 \leq 25$$

$$0 \leq |v - w|^2$$

$$|v|^2 \leq 9$$

$$|w|^2 \leq 25$$

## Cauchy-Schwarz inequality

DIVIDE BY 2

$$|v|^2 \leq 9$$

$$0 \leq$$

$$|v - w|^2 =$$

$$|v|^2 + |w|^2$$

$$- 2v \cdot w$$

$$|w|^2 \leq 25$$

•  $|^2$  is positive semidefinite

$$v \rightarrow \sqrt{t}v$$

$$w \rightarrow w/\sqrt{t}$$

$$v \cdot w = 3v \cdot \frac{w}{3} \leq \frac{9|v|^2 + (|w|^2/9)}{2} \leq \frac{377}{9}$$

$$v \cdot w = \sqrt{5}v \cdot \frac{w}{\sqrt{5}} \leq \frac{5|v|^2 + (|w|^2/5)}{2} \leq 25$$

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2} \leq \frac{9t + (25/t)}{2}$$

minimum value?

$$v \cdot w = \sqrt{t}v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2}$$

$$|v|^2 \leq 9$$

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$$\frac{1}{2}a + \frac{1}{2}b$$

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$$\begin{aligned}|v|^2 &\leq 9 \\ |w|^2 &\leq 25\end{aligned}$$

$$v \cdot w \leq \sqrt{|v|^2|w|^2} = |v||w|$$

$$\frac{1}{2}a + \frac{1}{2}b$$

$\uparrow$

$ta$

$b/t$

$$\begin{aligned}a &\rightarrow |v|^2 \\ b &\rightarrow |w|^2\end{aligned}$$

Fact:  $\forall a, b \in (0, \infty]$ ,  $\min_{t>0} \left[ \frac{t}{2}a + \frac{1}{2t}b \right] = \sqrt{ab}$

$$v \cdot w = \sqrt{t} v \cdot \frac{w}{\sqrt{t}} \leq \frac{t|v|^2 + (|w|^2/t)}{2}$$

$$\begin{aligned} |v|^2 &\leq 9 \\ |w|^2 &\leq 25 \end{aligned}$$

$v \rightarrow -v$

$$v \cdot w \leq \sqrt{|v|^2|w|^2} = |v||w|$$

$$v \cdot w \leq |v||w| \leq (3)(5) = 15$$

$$-(v \cdot w) = (-v) \cdot w \leq |-v||w| = |v||w|$$

$$|v \cdot w| = \max\{v \cdot w, -(v \cdot w)\} \leq |v||w|$$

$|x| = \max \{ x, -x \}$

Cauchy-Schwarz inequality:

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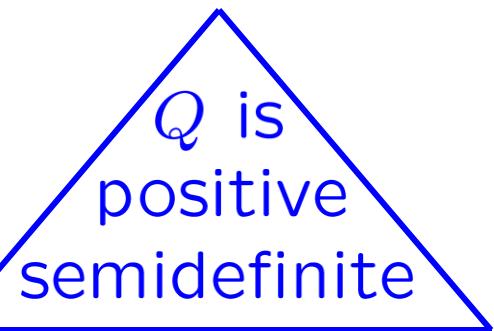
Theorem:

Let  $Q : V \rightarrow \mathbb{R}$  be a quadratic form.

Assume, for all  $v \in V$ , that  $Q(v) \geq 0$ .

Let  $B$  be the polarization of  $Q$ .

Then  $|B(v, w)| \leq \sqrt{Q(v)} \sqrt{Q(w)}$  .



What have I forgotten?

**Def'n:** A quadratic form  $Q : V \rightarrow \mathbb{R}$  is **positive semidefinite** if,  $\forall v \in V$ ,  $Q(v) \geq 0$ .

**Def'n:** A quadratic form  $Q : V \rightarrow \mathbb{R}$  is **positive definite** if,  $\forall v \in V \setminus \{0\}$ ,  $Q(v) > 0$ .

**Theorem:**

Let  $Q : V \rightarrow \mathbb{R}$  be a quadratic form.

Assume that  $Q$  is positive semidefinite.

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Cauchy-Schwarz inequality for  $B$  and  $Q$

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the geometric mean of  $Q(v)$  and  $Q(w)$

The absolute polarization at  $v, w$  of a Q. form  
positive semidefinite  
 $\wedge$   
the geometric mean of its values at  $v$  and  $w$



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