Financial Mathematics

Cayley’s Theorem

Fix, for this entire lecture, an integer $n \geq 1$.

Def'n:

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial, given by

$$p(x) = a_d x^d + \cdots + a_1 x \boxed{+ a_0}, \quad \forall x \in \mathbb{R},$$

for some $a_d, \ldots, a_0 \in \mathbb{R}$.

The **matrix extension of** p is

the function $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$P(X) = a_d X^d + \cdots + a_1 X \boxed{+ a_0 I}, \quad \forall X \in \mathbb{R}^{n\times n}.$$

$x :\to X$ scalar terms multiplied by I

Cayley's Theorem:

Fix $T \in \mathbb{R}^{n\times n}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the char. poly. of T.

Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be

the matrix extension of f.

Then $F(T) = 0$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the polynomial def'd by

$$f(x,y) = xy, \qquad \forall x,y \in \mathbb{R}.$$

Can we extend to $F : (\mathbb{R}^{n\times n})^2 \to \mathbb{R}^{n\times n}$? No.

$$F_1(X,Y) = XY, \qquad \forall X,Y \in \mathbb{R}^{n\times n}$$

$$f(x,y) = yx, \qquad \forall x,y \in \mathbb{R}. \qquad\qquad F_1 \neq F_2$$

$$F_2(X,Y) = YX, \qquad \forall X,Y \in \mathbb{R}^{n\times n}$$

Cayley's Theorem:

Fix $T \in \mathbb{R}^{n\times n}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the char. poly. of T.

Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the matrix extension of f.

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$$f(x, y) = yx, \quad \forall x, y \in \mathbb{R}. \qquad F_1 \neq F_2$$

$$F_2(X, Y) = YX, \quad \forall X, Y \in \mathbb{R}^{n\times n}$$

Let $\mathcal{C} \subseteq \mathbb{R}^{n\times n}$ be a commuting set.

Can we extend to $F : \mathcal{C}^2 \to \mathbb{R}^{n\times n}$? Yes.

$$F(X, Y) = XY, \quad \forall X, Y \in \mathcal{C}$$

$$\| \qquad\qquad$$

$$YX \qquad\qquad$$

Cayley's Theorem:

Fix $T \in \mathbb{R}^{n\times n}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the char. poly. of T.

Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the matrix extension of f.

Then $F(T) = 0$.

Proof: $\forall$integers $i, j \in [1, n]$, let $E^{ij} \in \mathbb{R}^{n\times n}$ be defined by $E^{ij}_{ab} = \delta^i_a \delta^j_b$.

Let $I := E^{11} + E^{22} + \cdots + E^{nn} \in \mathbb{R}^{n\times n}$ be the $n \times n$ identity matrix.

Let $\mathcal{P} := \{\text{polynomials in } T\}$ commuting set

$= \{\text{finite linear comb.s of } I, T, T^2, T^3, \ldots\}$.

Let $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ be the determinant.

Let $\mathrm{DET} : \mathcal{P}^{n\times n} \to \mathcal{P}$ be its matrix extension.

expl. . .

Say $n = 3$. Then

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{array}{r} aei + bfg + cdh \\ -ceg - fha - ibd, \end{array}$$

$$\forall a, b, c, d, e, f, g, h, i \in \mathbb{R}$$

$$\text{so DET} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{array}{r} AEI + BFG + CDH \\ -CEG - FHA - IBD, \end{array}$$

$$\forall A, B, C, D, E, F, G, H, I \in \mathcal{P}$$

Let $\mathcal{P} :=$ {polynomials in T} commuting set

$=$ {finite linear comb.s of $I, T, T^2, T^3, \ldots$}.

Let $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ be the determinant.

Let $\text{DET} : \mathcal{P}^{n \times n} \to \mathcal{P}$ be its matrix extension.

expl. . .

$\forall A, B \in \mathbb{R}^{n\times n}$,

$A \otimes B \in (\mathbb{R}^{n\times n})^{n\times n}$ is defined by:

$$(A \otimes B)_{ij} = A_{ij}B,$$

i.e.: $$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$$

Claim: $\forall A, B, C, D \in \mathbb{R}^{n\times n}$,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Pf: $\forall$integers $i, k \in [1, n]$,

$$[(A \otimes B)(C \otimes D)]_{ik} = \sum_j (A \otimes B)_{ij}(C \otimes D)_{jk}$$

$$= \sum_j (A_{ij}B)(C_{jk}D) = \left[\sum_j (A_{ij}C_{jk})\right](BD)$$

$$= [(AC)_{ik}](BD) = [(AC) \otimes (BD)]_{ik}$$

QED

$\forall A, B \in \mathbb{R}^{n\times n}$,

$\boxed{A \otimes B} \in (\mathbb{R}^{n\times n})^{n\times n}$ is defined by:

$$(A \otimes B)_{ij} = A_{ij} B,$$

i.e.: $$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$$

Claim: $\forall \mathcal{X} \in (\mathbb{R}^{n\times n})^{n\times n}, \quad \mathcal{X} = \sum_{i,j} E^{ij} \otimes \mathcal{X}_{ij}$.

Pf: Fix integers $a, b \in [1, n]$.

Want: $$\mathcal{X}_{ab} = \sum_{i,j} \underbrace{(E^{ij} \otimes \mathcal{X}_{ij})_{ab}}_{E^{ij}_{ab}\mathcal{X}_{ij} = \delta^i_a \delta^j_b \mathcal{X}_{ij}}$$

$\forall A, B \in \mathbb{R}^{n\times n}$,

$A \otimes B \in (\mathbb{R}^{n\times n})^{n\times n}$ is defined by:

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Claim: $\forall \mathcal{X} \in (\mathbb{R}^{n\times n})^{n\times n}, \quad \mathcal{X} = \sum_{i,j} E^{ij} \otimes \mathcal{X}_{ij}.$

Pf: Fix integers $a, b \in [1, n]$.

Want: $\mathcal{X}_{ab} = \sum_{i,j} \delta_a^i \delta_b^j \mathcal{X}_{ij}$

$\|$

$\mathcal{X}_{ab}$ QED

$\delta_a^i \delta_b^j \mathcal{X}_{ij}$

$\forall A, B \in \mathbb{R}^{n\times n}$,

$A \otimes B \in (\mathbb{R}^{n\times n})^{n\times n}$ is defined by:

$$(A \otimes B)_{ij} = A_{ij}B,$$

i.e.:

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$$

$$T \otimes I = \begin{bmatrix} T_{11}I & \cdots & T_{1n}I \\ \vdots & & \vdots \\ T_{n1}I & \cdots & T_{nn}I \end{bmatrix}$$

$\forall \Lambda \in \mathbb{R}^{n\times n}$, $$I \otimes \Lambda = \begin{bmatrix} \Lambda & & \\ & \ddots & \\ & & \Lambda \end{bmatrix}$$

$$T \otimes I = \begin{bmatrix} T_{11}I & \cdots & T_{1n}I \\ \vdots & & \vdots \\ T_{n1}I & \cdots & T_{nn}I \end{bmatrix} \in \mathcal{P}^{n\times n}$$

$\mathcal{P} =$ {polys in T}

$$\forall \Lambda \in \mathcal{P},\ I \otimes \Lambda = \begin{bmatrix} \Lambda & & \\ & \ddots & \\ & & \Lambda \end{bmatrix} \in \mathcal{P}^{n\times n}$$

$$T = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}$$

$\forall \lambda \in \mathbb{R},$ $$\lambda I = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

$f(\lambda) =$ $\det(T - \lambda I)$

$$T \otimes I = \begin{bmatrix} T_{11}I & \cdots & T_{1n}I \\ \vdots & & \vdots \\ T_{n1}I & \cdots & T_{nn}I \end{bmatrix} \in \mathcal{P}^{n \times n}$$

$\mathcal{P} =$ $\{\text{polys in } T\}$

$\forall \Lambda \in \mathcal{P},$ $$I \otimes \Lambda = \begin{bmatrix} \Lambda & & \\ & \ddots & \\ & & \Lambda \end{bmatrix} \in \mathcal{P}^{n \times n}$$

Pf in the 3×3 case ...

Claim: $\forall \Lambda \in \mathcal{P},$ $F(\Lambda) = \mathrm{DET}(T \otimes I - I \otimes \Lambda)$

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$\forall \lambda \in \mathbb{R},$ $$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$f(\lambda) = \det(T - \lambda I)$

$$T - \lambda I = \begin{bmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{bmatrix}$$

$$\begin{aligned} f(\lambda) &= (T_{11} - \lambda)(T_{22} - \lambda)(T_{33} - \lambda) & + \cdots + \cdots \\ &- T_{13}(T_{22} - \lambda)T_{31} & - \cdots - \cdots \end{aligned}$$

Pf in the 3×3 case ...

Claim: $\forall \Lambda \in \mathcal{P},$
$F(\Lambda) = \mathrm{DET}(T \otimes I - I \otimes \Lambda)$

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$\forall \lambda \in \mathbb{R}, \qquad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$f(\lambda) = \det(T - \lambda I)$$

$$T - \lambda I = \begin{bmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{bmatrix}$$

$$\begin{aligned} f(\lambda) \; = \; & (T_{11} - \lambda)(T_{22} - \lambda)(T_{33} - \lambda) && + \cdots + \cdots \\ - \; & T_{13}(T_{22} - \lambda)T_{31} && - \cdots - \cdots \end{aligned}$$

$$\begin{aligned} F(\Lambda) = \; & (T_{11}I - \Lambda)(T_{22}I - \Lambda)(T_{33}I - \Lambda) && + \cdots + \cdots \\ - \; & T_{13}I(T_{22}I - \Lambda)T_{31}I && - \cdots - \cdots \end{aligned}$$

$$T \otimes I = \begin{bmatrix} T_{11}I & T_{12}I & T_{13}I \\ T_{21}I & T_{22}I & T_{23}I \\ T_{31}I & T_{32}I & T_{33}I \end{bmatrix}$$

$$\forall \Lambda \in \mathcal{P},\ I \otimes \Lambda = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix}$$

$$\begin{matrix} T \otimes I \\ -I \otimes \Lambda \end{matrix} = \begin{bmatrix} T_{11}I - \Lambda & T_{12}I & T_{13}I \\ T_{21}I & T_{22}I - \Lambda & T_{23}I \\ T_{31}I & T_{32}I & T_{33}I - \Lambda \end{bmatrix}$$

$$\mathrm{DET}(T \otimes I - I \otimes \Lambda)$$

$$\|$$

$$\begin{aligned} F(\Lambda) = \ & (T_{11}I - \Lambda)(T_{22}I - \Lambda)(T_{33}I - \Lambda) \ + \cdots + \cdots \\ & - \ T_{13}I(T_{22}I - \Lambda)T_{31}I \qquad - \cdots - \cdots \end{aligned}$$

Cayley's Theorem:

Fix an integer $n > 0$ and $T \in \mathbb{R}^{n \times n}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the char. poly. of T.

Let $F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the matrix extension of f.

$\forall \Lambda \in \mathcal{P}$, Then $F(T) = 0$.

$\forall \Lambda \in \mathcal{P}$,

$$F(\Lambda) = \mathrm{DET}(T \otimes I - I \otimes \Lambda)$$

$$\mathrm{DET}(T \otimes I - I \otimes \Lambda)$$

$\|$

$$F(\Lambda) = (T_{11}I - \Lambda)(T_{22}I - \Lambda)(T_{33}I - \Lambda) + \cdots + \cdots$$
$$- \; T_{13}I(T_{22}I - \Lambda)T_{31}I \qquad - \cdots - \cdots$$

Cayley's Theorem:

Fix an integer $n > 0$ and $T \in \mathbb{R}^{n \times n}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the char. poly. of T.

Let $F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the matrix extension of f.

Then $F(T) = 0$.

$\forall \Lambda \in \mathcal{P}$, ☺

$$F(\Lambda) = \mathrm{DET}(T \otimes I - I \otimes \Lambda)$$

$$F(T) = \mathrm{DET}(T \otimes I - I \otimes T) = \mathrm{DET}(\mathcal{U})$$

$$\mathcal{U} := T \otimes I - I \otimes T$$

Want: $\mathrm{DET}\,\mathcal{U} = 0$

WARNING: DET is only defined on $\mathcal{P}^{n \times n}$.

$\mathcal{P} = \{\text{polynomials in } T\}$

$$T \otimes I = \begin{bmatrix} T_{11}I & \cdots & T_{1n}I \\ \vdots & & \vdots \\ T_{n1}I & \cdots & T_{nn}I \end{bmatrix} \in \mathcal{P}^{n\times n}$$

$$I \otimes T = \begin{bmatrix} T & & \\ & \ddots & \\ & & T \end{bmatrix} \in \mathcal{P}^{n\times n}$$

$$\mathcal{U} := T \otimes I - I \otimes T \in \mathcal{P}^{n\times n}$$

Want: $\mathrm{DET}\,\mathcal{U} = 0$

WARNING: DET is only defined on $\mathcal{P}^{n\times n}$.

$\forall M \in \mathbb{R}^{n \times n}$, let tr-cof$(M)$ be
the transposed cofactor matrix of M,

$$\mathcal{P} = \{\text{polynomials in } T\}$$

$\forall M \in \mathbb{R}^{n\times n}$, let tr-cof$(M)$ be the transposed cofactor matrix of M, def'd by $[\text{tr-cof}(M)]_{ij} = [(-1)^{i+j}][\det(\text{elim}_{ji}(M))]$.

$\text{elim}_{ji} :=$ eliminate jth row & ith column

not ij

$$\text{tr-cof}(M) = \begin{bmatrix} \det(\text{elim}_{11}(M)) & \cdots & \pm\det(\text{elim}_{n1}(M)) \\ \vdots & & \vdots \\ \pm\det(\text{elim}_{1n}(M)) & \cdots & \det(\text{elim}_{nn}(M)) \end{bmatrix}$$

$\mathcal{P} = \{\text{polynomials in } T\}$

$\text{tr-cof} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$

Let $\text{TR-COF} : \mathcal{P}^{n\times n} \to \mathcal{P}^{n\times n}$ be the matrix extension of tr-cof.

$\forall \mathcal{M} \in \mathcal{P}^{n\times n}$,

$$[\text{TR-COF}(\mathcal{M})]_{ij} = [(-1)^{i+j}][\text{DET}(\text{ELIM}_{ji}(\mathcal{M}))].$$

$\text{ELIM}_{ji} :=$ matrix extension of elim_{ji}

$$\text{TR-COF}(\mathcal{M}) = \begin{bmatrix} \text{DET}(\text{ELIM}_{11}(\mathcal{M})) & \cdots & \pm\text{DET}(\text{ELIM}_{n1}(\mathcal{M})) \\ \vdots & & \vdots \\ \pm\text{DET}(\text{ELIM}_{1n}(\mathcal{M})) & \cdots & \text{DET}(\text{ELIM}_{nn}(\mathcal{M})) \end{bmatrix}$$

$\mathcal{P} = \{\text{polynomials in } T\}$

tr-cof : $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$

Let TR-COF : $\mathcal{P}^{n\times n} \to \mathcal{P}^{n\times n}$ be the matrix extension of tr-cof.

$\forall M \in \mathbb{R}^{n\times n}$,

$$[\text{tr-cof}(M)]M = M[\text{tr-cof}(M)] = \begin{bmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{bmatrix} = (\det M)I$$

$\forall \mathcal{M} \in \mathcal{P}^{n\times n}$,

$$[\text{TR-COF}(\mathcal{M})]\mathcal{M} = \mathcal{M}[\text{TR-COF}(\mathcal{M})] = \begin{bmatrix} \text{DET}\,\mathcal{M} & & \\ & \ddots & \\ & & \text{DET}\,\mathcal{M} \end{bmatrix} = I \otimes (\text{DET}\,\mathcal{M})$$

Define mult : $(\mathbb{R}^{n\times n})^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$\text{mult}\begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix} = \sum_{i,j} E^{ij} X_{ij}$$

$X_{ij} :\to A_{ij}B$

Note: $\text{mult}(\mathcal{X} + \mathcal{Y}) = (\text{mult}(\mathcal{X})) + (\text{mult}(\mathcal{Y}))$

$\text{mult}(c\mathcal{X}) = c(\text{mult}(\mathcal{X}))$ (mult is linear.)

Claim: $\forall A, B \in \mathbb{R}^{n\times n}$, $\text{mult}(A \otimes B) = AB$.

Pf:

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$$

$$\text{mult}(A \otimes B) = \left[\sum_{ij} E^{ij} A_{ij}\right] B = AB$$ QED

$$\mathcal{U} = T \otimes I - I \otimes T \in \mathcal{P}^{n \times n}$$

Want: $\text{DET}\,\mathcal{U} = 0$

$$\mathcal{P} = \{\text{polynomials in } T\}$$

$$\text{TR-COF} : \mathcal{P}^{n \times n} \to \mathcal{P}^{n \times n}$$

$\forall \mathcal{M} \in \mathcal{P}^{n \times n}$,

$$[\text{TR-COF}(\mathcal{M})]\mathcal{M} = \mathcal{M}[\text{TR-COF}(\mathcal{M})] = I \otimes (\text{DET}\,\mathcal{M})$$

$\forall A, B \in \mathbb{R}^{n \times n}$, $\quad \text{mult}(A \otimes B) = AB$.

$$\mathcal{C} := \text{TR-COF}(\mathcal{U}) \in \mathcal{P}^{n \times n}$$

$$\mathcal{C}\mathcal{U} = [\text{TR-COF}(\mathcal{U})]\mathcal{U} = I \otimes (\text{DET}\,\mathcal{U})$$

$$\text{mult}(\mathcal{C}\mathcal{U}) = I(\text{DET}\,\mathcal{U}) = \text{DET}\,\mathcal{U}$$

Want: $\text{mult}(\mathcal{C}\mathcal{U}) = 0$ Want: $\text{mult}(\mathcal{C}(T \otimes I)) = \text{mult}(\mathcal{C}(I \otimes T))$

commuting set

$$\mathcal{P} = \{\text{polynomials in } T\}$$

$$\mathcal{C} := \text{TR-COF}(\mathcal{U}) \in \mathcal{P}^{n \times n}$$

$\forall$integers $i, j \in [1, n]$, $\mathcal{C}_{ij} T = T \mathcal{C}_{ij}$

Want: $\text{mult}(\mathcal{C}(T \otimes I)) = \text{mult}(\mathcal{C}(I \otimes T))$

$\forall \mathcal{X} \in (\mathbb{R}^{n\times n})^{n\times n}, \qquad \mathcal{X} = \sum_{i,j} E^{ij} \otimes \mathcal{X}_{ij}.$

$$\mathcal{C} = \sum_{i,j} E^{ij} \otimes \mathcal{C}_{ij}$$

$\forall A, B, C, D \in \mathbb{R}^{n\times n},$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

$$\mathcal{C}(T \otimes I) = \sum_{i,j} E^{ij} T \otimes \mathcal{C}_{ij}$$

$$\mathcal{C}(I \otimes T) = \sum_{i,j} E^{ij} \otimes \mathcal{C}_{ij} T$$

$\forall$integers $i, j \in [1, n]$, $\mathcal{C}_{ij} T = T \mathcal{C}_{ij}$

Want: $\text{mult}(\mathcal{C}(T \otimes I)) = \text{mult}(\mathcal{C}(I \otimes T))$

$\forall A, B \in \mathbb{R}^{n\times n}, \qquad \text{mult}(A \otimes B) = AB.$

$$\text{mult}(\mathcal{C}(T \otimes I)) = \sum_{i,j} E^{ij} T \mathcal{C}_{ij}$$

$$\|$$

STOP

$$\text{mult}(\mathcal{C}(I \otimes T)) = \sum_{i,j} E^{ij} \mathcal{C}_{ij} T$$

QED

$$\mathcal{C}(T \otimes I) = \sum_{i,j} E^{ij} T \otimes \mathcal{C}_{ij}$$

$$\mathcal{C}(I \otimes T) = \sum_{i,j} E^{ij} \otimes \mathcal{C}_{ij} T$$

$\forall$integers $i, j \in [1, n], \quad \mathcal{C}_{ij} T = T \mathcal{C}_{ij}$

Want: $\text{mult}(\mathcal{C}(T \otimes I)) = \text{mult}(\mathcal{C}(I \otimes T))$