# Financial Mathematics

## Cholesky decomposition

Let $Z_1, \ldots, Z_n$, be uncorrelated PCRVs,
all with variance 1.

Question: What is the variance-covariance matrix of $Z_1, \ldots, Z_n$?

Hint: The $(j,k)$-entry is $\mathrm{Cov}[Z_j, Z_k]$.

Answer: The $n \times n$ identity matrix.

Question: If we form new PCRVs by taking linear combinations of $Z_1, \ldots, Z_n$, what kinds of variance-covariance matrices can we get?

Necessary: Symmetric.
Positive semidefinite.

Question: Sufficient?

Problem: Suppose we are given a positive semidefinite symmetric matrix

$$C = [c_{jk}] \in \mathbb{R}^{n \times n}.$$

Find a matrix $A = [a_{jk}] \in \mathbb{R}^{n \times n}$ such that

if, $\forall$integers $j \in [1, n]$, we set $Y_j := \sum_{l=1}^{n} a_{jl} Z_l$,

then, $\forall$integers $j, k \in [1, n]$,

$$\text{Cov}[Y_j, Y_k] = c_{jk}.$$

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$$\mathrm{Cov}[Y_j, Y_k] = c_{jk}.$$

$$\mathrm{Cov}[Y_j, Y_k] = \underbrace{\sum_{l=1}^{n} \sum_{m=1}^{n} a_{jl} a_{km} \overbrace{\mathrm{Cov}[Z_l, Z_m]}^{\delta_l^m}}_{a_{jl} a_{kl}}$$

$$Y_k := \sum_{m=1}^{n} a_{km} Z_m,$$

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$$\mathrm{Cov}[Y_j, Y_k] = \sum_{l=1}^{n} a_{jl} a_{kl}$$

Want: $\forall j, k,$ $\quad c_{jk} = \sum_{l=1}^{n} a_{jl} a_{kl}$

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Let $B := A^t$, and,

$\forall j, k$, let $b_{jk}$ be the $(j, k)$-entry of $B$;

then $b_{jk} = a_{kj}$.

Want: $\forall j, k$, $\quad c_{jk} = \sum_{l=1}^{n} a_{jl} \boxed{a_{kl}}$

Problem: Suppose we are given a positive semidefinite symmetric matrix

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Let $B := A^t$, and, $\forall j, k$, let $b_{jk}$ be the $(j, k)$-entry of $B$; then $b_{jk} = a_{kj}$.

$(j, k)$-entry of $AB$

Want: $\forall j, k$, $c_{jk} = \boxed{\sum_{l=1}^{n} a_{jl} b_{lk}}$

Want: $C = AB$

Problem: Suppose we are given a positive semidefinite symmetric matrix

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Let $B := A^t$, and,

$\forall j,k$, let $b_{jk}$ be the $(j,k)$-entry of $B$;

then $b_{jk} = a_{kj}$.

Want: $C = AB$

Want: $C = AA^t$

Problem: Suppose we are given a positive semidefinite symmetric matrix

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Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$, find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

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$$Z := \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \qquad Y := \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$\mathrm{Cov}[Y, Y] = C \qquad\qquad A I A^t = A A^t$$

$$\mathrm{Cov}[Y, Y] = \mathrm{Cov}[AZ, AZ] \overset{\text{IOU}}{=} A(\mathrm{Cov}[Z, Z])A^t = A I A^t$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

$$
\begin{aligned}
\mathrm{Cov}[AZ, AZ] &= \mathrm{E}[(AZ)^\circ((AZ)^t)^\circ] \\
&= \mathrm{E}[(AZ)^\circ((Z^t A^t)^\circ] \quad \text{A is a const. matrix} \\
&= \mathrm{E}[AZ^\circ(Z^t)^\circ A^t] \\
&= A(\mathrm{E}[Z^\circ(Z^t)^\circ])A^t \\
&= A(\mathrm{Cov}[Z, Z])A^t
\end{aligned}
$$

$$
\mathrm{Cov}[AZ, AZ] \overset{\mathrm{IOU}}{=} A(\mathrm{Cov}[Z, Z])A^t
$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$, find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

Complicated Solution:

Spectral Thm requires finding eigval/eigvec

zeroes of char. poly.

Using the Spectral Theorem, find a rotation $R \in \mathbb{R}^{n\times n}$ and a diagonal $D \in \mathbb{R}^{n\times n}$ s.t. $\boxed{C = RDR^{-1}} = RDR^t$.

$D$ diag., with entries $\geq 0$, so find $D_* \in \mathbb{R}^{n\times n}$ s.t. $\boxed{D_*^2 = D}$.

Let $\boxed{A := RD_*R^{-1}} = RD_*R^t$. symmetric. Why??

☺

$$A^t = (R^t)^t(D_*)^tR^t = RD_*R^t = A$$

$$AA^t = AA = (RD_*\cancel{R^{-1}})(\cancel{R}D_*R^{-1})$$
$$= RD_*^2R^{-1} = RDR^{-1} = C$$

---

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$, find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

Easier Solution:

*e.g.*:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_1^t} =$$

$$\begin{bmatrix} 5 & 3 & -2 \\ 1 & 2 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$, find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

## Easier Solution:

*e.g.*: $E_1$ $\quad \overbrace{\begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}}^{E_1 C E_1^t}$ $\quad E_1^t$

$$\begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
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## Easier Solution:

e.g.: $$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{E_1 C E_1^t} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_2^t} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$, find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Easier Solution:

e.g.: $E_2$ $\qquad \overbrace{E_2 E_1 C E_1^t E_2^t}$ $\qquad E_2^t$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,

find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Easier Solution:

*e.g.*:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2E_1CE_1^tE_2^t} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3^t} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$E_3E_2E_1CE_1^tE_2^tE_3^t = I$$

---

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find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

## Easier Solution:

*e.g.*: $A = E_1^{-1} E_2^{-1} E_3^{-1}$

$C = AA^t$

$$C = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{\parallel \atop A} \cancel{I} \underbrace{(E_3^t)^{-1} (E_2^t)^{-1} (E_1^t)^{-1}}_{\parallel \atop A^t}$$

$$\boxed{E_3 E_2 E_1} C \boxed{E_1^t E_2^t E_3^t} = I$$

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$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} \\ \| \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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Easier Solution:

e.g.: $A = E_1^{-1} E_2^{-1} E_3^{-1}$

$$E_1^{-1} E_2^{-1} E_3^{-1} = A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,

find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Easier Solution:

$$E_1^{-1}E_2^{-1}E_3^{-1}$$

$$\parallel$$

e.g.: $A = E_1^{-1}E_2^{-1}E_3^{-1}$

$$C = \begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} = C$$

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Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

Mechanized Easier Solution:

e.g.:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{lower triangular}}^{\textstyle E_1} \overbrace{\begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}}^{\textstyle C} \overbrace{\begin{bmatrix} 1 & -3/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\textstyle E_1^t}$$

etc., using only lower triangular matrices

The Cholesky Decomposition

Theorem: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$, $\exists$lower triangular $A \in \mathbb{R}^{n\times n}$, s.t. $C = AA^t$.

upper $\triangle$?

positive semi-definite?

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$, find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

# Mechanized Easier Solution:

*e.g.*:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix} 13 & 7 & -6 \\ 7 & 6 & -3 \\ -6 & -3 & 3 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_1^t} =$$

$$\begin{bmatrix} 13 & 7 & -6 \\ 1 & 3 & 0 \\ -6 & -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 1 & -6 \\ 1 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix}$$

**Problem**: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$, find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Mechanized Easier Solution:

e.g.:

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 13 & 1 & -6 \\ 1 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix}}_{E_1 C E_1^t} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_2^t} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ -6 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$, find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Mechanized Easier Solution:

e.g.:

$$\underbrace{\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{E_2 E_1 C E_1^t E_2^{-1}}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3^t} =$$

$$\begin{bmatrix} 2/3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D^2$$

$$E_3 E_2 E_1 C E_1^t E_2^t E_3^t = D^2$$

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find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

## Mechanized Easier Solution:

*e.g.*: $C =$

$$\underbrace{(E_1)^{-1}(E_2)^{-1}(E_3)^{-1}D}_{\substack{\| \\ \because \\ A}}\underbrace{D(E_3^t)^{-1}(E_2^t)^{-1}(E_1^t)^{-1}}_{\substack{\| \\ A^t}}$$

Exercise: Compute $A$.

$$D = \begin{bmatrix} \sqrt{2/3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \qquad \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D^2$$

$$E_3 E_2 E_1 C E_1^t E_2^t E_3^t = D^2$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$, find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

## Problem if positive semidefinite??

*e.g.*:
$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

---

## The Cholesky Decomposition

Theorem: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$,
$\exists$lower triangular $A \in \mathbb{R}^{n\times n}$,
s.t. $C = AA^t$.

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Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
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## Problem if positive semidefinite??

*e.g.*: $$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

## The Cholesky Decomposition

Theorem: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$,
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Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
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## Problem if positive semidefinite??

*e.g.*: $$\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix}\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

## The Cholesky Decomposition

Theorem: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$,
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## Problem if positive semidefinite??

*e.g.*: $$\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

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## The Cholesky Decomposition

**Theorem**: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$,
$\exists$lower triangular $A \in \mathbb{R}^{n\times n}$,
s.t. $C = AA^t$.

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**Problem**: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

~~Problem~~ if positive semidefinite??

*e.g.*: $\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$

## The Cholesky Decomposition

Theorem: $\forall$pos. def. symm. $C \in \mathbb{R}^{n\times n}$,
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$$e.g.: \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

## Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
$\exists$lower triangular $A \in \mathbb{R}^{n \times n}$,
s.t. $C = AA^t$.

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Key fact: Let $C$ be a pos. s.def. symm. matrix.
Then (•) the (1,1)-entry of $C$ is $\geq 0$,
and (•) if the (1,1)-entry of $C$ is $= 0$,
then the first row & column of $C$
are both 0.

e.g.: $M := $
$$\begin{array}{cc} \begin{array}{cccc} x & 0 & y & 0 \end{array} & \\ \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 33 & 12 & 5 \\ 4 & 12 & 100 & -1 \\ 0 & 5 & -1 & 6 \end{bmatrix} & \begin{array}{c} x \\ 0 \\ y \\ 0 \end{array} \end{array}$$

$M$ is not positive semidefinite.

$$8xy + 100y^2$$
$$\|$$
$$Q_M(x, 0, y, 0) < 0,$$
for some $x, y$

Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
$\exists$lower triangular $A \in \mathbb{R}^{n \times n}$,
s.t. $C = AA^t$.

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then the first row & column of $C$
are both 0.

To prove the Improved Cholesky Decomp.:
Use the $(1,1)$-entry to kill the 1st row & col.
Then use the $(2,2)$-entry.
Then use the $(3,3)$-entry. *etc.*

Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
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s.t. $C = AA^t$.

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

In practice:

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad \begin{aligned} x &= 1 \\ y &= 3 \\ z &= 0 \end{aligned}$$

$$\|$$

$$\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy \\ xy & y^2+z^2 \end{bmatrix}$$

## Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$lower triangular $A \in \mathbb{R}^{n\times n}$,

diag. entries$\geq 0$

s.t. $C = AA^t$.

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

$$\begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} u & 0 & 0 \\ v & w & 0 \\ x & y & z \end{bmatrix} \begin{bmatrix} u & v & x \\ 0 & w & y \\ 0 & 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} u^2 & uv & ux \\ uv & v^2 + w^2 & vx + wy \\ ux & vx + wy & x^2 + y^2 + z^2 \end{bmatrix}$$

$$u = \sqrt{5} \qquad (9/5) + w^2 = 3 = 15/5$$

$$v = 3/\sqrt{5} \qquad w = \sqrt{6/5}$$

$$x = -2/\sqrt{5} \qquad -(6/5) + y\sqrt{6/5} = -1 = -5/5$$

$$y = (1/5)\sqrt{5/6} = 1/\sqrt{30}$$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
find $A \in \mathbb{R}^{n\times n}$ s.t. $C = AA^t$.

$$\begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} u & 0 & 0 \\ v & w & 0 \\ x & y & z \end{bmatrix} \begin{bmatrix} u & v & x \\ 0 & w & y \\ 0 & 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} u^2 & uv & ux \\ uv & v^2 + w^2 & vx + wy \\ ux & vx + wy & x^2 + y^2 + z^2 \end{bmatrix}$$

$u = \sqrt{5}$ $\qquad (4/5) + (1/30) + z^2 = 1$

$v = 3/\sqrt{5}$ $\qquad (24/30) + (\,w = \sqrt{6/5} = 30/30$

$x = -2/\sqrt{5}$ $\qquad z = \sqrt{5/30} = \sqrt{1/6} = 1/\sqrt{6}$

$w = \sqrt{6/5}$ $\qquad y = (1/5)\sqrt{5/6} = 1/\sqrt{30}$

Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,

find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

$$\begin{bmatrix} 5 & 3 & -2 \\ 3 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} u & 0 & 0 \\ v & w & 0 \\ x & y & z \end{bmatrix} \begin{bmatrix} u & v & x \\ 0 & w & y \\ 0 & 0 & z \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 3/\sqrt{5} & \sqrt{6/5} & 0 \\ -2/\sqrt{5} & 1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 3/\sqrt{5} & -2/\sqrt{5} \\ 0 & \sqrt{6/5} & 1/\sqrt{30} \\ 0 & 0 & 1/\sqrt{6} \end{bmatrix}$$

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$$u = \sqrt{5} \qquad (4/5) + (1/30) + z^2 = 1$$

$$v = 3/\sqrt{5} \qquad (24/30) + (1/30) + z^2 = 30/30$$

$$x = -2/\sqrt{5} \qquad z = \sqrt{5/30} = \sqrt{1/6} = 1/\sqrt{6}$$

$$w = \sqrt{6/5} \qquad y = (1/5)\sqrt{5/6} = 1/\sqrt{30}$$

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Problem: Given pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,

find $A \in \mathbb{R}^{n \times n}$ s.t. $C = AA^t$.

Key fact: Let $C$ be a pos. s.def. symm. matrix.

Then (•) the $(1,1)$-entry of $C$ is $\geq 0$,

and (•) if the $(1,1)$-entry of $C$ is $= 0$, then the first row & column of $C$ are both 0.

---

Key fact: Let $C$ be a pos. s.def. symmetric $n \times n$ matrix.

Then (•) the $(n,n)$-entry of $C$ is $\geq 0$,

and (•) if the $(n,n)$-entry of $C$ is $= 0$, then the $n$th row & column of $C$ are both 0.

## Second Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,
$\exists$upper triangular $A \in \mathbb{R}^{n\times n}$,
s.t. $C = AA^t$.

diag. entries$\geq$ 0

Key fact: Let $C$ be a pos. s.def. symmetric $n \times n$ matrix.

Then (•) the $(n,n)$-entry of $C$ is $\geq 0$,

and (•) if the $(n,n)$-entry of $C$ is $= 0$, then the $n$th row & column of $C$ are both 0.

## Improved Cholesky Decomposition

**Theorem:** $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
$\exists$lower triangular $A \in \mathbb{R}^{n \times n}$,
s.t. $C = AA^t$.

## Second Improved Cholesky Decomposition

**Theorem:** $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
$\exists$upper triangular $A \in \mathbb{R}^{n \times n}$,
s.t. $C = AA^t$.

diag. entries $\geq 0$

$B := A^t$ $\qquad$ $C = B^t B$

$A$ upper triangular iff $B$ lower triangular

## Third Improved Cholesky Decomposition

**Theorem:** $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n \times n}$,
$\exists$lower triangular $B \in \mathbb{R}^{n \times n}$,
s.t. $C = B^t B$.

diag. entries $\geq 0$

## Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$lower triangular $A \in \mathbb{R}^{n\times n}$,

s.t. $C = AA^t$.

## Second Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$upper triangular $A \in \mathbb{R}^{n\times n}$,

s.t. $C = AA^t$.

diag. entries$\geq 0$

$B := A^t$ $\qquad$ $C = B^tB$

$A$ lower triangular iff $B$ upper triangular

## Fourth Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$upper triangular $B \in \mathbb{R}^{n\times n}$,

s.t. $C = B^tB$.

diag. entries$\geq 0$

## Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$lower triangular $A \in \mathbb{R}^{n\times n}$,

s.t. $C = AA^t$.

---

## Second Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$upper triangular $A \in \mathbb{R}^{n\times n}$,

diag. entries$\geq 0$

s.t. $C = AA^t$.

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SKILL:

Given pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

find the four Cholesky decompositions of $C$.

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## Fourth Improved Cholesky Decomposition

Theorem: $\forall$pos. semidef. symm. $C \in \mathbb{R}^{n\times n}$,

$\exists$upper triangular $B \in \mathbb{R}^{n\times n}$,

diag. entries$\geq 0$

s.t. $C = B^tB$.

# END OF LINEAR ALGEBRA