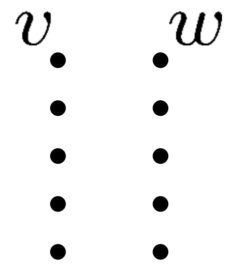
Financial Mathematics

Functional analysis

$$v \cdot w = \sum_{j=1}^{5} v_j w_j$$

Template for $\mathbb{R}^5$:



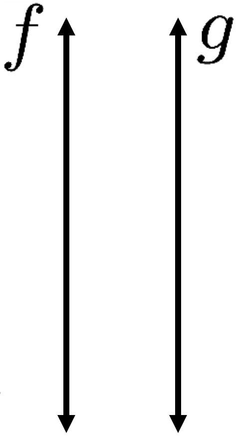
no good dot product

Template for $\{\text{functions } \mathbb{R} \to \mathbb{R}\}$:

$$f \cdot g = \int_{-\infty}^{\infty} f(x) g(x)\, dx$$

Template for $L^2(\mathbb{R})$:

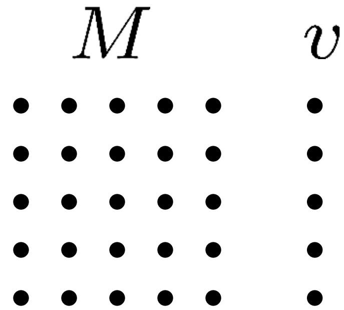
$$L^2(\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} h^2 < \infty\}$$



$M \in \mathbb{R}^{5\times 5}$

$$(L_M(v))_j = M_{j\bullet} \cdot v$$

Template for linear maps $\mathbb{R}^5 \to \mathbb{R}^5$:



no good dot product

Template for linear maps

$\{\text{fns } \mathbb{R} \to \mathbb{R}\} \to \{\text{fns } \mathbb{R} \to \mathbb{R}\}$:

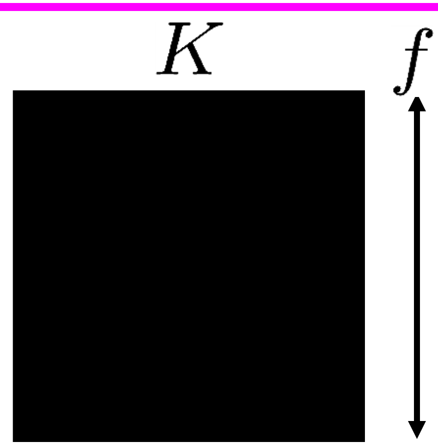
?

$K : \mathbb{R}^2 \to \mathbb{R}$

$$(L_K(f))(x) = K(x, \bullet) \cdot f$$

Template for linear maps

$L^2(\mathbb{R}) \to L^2(\mathbb{R})$:



$M \in \mathbb{R}^{5\times 5}$

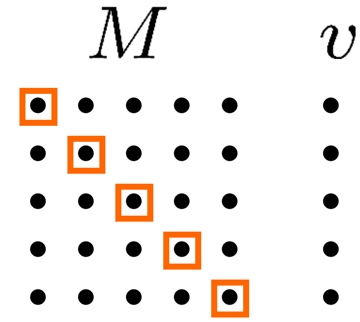
$(L_M v)_j$

$\|$

$M_{j\bullet} \cdot v$

Template for

linear maps $\mathbb{R}^5 \to \mathbb{R}^5$:



$$(L_M v)_j = \sum_k M_{jk} v_k$$

$(L_d v)_j =$
$d_j v_j$

$$(L_K f)(x) = \int_{-\infty}^{\infty} [K(x,y)][f(y)]\, dy$$

$(L_r f)(x) =$
$[r(x)][f(x)]$

$K : \mathbb{R}^2 \to \mathbb{R}$

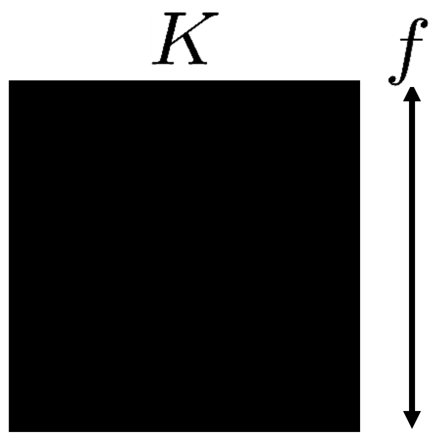
$(L_K(f))(x)$

$\|$

$K(x, \bullet) \cdot f$

Template for
linear maps

$L^2(\mathbb{R}) \to L^2(\mathbb{R})$:



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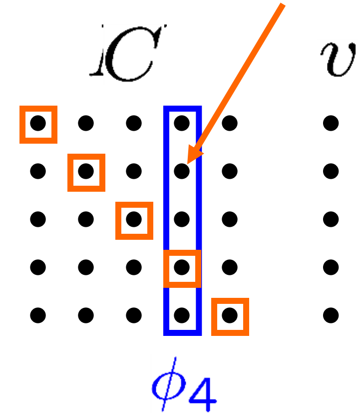


$$(L_M v)_j = \sum_k M_{jk} v_k$$

$$(L_d v)_j = d_j v_j$$

$$\text{orthogonal } M = \exp \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 5 & 6 & 7 \\ -2 & -5 & 0 & 8 & 9 \\ -3 & -6 & -8 & 0 & 10 \\ -4 & -7 & -9 & -10 & 0 \end{bmatrix}$$

Goal: $L_C^{-1} L_M L_C$ diagonal

The $(2,4)$ entry of C is $(\phi_4)_2$

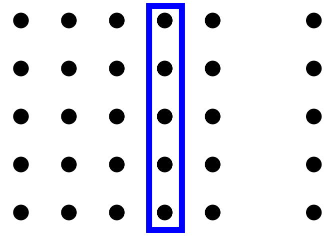
$$L_M^{\mathbb{C}} \phi_1 = d_1 \phi_1, \ldots, L_M^{\mathbb{C}} \phi_5 = d_5 \phi_5$$

$$d_1, \ldots, d_5 \in \mathbb{T}$$

$M \in \mathbb{R}^{5\times 5}$

$(L_M v)_j$ $\parallel$ $M_{j\bullet} \cdot v$

Template for linear maps $\mathbb{R}^5 \to \mathbb{R}^5$:



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orthogonal (M)

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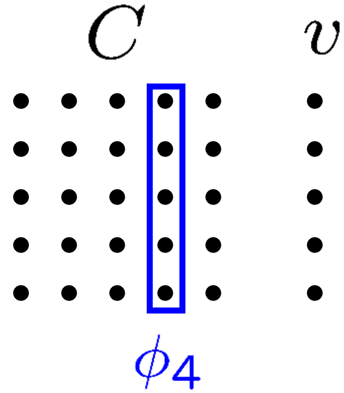
$$d_1, \ldots, d_5 \in \mathbb{T}$$

$v \in \mathbb{C}^5$ $\cup$ $L_C(v)$

$M \in \mathbb{R}^{5\times 5}$

$$(L_M v)_j \;\|\; M_{j\bullet} \cdot v$$

Template for linear maps $\mathbb{R}^5 \to \mathbb{R}^5$:



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orthogonal → M

Goal: $L_C^{-1} L_M^{\mathbb{C}} L_C$ diagonal

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$$d_1, \ldots, d_5 \in \mathbb{T}$$

$v \in \mathbb{C}^5$

$\cup$

$L_C(v)$

"orthogonal"

$$(Tf)(x) = f(x+1)$$

$$\forall y \in \mathbb{R}, \phi_y(x) = e^{-ixy}$$

$$(T^{\mathbb{C}}\phi_y)(x) = e^{-i(x+1)y}$$

$$= e^{-iy}e^{-ixy} = e^{-iy} \cdot \phi_y(x)$$

$$T^{\mathbb{C}}\phi_y = e^{-iy} \cdot \phi_y \; (e^{-iy} \in \mathbb{T})$$

ϕ_y is an eigenvector with eigenvalue e^{-iy}

Goal: $L_C^{-1} T L_C$ diagonal

$$C(x,y) = \phi_y(x)$$

$$= e^{-ixy}$$

The yth "column" of C should by ϕ_y

The xy "entry" of C should be $\phi_y(x)$

$$(L_K f)(x) = \int_{-\infty}^{\infty} [K(x,y)][f(y)]\, dy$$

$$(L_r f)(x) = [r(x)][f(x)]$$

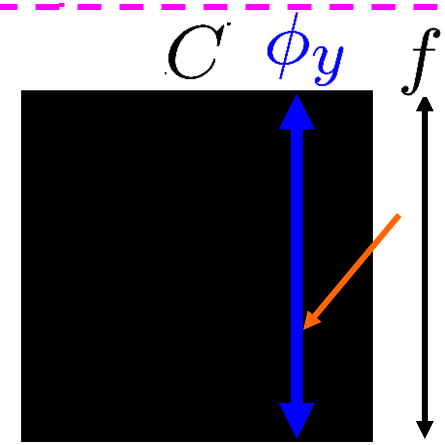
$$K : \mathbb{R}^2 \to \mathbb{R}$$

$$(L_K(f))(x) \;\|\; K(x,\bullet) \cdot f$$

Template for

$$L^2(\mathbb{R})^{\mathbb{C}} \to L^2(\mathbb{R})^{\mathbb{C}}:$$

$$L^2(\mathbb{R})^{\mathbb{C}} \stackrel{..}{=} \{f : \mathbb{R} \to \mathbb{C} \mid \textstyle\int_{-\infty}^{\infty} |f|^2 < \infty\}$$



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$(L_r f)(x) = [r(x)][f(x)]$

$$(L_C f)(x) = \int_{-\infty}^{\infty} [C(x,y)][f(y)]\, dy$$

$$= \int_{-\infty}^{\infty} [e^{-ixy}][f(y)]\, dy$$

Def'n: The Fourier transform of $f \in L^2(\mathbb{R})^{\mathbb{C}}$ is $L_C f^{\in}$

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Goal: $L_C^{-1} T^{\mathbb{C}} L_C$ diagonal

$$C(x,y) = \phi_y(x) = e^{-ixy}$$

Problem: Find $r : \mathbb{R} \to \mathbb{C}$ s.t. $L_C^{-1} T^{\mathbb{C}} L_C = L_r$

$$(L_K f)(x) = \int_{-\infty}^{\infty} [K(x,y)][f(y)]\, dy$$

$$(L_r f)(x) = [r(x)][f(x)]$$

$$(L_C f)(x) = \int_{-\infty}^{\infty} [C(x,y)][f(y)]\, dy = \int_{-\infty}^{\infty} [e^{-ixy}][f(y)]\, dy$$

Def'n: The Fourier transform of $f \in L^2(\mathbb{R})^{\mathbb{C}}$ is $L_C f$