Financial Mathematics

Pricing/hedging in many subperiods

Part 2

$f(S) = (5000S - 5000)_+$

probability problems,
then expected value problems

Goal:

approximately

E

Compute the expected value of $f(e^{Hu+Td})$.

Then multiply by $\boxed{e^{-rN}} = (e^r)^{-N}$.

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(e^{Hu+Td})$,
30 days from now.

$e^r = 1.000000001$
$N = 2,592,000$

$e^{rN}V =$ expected payout $=: E =$???

$\boxed{V} = \boxed{e^{-rN}}E$

$=$ discounted expected payout

$$f(S) = (5000S - 5000)_+$$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

Easier problem:

Compute the expected value of $f(D_2)$.

$D_2 = H_2 - T_2$:

0.25	2	$f(2)$
0.5	0	$f(0)$
0.25	−2	$f(-2)$

works for any function

$f :\rightarrow g$

$$[0.25][f(2)] + [0.5][f(0)] + [0.25][f(-2)] = 1,250$$

Define: $g(S) = 5e^S + S^2$

Easier problem:

Compute the expected value of $g(D_2)$.

$D_2 = H_2 - T_2$:

0.25	2	$g(2)$
0.5	0	$g(0)$
0.25	-2	$g(-2)$

$[0.25][g(2)] + [0.5][g(0)] + [0.25][g(-2)] =$ Exercise

Recall: $\underline{f(\boxed{S}) = (5000\boxed{S} - 5000)_+}$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute $\underline{\text{the expected value of } f(Z)}$.

Z:

$\underline{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx}$ | $\underline{x}$ $f(x)$

Do this for $\underline{\textit{all } x \in \mathbb{R}}$

$$\int_{-\infty}^{\infty} [f(x)] \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$\boxed{f(\boxed{x}) = (5000\boxed{x} - 5000)_+}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[f(x)]}\, e^{-x^2/2}\,dx = \text{exercise}$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute the expected value of $f(Z)$.

Z:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \;\Big|\; x \quad f(x)$$

Do this for *all* $x \in \mathbb{R}$

$$E[f(Z)] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} [f(x)] \int_{-\infty}^{\infty} e^{-x^2/2}\,dx^{-x^2/2}\,dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute the expected value of $f(Z)$.

Z:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \quad \vdash x \qquad f(x)$$

Do this for all $x \in \mathbb{R}$

works for any exp-bdd function $f :\to g$

$$E[f(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute the expected value of $g(Z)$.

Z:

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ — x $g(x)$

Do this for all $x \in \mathbb{R}$

works for any exp-bdd function g

g

temporary change of color. . .

$$E[g(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute the expected value of $g(Z)$.

Z:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ $\vert$ x $g(x)$

Do this for *all* $x \in \mathbb{R}$

works for
any exp-bdd
function g

temporary change of color. . .

$$E[g(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute the expected value of $g(Z)$.

Z:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ — x $g(x)$

Do this for *all* $x \in \mathbb{R}$

Change every Z to x and then integrate against $h(x)\,dx$, from $-\infty$ to ∞.

$$\boxed{h(x)} := \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

works for any exp-bdd function g

$$E[g(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem: $g(X)$ $X = D_N/\sqrt{N}$

Compute the expected value of $g(Z)$.

Z:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ | x $g(x)$

Do this for all $x \in \mathbb{R}$

works for any exp-bdd function g

$$E[g(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem: $X = D_N/\sqrt{N}$

Compute $\underset{\text{approximately}}{^\vee}$ the expected value of $g(X)$.

Z:

$\approx$

X

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$	x	$g(x)$

Do this for all $x \in \mathbb{R}$

works for any exp-bdd function g

$$E[g(X)] \approx$$

$$E[g(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $\boxed{f(e^{Hu+Td})}$.

New easier problem: Subgoal: Choose g s.t.: $||$

Compute $\underset{\text{approximately}}{^\vee}$ the expected value of $\boxed{g(X)}$.

$$E[g(X)] \approx$$

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

write H, T
as expr.s of X

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\underbrace{\text{Compute}}_{}$ approximately the $\overset{E}{\text{expected}}$ value of $\boxed{f(e^{Hu+Td})}$.

New easier problem:

Subgoal: Choose g s.t.: || g exp-bdd?

Compute approximately the expected value of $\boxed{g(X)}$.

works for
any exp-bdd
function g

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

write H, T as expr.s of X

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{\overset{\downarrow}{H}u+\overset{\downarrow}{T}d})$.

New easier problem:

Compute $\underset{\text{approximately}}{\vee}$ the expected value of $g(X)$.

$\boxed{X} = \boxed{(H-T)/\sqrt{N}}$ $\qquad N = 2{,}592{,}000$

$\times\sqrt{N}$ $\qquad \times\sqrt{N}$

$$\boxed{H \cancel{+T} = N} \xrightarrow{\hspace{2cm}} \boxed{\begin{array}{l} \cancel{H} + T = N \\ \cancel{-H} + T = -X\sqrt{N} \end{array}} \text{ ADD}$$

$$\boxed{H \cancel{-T} = X\sqrt{N}} \text{ ADD NEGATE}$$

$$2H = N + X\sqrt{N} \qquad 2T = N - X\sqrt{N}$$

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

write H, T as expr.s of X

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute $\underset{\text{approximately}}{^\vee}$ the expected value of $g(X)$.

$\boxed{H = [N + X\sqrt{N}]/2}$ $\boxed{T = [N - X\sqrt{N}]/2}$

$N := 30 \times 24 \times 60 \times 60 = 2,592,000$

$\boxed{2H = N + X\sqrt{N}}$
DIVIDE BY 2

$\boxed{2T = N - X\sqrt{N}}$
DIVIDE BY 2

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

write H, T as expr.s of X

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute $\underset{\text{approximately}}{^\vee}$ the expected value of $g(X)$.

$$\boxed{H = [N + X\sqrt{N}]/2} \qquad \boxed{T = [N - X\sqrt{N}]/2}$$

$\times u$ $\qquad$ $\times d$

$$\left.\begin{aligned} Hu &= [Nu + X\sqrt{N}u]/2 \\ Td &= [Nd - X\sqrt{N}d]/2 \end{aligned}\right\}\text{ADD}$$

$$\begin{aligned} Hu + Td &= [N(u+d) + X\sqrt{N}(u-d)]/2 \\ &= [N(u+d)/2] + [X\sqrt{N}(u-d)/2] \end{aligned}$$

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

write H, T
as expr.s of X

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute $^\vee_{\text{approximately}}$ the expected value of $g(X)$.

$$H = [N + X\sqrt{N}]/2 \qquad T = [N - X\sqrt{N}]/2$$

$$e^{Hu+Td} = [e^{N(u+d)/2}][e^{\boxed{X}\sqrt{N}(u-d)/2}]$$

$$= [e^{N(u+d)/2}][e^{(\sqrt{N}(u-d)/2)\boxed{X}}]$$

$$Hu + Td$$

$$Hu + Td = [N(u+d)/2] + [X\sqrt{N}(u-d)/2]$$

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

write H, T as expr.s of X

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $f(e^{Hu+Td})$.

New easier problem:

Compute approximately the expected value of $g(X)$.

$$H = [N + X\sqrt{N}]/2 \qquad\qquad T = [N - X\sqrt{N}]/2$$

$$e^{Hu+Td}$$

$$e^{Hu+Td} = [\overbrace{e^{N(u+d)/2}}^{C}][e^{\overbrace{(\sqrt{N}(u-d)/2)}^{k}X}] = Ce^{kX}$$

$$Hu + Td = [N(u+d)/2] + [X\sqrt{N}(u-d)/2]$$

$$E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

write H, T as expr.s of X

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $\boxed{f(e^{Hu+Td})}$.

$\|$

Restatement of goal:

Compute $\underset{\text{approximately}}{\vee}$ the expected value of $\boxed{g(X)}$.

$$H = [N + X\sqrt{N}]/2 \qquad T = [N - X\sqrt{N}]/2$$

$$e^{Hu+Td} = [\overbrace{e^{N(u+d)/2}}^{C}][e^{(\overbrace{\sqrt{N}(u-d)/2}^{k})X}] = Ce^{kX}$$

$$g(x) := f(Ce^{kx})$$

g exp-bdd? YES

$$f(e^{Hu+Td}) = f(Ce^{kX}) = g(X)$$

$$E = E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

$\overbrace{\text{Compute}}^{\text{approximately}}$ the $\overset{E}{\text{expected}}$ value of $\boxed{f(e^{Hu+Td})}$.

$$\|$$

Restatement of goal:

Compute $\underset{\text{approximately}}{\vee}$ the expected value of $\boxed{g(X)}$.

$N = 2,592,000$

$u = 0.00003561536577$

$d = -0.00003561463419$

$1.00094857729 = C$

$k = 0.0573390439012$

$$e^{Hu+Td} = [\overbrace{e^{N(u+d)/2}}^{C}][e^{\overbrace{(\sqrt{N}(u-d)/2)}^{k}X}] = Ce^{kX}$$

$$g(x) := f(Ce^{kx})$$

$$f(e^{Hu+Td}) = f(Ce^{kX}) = g(X)$$

$$E = E[g(X)] \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(\boxed{S}) = (5000S - 5000)_+$

$$= 5000(\boxed{S} - 1)_+$$

$$g(x) := f(\boxed{Ce^{kx}}) = 5000(\boxed{Ce^{kx}} - 1)_+$$

$$E \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$1.00094857729 = C$

$k = 0.0573390439012$

$$g(x) := f(Ce^{kx})$$

$$E = E[g(X)] \approx \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)] \, e^{-x^2/2} \, dx}$$

$$E \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$$E \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$1.00094857729 = C$

$k = 0.0573390439012$

$k = 0.0573390439012$

$1.00094857729 = C$

$$E \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx}-1)_+ \, e^{-x^2/2}\,dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\underline{Ce^{\overset{a}{kx}}-1})_{\boxed{+}} \, e^{-x^2/2}\,dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{a}^{\infty} (Ce^{kx}-1) \, e^{-x^2/2}\,dx$$

$$\begin{aligned} Ce^{ka}-1 &= 0 \\ Ce^{ka} &= 1 \\ e^{ka} &= 1/C \\ ka &= \ln(1/C) = -\ln C \\ a &= -(\ln C)/k \\ &= -0.0165354585751 \end{aligned}$$

$k = 0.0573390439012$

$1.00094857729 = C$

$$E \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_{+} \, e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_{+} \, e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{a}^{\infty} (Ce^{kx} - 1) \, e^{-x^2/2} \, dx$$

$$k = 0.05$$

$$= \frac{5000}{\sqrt{2\pi}} \left[C \int_{a}^{\infty} e^{kx} \, e^{-x^2/2} \, dx - \int_{a}^{\infty} e^{-x^2/2} \, dx \right]$$

$1.00094857729 = C$

$k = 0.0573390439012$

$a = -0.0165354585751$

$E \approx$

$$E \approx \frac{5000}{\sqrt{2\pi}}\left[C\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx \;-\; \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$\frac{5000}{\sqrt{2\pi}}\left[C\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx \;-\; \int_a^\infty e^{-x^2/2}\, dx \right]$$

$1.00094857729 = C$

$k = 0.0573390439012$

$a = -0.0165354585751$

NEGATE THE LOWER LIMIT

DON'T FORGET

$$\boxed{\sqrt{2\pi}}\,\Phi(-a)$$

$x :\to x + k$

$$E \approx \frac{5000}{\sqrt{2\pi}}\left[C \underbrace{\int_a^\infty \boxed{e^{kx}}\, e^{-x^2/2}\,dx} - \overbrace{\int_a^\infty e^{-x^2/2}\,dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

THE LOWER LIMIT

$$\underbrace{\int_{\boxed{a-k}}^\infty e^{k(x+k)}\, e^{-(x+k)^2/2}\,dx}$$

$$\underbrace{\cancel{e^{kx}}\,\boxed{e^{k^2}}\,e^{-x^2/2}\,\boxed{e^{-k^2/2}}\,\cancel{e^{-kx}}}$$

$$\underbrace{e^{k^2/2}\int_{a-k}^\infty e^{-x^2/2}\,dx}$$

THE LOWER LIMIT

$$\boxed{\sqrt{2\pi}}\,\Phi(\underline{k-a})$$

DON'T FORGET

NEGATE THE LOWER LIMIT

$1.00094857729 = C$

$k = 0.0573390439012$

$a = -0.0165354585751$

$$E \approx \frac{5000}{\sqrt{2\pi}}\left[C \underbrace{\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$\int_{a-k}^\infty \underbrace{e^{k(x+k)}\, e^{-(x+k)^2/2}}\, dx$$

$$\underbrace{e^{k^2}\, e^{-x^2/2}\, e^{-k^2/2}}$$

$$e^{k^2/2}$$

$$\underbrace{\sqrt{2\pi}\,\Phi(k-a)}$$

$1.00094857729 = C$ $\quad k = 0.0573390439012$

$a = -0.0165354585751$

$$E \approx \frac{5000}{\sqrt{2\pi}}\left[C \underbrace{\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx}_{} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$e^{k^2/2}\sqrt{2\pi}\,\Phi(k-a)$$

$1.00094857729 = C$

$k = 0.0573390439012$

$a = -0.0165354585751$

$$E \approx \frac{5000}{\cancel{\sqrt{2\pi}}}\left[C \underbrace{\int_a^{\infty} e^{kx}\, e^{-x^2/2}\, dx}_{e^{k^2/2}\cancel{\sqrt{2\pi}}\,\Phi(k-a)} - \overbrace{\int_a^{\infty} e^{-x^2/2}\, dx}^{\cancel{\sqrt{2\pi}}\,\Phi(-a)} \right]$$

$$= 5000\left[\overbrace{C\, e^{k^2/2}}^{1.00259537252} \Big[\overbrace{\Phi(\underbrace{k-a}_{0.07387450247})}^{0.52944}\Big] - \Big[\overbrace{\Phi(\underbrace{-a}_{0.0165354585751})}^{0.5066}\Big] \right]$$

$$= 121.07046876$$

$1.00094857729 = C$

$k = 0.0573390439012$

$a = -0.0165354585751$

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(u^H d^T)$,
30 days from now.

$$e^r = 1.000000001$$
$$N = 2,592,000$$
$$E \approx$$

$$e^{rN}V = \text{expected payout} =: \underbrace{E = ???}_{\text{approx.}}$$

$$V = e^{-rN}E$$
$$= \text{discounted expected payout}$$

$$V = e^{-rN}E \approx 120.757060394$$

$$1.000000001 =: e^r$$

$$e^{-rN} = 0.997411356336$$

121.07046876

exact answer? soon...

$$E \approx 121.07046876$$

$$N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

Scenarios with j upticks and $N-j$ downticks:

There are $\binom{N}{j}$ of them.

Each has (risk-neutral) probability: $(0.5)^j(0.5)^{N-j}$

$$\boxed{\text{prob.}_{j,N-j} = \binom{N}{j}(0.5)^j(0.5)^{N-j}}$$

$$V = e^{-rN}E \approx 120.757060394$$

1.000000001

$=: e^r$

$$e^{-rN} = 0.997411356336$$

exact answer?

$$E \approx 121.07046876$$

$$N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

Scenarios with j upticks and $N-j$ downticks:

ln(stock price) starts at 0,

$f(S) = 5000(S-1)_+$

ends at $ju+(N-j)d$

stock price ends at $\boxed{e^{ju+(N-j)d}}$

option pays $\underbrace{5000(e^{ju+(N-j)d}-1)_+}_{\text{ending value of hedge}}$

$$\boxed{\text{prob.}_{j,N-j} = \binom{N}{j}(0.5)^j(0.5)^{N-j}}$$

$$V = e^{-rN}E \approx 120.757060394$$

$$1.000000001 =: e^r$$

$$e^{-rN} = 0.997411356336$$

$\boxed{\text{exact answer?}}$

$$E \approx 121.07046876$$

$$N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

Scenarios with j upticks and $N-j$ downticks:

$E :=$ (risk-neutral) expected ending value of hedge

To compute it, multiply this by this,
then add over $j = 0, \ldots, N$.

option pays $\underbrace{\boxed{5000(e^{ju+(N-j)d} - 1)_+}}_{\text{ending value of hedge}}$

$$\boxed{\text{prob.}_{j,N-j} = \boxed{\binom{N}{j}(0.5)^j(0.5)^{N-j}}}$$

$V = e^{-rN}E \approx 120.757060394$

$1.000000001 =: e^r$

$e^{-rN} = 0.997411356336$

exact answer?

$E \approx 121.07046876$

$N := 30 \times 24 \times 60 \times 60 = 2,592,000$

Scenarios with j upticks and $N-j$ downticks:

$E :=$ (risk-neutral) expected ending value of hedge

To compute it, multiply this by this,
then add over $j = 0, \ldots, N$.

option pays $\underbrace{\boxed{5000(e^{ju+(N-j)d} - 1)_+}}_{\text{ending value of hedge}}$

$$\boxed{\text{prob.}_{j,N-j} = \boxed{\binom{N}{j}(0.5)^j(0.5)^{N-j}}}$$

$$V = e^{-rN}E \approx 120.757060394$$

Exact answer:

$$E = \sum_{j=0}^{N} \boxed{\binom{N}{j}[(0.5)^j(0.5)^{N-j}]}\boxed{5000(e^{ju+(N-j)d} - 1)_+}$$

exact answer?

$$E \approx 121.07046876$$

$$N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

$$V = e^{-rN} E \approx 120.757060394$$

$$\boxed{\begin{aligned} u &= 0.00003561536577 \\ d &= -0.00003561463419 \end{aligned}}$$

Exact answer:

$$E = \sum_{j=0}^{N} \binom{N}{j} [\overbrace{(0.5)^j (0.5)^{N-j}}^{(0.5)^N}] 5000 (e^{ju+(N-j)d} - 1)_+$$

$$= 121.1129585417487 \quad \boxed{e^{-rN} = \ 0.997411356336}$$

$$E \approx 121.07046876$$

$$N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

Another Central Limit Theorem application:

Recall: $S_0 = 1$ is the initial price of the stock.

Let S_1 denote the price after one year.

Exercise: Compute $\mathsf{E}[S_1]$, approximately.

$$V = e^{-rN} E \approx 120.757060394$$

Exact answer:

$$\boxed{\begin{aligned} u &= 0.00003561536577 \\ d &= -0.00003561463419 \end{aligned}}$$

$$E = \sum_{j=0}^{N} \binom{N}{j} [\overbrace{(0.5)^j (0.5)^{N-j}}^{(0.5)^N}] 5000 (e^{ju+(N-j)d} - 1)_+$$

$$= 121.1129585417487 \quad \boxed{e^{-rN} = 0.997411356336}$$

$$E \approx 121.07046876$$

$$V = e^{-rN} E = 120.7994402 \ ■$$

Market analyst: annual vol = 0.200002881086
annual drift = 0.03399864624

Recall: $S_0 = 1$ is the initial price of the stock.
Let S_1 denote the price after one year.
Exercise: Compute $\mathrm{E}[S_1]$, approximately.

$$V = e^{-rN} E \approx 120.757060394$$

$$\boxed{\begin{aligned} u &= 0.00003561536577 \\ d &= -0.00003561463419 \end{aligned}}$$

Exact answer:

$$E = \sum_{j=0}^{N} \binom{N}{j} [\overbrace{(0.5)^j (0.5)^{N-j}}^{(0.5)^N}] 5000 (e^{ju+(N-j)d} - 1)_+$$

$$= 121.1129585417487 \quad \boxed{e^{-rN} = 0.997411356336}$$

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Solution: By the Central Limit Theorem, $\ln S_1$, being a large sum of iid PCRVs, is approximately normal.

Then exponentiation nearly almost commutes with expectation.

That is, $\mathrm{E}[\overset{S_1 \,=}{\boxed{e^{\ln S_1}}}] \approx e^{\overset{\mathrm{E}[\ln S_1] + \frac{1}{2}\mathrm{Var}[\ln S_1] \;\parallel}{\boxed{\mathrm{AE}[\ln S_1]}}}$.

That is, $\mathrm{E}[S_1] \approx e^{\mathrm{E}[\ln S_1] + \frac{1}{2}\mathrm{Var}[\ln S_1]}$.

Market analyst: annual vol $= 0.200002881086$
annual drift $= 0.03399864624$

Recall: $S_0 = 1$ is the initial price of the stock.
Let S_1 denote the price after one year.
Exercise: Compute $\mathrm{E}[S_1]$, approximately.

Solution:

$$\mathrm{E}[S_1] \approx e^{\overset{\text{drift}}{\boxed{\mathrm{E}[\ln S_1]}}+\frac{1}{2}\overset{\text{vol squared}}{\boxed{\mathrm{Var}[\ln S_1]}}}$$

$$= e^{(0.03399864624)+\frac{1}{2}(0.200002881086)^2}$$

$$\mathrm{E}[S_1] \approx e^{\mathrm{E}[\ln S_1]+\frac{1}{2}\mathrm{Var}[\ln S_1]}$$

Market analyst: annual vol $=$ 0.200002881086
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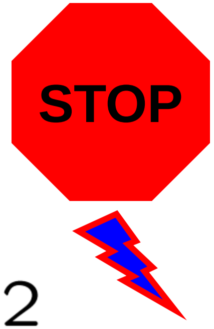
$$= e^{(0.03399864624)+\frac{1}{2}(0.200002881086)^2}$$

$$= 1.05548378145 \;\blacksquare$$



Expected annual return is about 5.5%.
Annual drift is 0.03399864624.
Annual augmented drift is, by definition,

$$(0.03399864624) + \frac{1}{2}(0.200002881086)^2$$

annual risk-free factor $= e^{\text{annual augmented drift}}$