Financial Mathematics

Testing
the Black-Scholes formula

inputs:
$T,\ \sigma_*,\ r_*,\ S_0,\ K$

PRESENT FORMULA

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

inputs:
$\sigma,\ r,\ S_0,\ K$

Let $K' := \dfrac{K}{e^{r}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma} \pm \dfrac{\sigma}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

version zero of Black-Scholes

PRESENT FORMULA
TIME NORMALIZED

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$

second version of Black-Scholes

NEUTRAL FORMULA

forward price on stock

Let $\boxed{F} := Se^{r_* T}$.

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

FORWARD FORMULA

Do these formulas really approximate the CRR price?

Kyle wants right, but not obligation, to buy 5000 shares of ABC for \$5000, 30 days from now.

Gail, seller

$N :=$ number of seconds in 30 days

Gail selects:

N-subperiod 50.001-49.999 CRR model

$S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

PRESENT FORMULA
TIME NORMALIZED

PRESENT FORMULA
TIME NORMALIZED

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

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$$V = e^{-rN} E = 120.7994402$$

close?

PRESENT FORMULA
TIME NORMALIZED

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

Do these formulas really approximate the CRR price?

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Gail, seller

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N-subperiod 50.001-49.999 CRR model

Market analyst: (ann) vol = 0.200002881086

Banker:

(annual) continuous compounding nominal rate

$= 0.0315$[illegible]

$V = 120.7994402$

close?

$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$

$V = 120.7994402$

close?

$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$

Kyle wants right, but not obligation, to buy 5000 shares of ABC for \$5000, 30 days from now. $K = 1$

Gail, seller

$T = 30/365$

$N :=$ number of seconds in 30 days

Gail selects:

N-subperiod 50.001-49.999 CRR model

Market analyst: (ann) vol $= 0.200002881086$

Banker:

(annual) continuous compounding nominal rate $= 0.031535998802$

$$V = 120.7994402$$

close?

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

$T = 30/365$ Assume: Initial price $= \$1$/share.

$r = r_* T$

$S_0 = 1$

$\sigma = \sigma_* \sqrt{T}$

$K = 1$ $T = 30/365$

$K = 1$

$$K' = \frac{K}{e^r}$$

$$d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2}$$

nn) vol = 0.200002881086

$$d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2}$$

s compounding nominal rate

315359998802

Market analyst: (ann) vol $= 0.200002881086$

Banker:

(annual) continuous compounding nominal rate $= 0.0315359998802$

$V = 120.7994402$

close?

$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$

$T = 30/365$ $\qquad S_0 = 1$

$r = r_* T = 0.00259199999014 \qquad S_0 = 1$

$\sigma = \sigma_* \sqrt{T} = 0.057339043865$

$K = 1$

$$K' = \frac{K}{e^r}$$

$$d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2}$$

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$r = r_* T = 0.00259199999014$

$\sigma = \sigma_* \sqrt{T} = 0.057339043865$

$K = 1$

$$K' = \frac{K}{e^r} = 0.997411356345$$

$$d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2}$$

$$d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2}$$

Market analyst: (ann) vol = 0.200002881086

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$$d_+ = \frac{\boxed{\ln(S_0/K')}}{\sigma} + \frac{\sigma}{2}$$

$$d_- = \frac{\boxed{\ln(S_0/K')}}{\sigma} - \frac{\sigma}{2}$$

The option is (bogus) "at the money".

Whenever $S_0 = K$,

$\ln(S_0/K') = r$

$$\boxed{\ln(S_0/K')} = 0.00259199999014$$

$V = 120.7994402$

close?

$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$

$T = 30/365$ $\quad S_0 = 1$

$r = r_* T = 0.00259199999014$

$\sigma = \sigma_* \sqrt{T} = 0.057339043865$

$K = 1$

$$K' = \frac{K}{e^r} = 0.997411356345$$

$$d_+ = \frac{\ln(S_0/K')}{\sigma} + \frac{\sigma}{2} = \left(\begin{array}{r} 0.0452047996532 \\ +0.0286695219325 \end{array}\right)$$

$$d_- = \frac{\ln(S_0/K')}{\sigma} - \frac{\sigma}{2} = \left(\begin{array}{r} 0.0452047996532 \\ -0.0286695219325 \end{array}\right)$$

$$\ln(S_0/K') = 0.00259199999014$$

$$V = 120.7994402$$

close?

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

$T = 30/365$ $\qquad S_0 = 1$

$r = r_* T = 0.00259199999014$

$\sigma = \sigma_* \sqrt{T} = 0.057339043865$

$K = 1$

$$K' = \frac{K}{e^r} = 0.997411356345$$

$$d_+ = \begin{pmatrix} 0.0452047996532 \\ +0.0286695219325 \end{pmatrix} = \begin{pmatrix} 0.0452047996532 \\ +0.0286695219325 \end{pmatrix} = 0.0738743215857$$

$$d_- = \begin{pmatrix} 0.0452047996532 \\ -0.0286695219325 \end{pmatrix} = \begin{pmatrix} 0.0452047996532 \\ -0.0286695219325 \end{pmatrix} = 0.0165352777207$$

$$V = 120.7994402$$

close?

$$5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])$$

$T = 30/365$ $\qquad$ $S_0 = 1$

$r = r_* T = 0.00259199999014$

$\sigma = \sigma_* \sqrt{T} = 0.057339043865$

$K = 1$

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$$d_+ = \begin{pmatrix} 0.0452047996532 \\ +0.0286695219325 \end{pmatrix} = 0.0738743215857$$

$$d_- = \begin{pmatrix} 0.0452047996532 \\ -0.0286695219325 \end{pmatrix} = 0.0165352777207$$

$$\boxed{\Phi(d_+)} = 0.52944$$

$$\boxed{\Phi(d_-)} = 0.50660$$

$V = 120.7994402$

close?

$5000(S_0[\boxed{\Phi(d_+)}] - K'[\boxed{\Phi(d_-)}])$

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$K = 1$

$$K' = \frac{K}{e^r} = 0.997411356345$$

$$\Phi(d_+) = 0.52944$$

$$\Phi(d_-) = 0.50660$$

$$\boxed{S_0}[\boxed{\Phi(d_+)}] - K'[\boxed{\Phi(d_-)}] = 0.024151406898$$

$$\Phi(d_+) = 0.52944$$

$$\Phi(d_-) = 0.50660$$

$$V = 120.7994402$$

$$5000(\boxed{S_0[\Phi(d_+)] - K'[\Phi(d_-)]})$$

close?

$$T = 30/365 \qquad S_0 = 1$$

$$r = r_* T = 0.0025919999014$$

$$\sigma = \sigma_* \sqrt{T} = 0.057339043865$$

$$K = 1$$

$$K' = \frac{K}{e^r} = 0.997411356345$$

$$\Phi(d_+) = 0.52944$$

$$\Phi(d_-) = 0.50660$$

$$S_0[\Phi(d_+)] - K'[\Phi(d_-)] = 0.024151406898$$

$$\boxed{5000\left(S_0[\Phi(d_+)] - K'[\Phi(d_-)]\right)}$$

$$= 120.7570345$$

$$V = 120.7994402$$

YES

close?

$$\boxed{5000(S_0[\Phi(d_+)] - K'[\Phi(d_-)])}$$

inputs:
$T,\ \sigma_*,\ r_*,\ S_0,\ K$

PRESENT FORMULA

Let $K' := \dfrac{K}{e^{r_* T}}$.

Let $d_\pm := \dfrac{\ln(S_0/K')}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

first version of Black-Scholes

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$\sigma,\ r,\ S_0,\ K$

Let $K' := \dfrac{K}{e^{r}}$.

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output: $S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

version zero of Black-Scholes

PRESENT FORMULA
TIME NORMALIZED

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_* T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

output: $S_0[\Phi(d_+)] - [Ke^{-r_* T}][\Phi(d_-)]$

second version of Black-Scholes

NEUTRAL FORMULA

forward price on stock

Let $\boxed{F} := Se^{r_* T}$.

Let $d_\pm := \dfrac{[\ln(F/K)]}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$

forward price on option

output: $e^{-r_* T}\left(\boxed{F[\Phi(d_+)] - K[\Phi(d_-)]}\right)$

third version of Black-Scholes

FORWARD FORMULA

YES

Do these formulas really approximate the CRR price?

YES

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$T,\ \sigma_*,\ r$

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NEUTRAL FORMULA

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

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output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

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Let $d_\pm := \dfrac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} \pm \dfrac{\sigma_*\sqrt{T}}{2}$.

NEUTRAL FORMULA

output: $S_0[\Phi(d_+)] - [Ke^{-r_*T}][\Phi(d_-)]$

second version of Black-Scholes

$$\mathrm{BlSch}(T,\sigma_*,r_*,S_0,K) :=$$

$$S_0\left[\Phi\left(\boxed{\frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} + \frac{\sigma_*\sqrt{T}}{2}}\right)\right]$$

$$-\left[Ke^{-r_*T}\right]\left[\Phi\left(\boxed{\frac{[\ln(S_0/K)] + r_*T}{\sigma_*\sqrt{T}} - \frac{\sigma_*\sqrt{T}}{2}}\right)\right]$$

Fact: For all $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$,

$\sigma_* \mapsto \mathrm{BlSch}(T,\sigma_*,r_*,S_0,K) : (0,\infty) \to (0,\infty)$

is increasing.

Exercise: Prove this.

Definition:

For all $V > 0$, $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$,

if $\exists \sigma_* > 0$ such that

$$V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)$$

then this solution σ_* is unique and is called **the implied volatility associated to** V**,** T**,** r_***,** S_0 **and** K.

$$\text{BlSch}(T, \sigma_*, r_*, S_0, K) :=$$

$$S_0 \left[\Phi \left(\frac{[\ln(S_0/K)] + r_* T}{\sigma_* \sqrt{T}} + \frac{\sigma_* \sqrt{T}}{2} \right) \right] - \left[K e^{-r_* T} \right] \left[\Phi \left(\frac{[\ln(S_0/K)] + r_* T}{\sigma_* \sqrt{T}} - \frac{\sigma_* \sqrt{T}}{2} \right) \right]$$

Fact: For all $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$,

$\sigma_* \mapsto \text{BlSch}(T, \sigma_*, r_*, S_0, K) : (0, \infty) \to (0, \infty)$ is increasing.

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Fiction: Black-Scholes works, So why teach BS??
i.e., volatility, drift and risk-free rates are constant.

Fiction: Home mortgage interest rates stay constant over thirty year periods.

Nevertheless: They're useful, because . . .
they give a way of comparing mortgages.

dimensionless

Similar for Black-Scholes.

Next subtopic: Volatitility smiles and skews and volatility surfaces

Definition:

For all $V > 0$, $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$,

if $\exists \sigma_* > 0$ such that

$$V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)$$

then this solution σ_* is unique and is called **the implied volatility associated to** V**,** T**,** r_***,** S_0 **and** K.

Fiction: Black-Scholes works,
i.e., volatility, drift and risk-free rates are constant.

Pick a financial instrument (*e.g.*, a stock).
Look up S_0. Look up r_*.
Fix T.
For various choices of K,
look up V, compute σ_* and plot (K, σ_*).
The result is called
the **volatility smile** or the **volatility skew**,
depending on whether it's
concave up or concave down.

Definition:

For all $V > 0$, $T > 0$, $r_* > 0$, $S_0 > 0$ and $K > 0$,

if $\exists \sigma_* > 0$ such that

$$V = \text{BlSch}(T, \sigma_*, r_*, S_0, K)$$

then this solution σ_* is unique and is called **the implied volatility associated to V, T, r_*, S_0 and K.**

Fiction: Black-Scholes works,
i.e., volatility, drift and risk-free rates are constant.

Pick a financial instrument (*e.g.*, a stock).
Look up S_0. Look up r_*.
For various choices of K and T,
look up V, compute σ_* and plot (K, T, σ_*).
The result is called
the **volatility surface**.

STOP