Financial Mathematics

The Triangular Central Limit Theorem

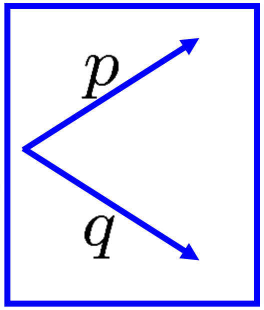
Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Def'n: $\forall p, q \in [0,1]$, s.t. $p + q = 1$ $\boxed{\mathcal{B}^p_q} := \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d}$



means: $E[\phi(Y_n)] \to E[\phi(\boxed{Z})]$,

$\forall$ bdd contin. $\phi : \mathbb{R} \to \mathbb{R}$

equivalent: $\mathcal{F}\delta[Y_n] \to \mathcal{F}\delta[\boxed{Z}]$

Z not yet def'd, so. . .

Th'm (ΔCLT): (Triangular Central Limit Theorem)

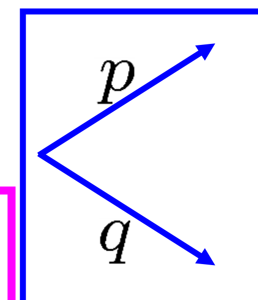
Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1 - \varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

$$
\begin{aligned}
Y_1 &= X_{1,1} \\
Y_2 &= X_{2,1} + X_{2,2} \\
Y_3 &= X_{3,1} + X_{3,2} + X_{3,3} \\
&\ \ \vdots
\end{aligned}
$$

Def'n: $\forall p, q \in [0,1]$, s.t. $p + q = 1$, $\boxed{\mathcal{B}^p_q} := \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d}$



means: $\mathrm{E}[\phi(Y_n)] \to \frac{1}{\sqrt{2\pi}} \int [\phi(x)] e^{-x^2/2}\, dx$,

$\forall$ bdd contin. $\phi : \mathbb{R} \to \mathbb{R}$

equivalent: $\mathcal{F}\delta[Y_n] \to e^{-t^2/2}$

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

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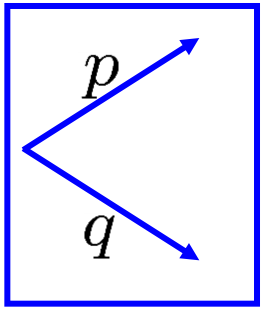


The iid sum of general binaries is the union of the iid sums of specific binaries.

$$\sum^n \mathcal{B}^p_q = \sum^n \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d} = \bigcup_{d<u} \sum^n \mathcal{B}^{p,u}_{q,d}$$

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

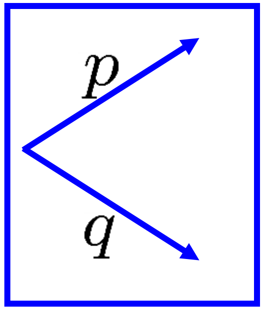
For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

$$\boxed{q_n := 1 - p_n}$$

Def'n: $\forall p, q \in [0,1]$, s.t. $p + q = 1$

$$\boxed{\mathcal{B}^p_q} := \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d}$$



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Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\underbrace{\Sigma^n \mathcal{B}^{p_n}_{1-p_n}}_{\bigcup_{d<u} \Sigma^n \mathcal{B}^{p_n,u}_{q_n,d}}] \cap \mathcal{S}$.

$u :\rightarrow \hat{u}_n$

$d :\rightarrow \hat{d}_n$

Then $Y_n \rightarrow Z$ in distribution.

Pf: $\forall n$, choose $\hat{d}_n < \hat{u}_n$ s.t.

$\boxed{q_n := 1 - p_n}$

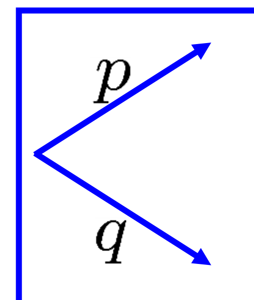
$$Y_n \in \Sigma^n \mathcal{B}^{p_n,\hat{u}_n}_{q_n,\hat{d}_n}$$

$u_n := \hat{u}_n\sqrt{n}$

$d_n := \hat{d}_n\sqrt{n}$

Def'n: $\forall p, q \in [0,1]$, s.t. $p + q = 1$

$$\boxed{\mathcal{B}^p_q} := \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d}$$



$p :\rightarrow p_n$

$q :\rightarrow q_n$

$$\Sigma^n \mathcal{B}^p_q = \Sigma^n \bigcup_{d<u} \mathcal{B}^{p,u}_{q,d} = \bigcup_{d<u} \Sigma^n \mathcal{B}^{p,u}_{q,d}$$

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in \underbrace{\left[\sum^n \mathcal{B}^{p_n}_{1-p_n}\right]}_{\bigcup_{d<u} \sum^n \mathcal{B}^{p_n,u}_{q_n,d}} \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $\forall n$, choose $\overbrace{\left[\hat{d}_n < \hat{u}_n\right]}^{\times \sqrt{n}}$ s.t. $\boxed{q_n := 1 - p_n}$

$[u_n := \hat{u}_n \sqrt{n}] \times [1/\sqrt{n}]$

$[d_n := \hat{d}_n \sqrt{n}] \times [1/\sqrt{n}]$

$$Y_n \in \sum^n \mathcal{B}^{p_n, \hat{u}_n}_{q_n, \hat{d}_n}$$

$u_n/\sqrt{n} = \hat{u}_n$ $\qquad u_n > d_n$

$$= \sum^n [\mathcal{B}^{p_n, u_n/\sqrt{n}}_{q_n, d_n/\sqrt{n}}]$$

$d_n/\sqrt{n} = \hat{d}_n$

$$= \sum^n [\mathcal{B}^{p_n, u_n}_{q_n, d_n} / \sqrt{n}]$$

$$= \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}$$

Next step:
Find u_n, d_n.

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in \left[\sum^n \mathcal{B}^{p_n}_{1-p_n}\right] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$ $\qquad$ $u q_n := 1 - p_n$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n} \quad \cap \quad \mathcal{S}$$

$$\left(\left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}\right) \cap \mathcal{S} \neq \emptyset \qquad u_n > d_n$$

$$\left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}$$

Next step:
Find u_n, d_n.

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1 - \varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$ $\qquad u_n > d_n$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n} \quad \cap \quad \mathcal{S}$$

$$\boxed{\left(\left[\boxed{\sum^n}\boxed{\mathcal{B}^{p_n, u_n}_{q_n, d_n}}\right] / \sqrt{n}\right)} \cap \mathcal{S} \neq \emptyset$$

identically distributed

$$\left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \boxed{\sqrt{n}} \quad \subseteq \quad \mathcal{S}$$

renormalized
i.i.d. sum preserves and
reflects standardness

$$\mathcal{B}^{p_n, u_n}_{q_n, d_n} \quad \subseteq \quad \mathcal{S}$$

Next step:
Find u_n, d_n.

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $\qquad q_n := 1 - p_n \qquad\qquad u_n > d_n$

$$Y_n Y \in \left[\in \sum^r \left[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] / \sqrt{n}\right.$$

$$\mathcal{B}^{p_n,u_n}_{q_n,d_n} \quad \subseteq \quad \mathcal{S}$$

$$\mathrm{E}\left[\mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] = 0$$

$$\mathrm{Var}\left[\mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] = 1 \qquad \mathcal{S}$$

q_n, u_n

Next step:
Find u_n, d_n.

Th'm (ΔCLT): (Triangular Central Limit Theorem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1 - \varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}$$

$$\mathcal{B}^{p_n, u_n}_{q_n, d_n} \subseteq \mathcal{S}$$

$$\mathrm{E}\left[\mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] = 0$$

$$\mathrm{Var}\left[\mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] = 1$$

$$\boxed{\begin{gathered} u_n > d_n \\ p_n u_n + q_n d_n = 0 \\ p_n u_n^2 + q_n d_n^2 = 1 \end{gathered}}$$

Solve for u_n, d_n.

exercise:

$$u_n = \sqrt{q_n / p_n}$$

$$d_n = -\sqrt{p_n / q_n}$$

Next step: Show u_n, d_n are bdd.

Th'm (ΔCLT): (Tri $q_1, q_2, q_3, \ldots \in (\varepsilon, 1-\varepsilon)$ orem)

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}$$

$$0 < u_n < \sqrt{(1-\varepsilon)/\varepsilon}$$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$-\sqrt{(1-\varepsilon)/\varepsilon} < d_n < 0$$

$$u_n > d_n$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$u_n = \sqrt{q_n / p_n} > 0$$

$$d_n = -\sqrt{p_n / q_n}$$

Next step: Show u_n Find $\mathcal{F}\delta[Y_n]$!.

Th'm (ΔCLT):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}$$

$$\mathcal{F}\delta\left[\mathcal{B}^{p_n, u_n}_{q_n, d_n}\right]$$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

Next step:
Find $\mathcal{F}\delta[Y_n]$.

Th'm (ΔCLT):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}_{1-p_n}^{p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$Y_n \in \left[\sum^n \mathcal{B}_{q_n, d_n}^{p_n, u_n}\right] / \sqrt{n}$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

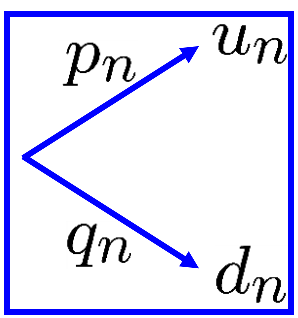
$$d_n,\ u_n \in (-M, M)$$

$$\mathcal{F}\delta\left[\mathcal{B}_{q_n, d_n}^{p_n, u_n}\right] = \phi(t)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

$$/\!/$$

$$[\ p_n z^{u_n} + q_n z^{d_n}\ ]_{z :\to e^{-it}}$$



z^{u_n}

z^{d_n}

Next step:
Find $\mathcal{F}\delta[Y_n]$.

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Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] / \sqrt{n}$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$\mathcal{F}\delta\left[\mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] = \phi(t)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

$$\mathcal{F}\delta\left[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] = [\phi(t)]^n$$

$$t :\to t/\sqrt{n}$$

$$\mathcal{F}\delta\left[\left[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] / \sqrt{n}\right] = [\phi(t/\sqrt{n})]^n$$

Next step:
Find $\mathcal{F}\delta[Y_n]$.

Th'm (ΔCLT):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$Y_n \in \boxed{\left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}}$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

$$\mathcal{F}\delta[Y_n] = [\phi(t/\sqrt{n})]^n$$

$$\mathcal{F}\delta\left[\boxed{\left[\sum^n \mathcal{B}^{p_n, u_n}_{q_n, d_n}\right] / \sqrt{n}}\right] = [\phi(t/\sqrt{n})]^n$$

Next step:
Find $\mathcal{F}\delta[Y_n]$.

Th'm (ΔCLT):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

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Then $Y_n \to Z$ in distribution.

Pf: $q_n := 1 - p_n$

$$M := \sqrt{(1-\varepsilon)/\varepsilon}$$

$$Y_n \in \left[\sum^n \mathcal{B}^{p_n,u_n}_{q_n,d_n}\right] / \sqrt{n}$$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

$$\mathcal{F}\delta[Y_n] = [\phi(t/\sqrt{n})]^n$$

INDETERMINATE FORMS PROBLEM

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

i.e., $\mathcal{F}\delta[Y_n] \to \mathcal{F}\delta[Z]$

Pf:

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$q_n := 1 - p_n$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Pf:

Want: $q_n := 1 - p_n$ $e^{-5^2/2}$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Pf:

$q_n := 1 - p_n$

$$p_n u_n + q_n d_n = 0$$

$$p_n u_n^2 + q_n d_n^2 = 1$$

$$d_n,\ u_n \in (-M, M)$$

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Want: $[\phi(5/\sqrt{n})]^n \to e^{-5^2/2}$

$\alpha_n \to 0$

2nd order Maclaurin approx.

$-5iu_n/\sqrt{n} \to 0$

$$\phi(\boxed{5/\sqrt{n}}) = p_n(e^{\boxed{-5iu_n/\sqrt{n}}}) + q_n(e^{-5id_n/\sqrt{n}})$$

$$p_n \left(\boxed{1 - \frac{5iu_n}{\sqrt{n}} - \frac{5^2 u_n^2}{2n} + \frac{\alpha_n u_n^2}{n}} \right)$$

$x_n :\to -5iu_n/\sqrt{n}$

$$\boxed{x_n \to 0 \quad \Rightarrow \quad e^{x_n} = 1 + x_n + \frac{x_n^2}{2} + o(x_n^2)}$$

Pf:

$q_n := 1 - p_n$

$$p_n u_n + q_n d_n = 0$$
$$p_n u_n^2 + q_n d_n^2 = 1$$
$$d_n,\ u_n \in (-M, M)$$

$$\phi(t) := p_n e^{-iu_nt} + q_n e^{-id_nt}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Want: $[\phi(5/\sqrt{n})]^n \to e^{-5^2/2}$

$\alpha_n \to 0$
2nd order Maclaurin approx.

$-5id_n/\sqrt{n} \to 0$

$$\phi(5/\sqrt{n}) = p_n(e^{-5iu_n/\sqrt{n}}) + q_n(e^{\boxed{-5id_n/\sqrt{n}}})$$

$$= \quad p_n\left(1 - \frac{5iu_n}{\sqrt{n}} - \frac{5^2u_n^2}{2n} + \frac{\alpha_n u_n^2}{n}\right)$$
$$+q_n\left(\boxed{1 - \frac{5id_n}{\sqrt{n}} - \frac{5^2d_n^2}{2n} + \frac{\beta_n d_n^2}{n}}\right)$$

$\beta_n \to 0$
2nd order Maclaurin approx.

$x_n :\to -5id_n/\sqrt{n}$

$$\boxed{x_n \to 0 \quad \Rightarrow \quad e^{x_n} = 1 + x_n + \frac{x_n^2}{2} + o(x_n^2)}$$

$\delta_n := \boxed{\alpha_n p_n u_n^2} + \boxed{\beta_n q_n d_n^2} \to 0$

$\boxed{\alpha_n p_n u_n^2} \to 0$, $\boxed{\beta_n q_n d_n^2} \to 0$

Pf:

$$\boxed{p_n u_n + q_n d_n = 0}$$

$$\boxed{p_n u_n^2 + q_n d_n^2 = 1} \Rightarrow \underline{p_n u_n^2 \le 1}, \underline{q_n d_n^2 \le 1}$$

$q_n := 1 - p_n$

$$d_n,\ u_n \in (-M, M)$$

$\boxed{\alpha_n \to 0}$ 2nd order Maclaurin approx.

$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Want: $[\phi(5/\sqrt{n})]^n \to e^{-5^2/2}$

$$\phi(5/\sqrt{n}) = p_n(e^{-5iu_n/\sqrt{n}}) + q_n(e^{-5id_n/\sqrt{n}})$$

$\boxed{\beta_n \to 0}$ 2nd order Maclaurin approx.

$$= \boxed{p_n}\left(\boxed{1} - \cancel{\frac{5i\boxed{u_n}}{\sqrt{n}}} - \frac{5^2\boxed{u_n^2}}{2n} + \frac{\boxed{\alpha_n u_n^2}}{n}\right)$$

$$+\boxed{q_n}\left(\boxed{1} - \cancel{\frac{5i\boxed{d_n}}{\sqrt{n}}} - \frac{5^2\boxed{d_n^2}}{2n} + \frac{\boxed{\beta_n d_n^2}}{n}\right)$$

$p_n + q_n = 1$

$$= 1 - \frac{5^2}{2n} + \frac{\delta_n}{n}$$

δ_n $\rightarrow 0$

Pf:

$$p_n u_n + q_n d_n = 0$$
$$p_n u_n^2 + q_n d_n^2 = 1$$
$$d_n,\, u_n \in (-M, M)$$
$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \rightarrow e^{-t^2/2}$

Want: $\underline{[\phi(5/\sqrt{n})]^n} \rightarrow e^{-5^2/2}$

$$\boxed{\phi(5/\sqrt{n})} = 1 - \frac{5^2}{2n} + \frac{\delta_n}{n}$$

RAISE TO n^{th} POWER

$\delta_n \rightarrow 0$

$$1 \qquad - \frac{5^2}{2n} \quad + \frac{\delta_n}{n}$$

Pf: $$p_n u_n + q_n d_n = 0$$
$$p_n u_n^2 + q_n d_n^2 = 1$$
$$d_n,\ u_n \in (-M, M)$$
$$\phi(t) := p_n e^{-iu_n t} + q_n e^{-id_n t}$$

Want: $[\phi(t/\sqrt{n})]^n \to e^{-t^2/2}$

Want: $[\phi(5/\sqrt{n})]^n \to e^{-5^2/2}$

$$\phi(5/\sqrt{n}) = 1 - \frac{5^2}{2n} + \frac{\delta_n}{n}$$

RAISE TO nth POWER

$\delta_n \to 0$

$$[\phi(5/\sqrt{n})]^n = \left[1 - \frac{5^2}{2n} + \frac{\delta_n}{n}\right]^n \to e^{-5^2/2}$$

QED

$x :\to -5^2/2$

Fact: $\forall x \in \mathbb{R}, \quad \forall \delta_n \to 0, \quad \left[1 + \frac{x}{n} + \frac{\delta_n}{n}\right]^n \to e^x$

The exponential limit

Th'm (ΔCLT):

$\boxed{\text{Let } \varepsilon > 0 \text{ and let } p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon).}$

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Th'm (ΔCLT2):

What if the sequence is not standard?

$\boxed{\text{Say } p_1, p_2, p_3, \ldots \to p \in (0,1).}$

For all integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

against contin, exp-bdd

pf omitted

Then $Y_n \to Z$ in distribution.

Pf: Choose $\varepsilon > 0$ s.t. $p \in (\varepsilon, 1-\varepsilon)$.

Choose an integer $N \geq 1$ s.t.

$$p_N, p_{N+1}, p_{N+2}, \ldots \in (\varepsilon, 1-\varepsilon).$$

Then $Y_N, Y_{N+1}, Y_{N+2}, \ldots \to Z$ in distr.

Then $Y_1, Y_2, Y_3, \ldots \to Z$ in distr.,

i.e., $Y_n \to Z$ in distr. QED

Def'n: $X_n \to \sigma Z + \mu$ **in distribution means**

$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution.}$$

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What if the sequence is not standard?

Say $p_1, p_2, p_3, \ldots \to p \in (0,1)$.

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Def'n: $X_n \to \sigma Z + \mu$ **in distribution** **means**

$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution.}$$

I.e., $\forall$continuous, bounded ϕ,

$$\mathrm{E}\left[\phi\left(\frac{X_n - \mu}{\sigma}\right)\right] \to \mathrm{E}[\phi(\boxed{Z})],$$ Z not yet def'd, so. . .

Def'n: $X_n \to \sigma Z + \mu$ **in distribution** **means**

$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution.}$$

I.e., $\forall$continuous, bounded ϕ,

$$\mathrm{E}\left[\phi\left(\frac{X_n - \mu}{\sigma}\right)\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\, dx,$$

Def'n: $X_n \to \sigma Z + \mu$ **in distribution** **means**

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I.e., $\forall$continuous, bounded ϕ,

$$\mathrm{E}\left[\boxed{\phi\left(\frac{X_n-\mu}{\sigma}\right)}\right] \xrightarrow{\text{KNOW}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\boxed{\phi(x)}][e^{-x^2/2}]\, dx,$$

$$\Updownarrow$$

i.e., $\forall$continuous, bounded ψ,

$$\mathrm{E}[\boxed{\psi(X_n)}] \xrightarrow{\text{WANT}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\boxed{\psi(\sigma x + \mu)}][e^{-x^2/2}]\, dx.$$

☺

Pf of $\Downarrow$: Given contin., bdd ψ.

Exercise: Prove $\Uparrow$

$$\phi(x) := \psi(\sigma x + \mu)$$

$$\phi\left(\boxed{\frac{X_n-\mu}{\sigma}}\right) = \psi\left(\cancel{\sigma}\left(\boxed{\frac{X_n-\cancel{\mu}}{\cancel{\sigma}}}\right)+\cancel{\mu}\right) = \psi(X_n)$$

QED

Def'n: $X_n \to \sigma Z + \mu$ **in distribution means**

$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution.}$$

I.e., $\forall$continuous, bounded ϕ,

$$\mathsf{E}\left[\phi\left(\frac{X_n - \mu}{\sigma}\right)\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\phi(x)][e^{-x^2/2}]\, dx,$$

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$$\mathsf{E}[\psi(X_n)] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\psi(\sigma x + \mu)][e^{-x^2/2}]\, dx.$$

Th'm (ΔCLT3):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1 - \varepsilon)$.

For all integers $n \geq 1$, let $X_n \in \sum^n \mathcal{B}^{p_n}_{1-p_n}$.

NOT necessarily in $\mathcal{S}$

Def'n: $X_n \to \sigma Z + \mu$ **in distribution means**

$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution.}$$

I.e., $\forall$continuous, bounded ϕ,

$$\mathrm{E}\left[\phi\left(\frac{X_n - \mu}{\sigma}\right)\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\phi(x)][e^{-x^2/2}]\,dx,$$

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For all integers $n \geq 1$, let $X_n \in \sum^n \mathcal{B}^{p_n}_{1-p_n}$.
Assume $\mathrm{E}[X_n] \to \mu$ and $\mathrm{SD}[X_n] \to \sigma$.
Then $X_n \to \sigma Z + \mu$ in distribution.

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Then $X_n \to \sigma Z + \mu$ in distribution.

Pf: $\mu_n := \mathrm{E}[X_n]$

$\sigma_n := \mathrm{SD}[X_n]$

$$Y_n := \frac{X_n - \mu_n}{\sigma_n} \in \left[\sum^n \mathcal{B}^{p_n}_{1-p_n}\right] \cap \mathcal{S}$$

Uptick/downtick *VALUES* change,

but uptick/downtick *PROBABILITIES* do not.

First step:

renormalize X_n

Th'm (ΔCLT3):

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Assume E$[X_n] \to \mu$ and SD$[X_n] \to \sigma$.

Then $X_n \to \sigma Z + \mu$ in distribution.

Pf: $\mu_n :=$ E$[X_n]$

$\sigma_n :=$ SD$[X_n]$

$$Y_n := \frac{X_n - \mu_n}{\sigma_n} \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$$

$$\frac{X_n - \mu_n}{\sigma_n} = Y_n \to Z \text{ in distribution}$$

Th'm (ΔCLT):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

$\forall$integers $n \geq 1$, let $Y_n \in [\sum^n \mathcal{B}^{p_n}_{1-p_n}] \cap \mathcal{S}$.

Then $Y_n \to Z$ in distribution.

Th'm (ΔCLT3):

Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1 - \varepsilon)$.

For all integers $n \geq 1$, let $X_n \in \sum^n \mathcal{B}^{p_n}_{1-p_n}$.

Assume $\mathrm{E}[X_n] \to \mu$ and $\mathrm{SD}[X_n] \to \sigma$.

Then $X_n \to \sigma Z + \mu$ in distribution.

Pf: $\mu_n := \mathrm{E}[X_n]$

$\sigma_n := \mathrm{SD}[X_n] \to \sigma$

$$\frac{X_n - \mu_n}{\sigma_n}\frac{\imath - \mu_n}{\sigma_n} \to Z \text{ in distribution}$$

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Pf: $\mu_n := \mathrm{E}[X_n]$

$\sigma_n := \mathrm{SD}[X_n] \to \sigma$

$$\underbrace{\frac{\cancel{\sigma_n}}{\sigma}}_{\to 1} \times \frac{X_n - \mu_n}{\cancel{\sigma_n}} \to Z \text{ in distribution}$$

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Pf: $\mu_n := \mathrm{E}[X_n] \to \mu$

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Pf: $\mu_n := \mathrm{E}[X_n] \to \mu$

$\sigma_n := \mathrm{SD}[X_n] \to \sigma$

$$\underbrace{\frac{\cancel{\mu_n} - \mu}{\sigma}}_{\to 0} + \frac{X_n - \cancel{\mu_n}}{\sigma} \to Z \text{ in distribution}$$

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Let $\varepsilon > 0$ and let $p_1, p_2, p_3, \ldots \in (\varepsilon, 1-\varepsilon)$.

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$\sigma_n := \mathrm{SD}[X_n] \to \sigma$

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Def'n: $X_n \to \sigma Z + \mu$ **in distribution means**

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$$\frac{X_n - \mu}{\sigma} \to Z \text{ in distribution}$$

$$X_n \to \sigma Z + \mu \text{ in distribution} \quad \text{QED}$$

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