$\begin{array}{c} {\rm FM~5011~Fall~2011,~Final~Exam} \\ {\rm Handout~date:~Thursday~15~December~2011} \end{array}$

PRINT NAME:

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

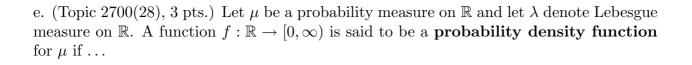
REMEMBER TO SIGN YOUR NAME:

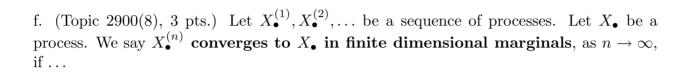
- I. Definitions: Complete the following sentences.
- a. (Topic 1500(28), 3 pts.) Let M be a set. A subset \mathcal{S} of 2^M is called a σ -algebra on M if . . .

b. (Topic 3000(37), 3 pts.) Let U be a random variable on the probability space $(\Omega, \mathcal{B}, \mu)$ and let \mathcal{S} be a σ -subalgebra of \mathcal{B} . A random variable X represents $E[U|\mathcal{S}]$ if ...

c. (Topic 2900(20), 3 pts.) Let $X: \Omega \to \mathbb{R}$ be a random variable on $(\Omega, \mathcal{B}, \mu)$. The σ -algebra of X is $\mathcal{S}_X = \cdots$

d. (Topic 2330(23), 3 pts.) Let μ and ν be probability measures on a Borel space (Ω, \mathcal{B}) . We say that μ is **absolutely continuous** with respect to μ , and write $\mu << \nu$, if . . .





g. (Topic 2900(23), 3 pts.) Two random variables X and Y are said to be **independent** if . . .

h. (Topic 2900(35), 3 pts.) Let μ and ν be two probability measures on \mathbb{R} . Define $A: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by A(x,y) = x+y. Then the **convolution** of μ and ν is given by $\mu * \nu = \cdots$

- II. True or False. (No partial credit.)
- a. (Topic 2700(47), 2 pts.) Let X and Y be random variables. If their distributions have the same Fourier transforms, then X = Y a.s.
- b. (Topic 2900(9), 2 pts.) If X and Y are identically distributed random variables, then X^2 and Y^2 are also identically distributed.
- c. (Topic 3600(21), 2 pts.) If V_{\bullet} and W_{\bullet} are Brownian motions, then V=W in finite dimensional marginals.
- d. (Topic 3000(37), 2 pts.) Let X and Y be random variables. Assume that X and Y are independent. Then $\mathrm{E}[X|Y]=X$.
- e. (Topic 2900(22), 2 pts.) If X is a random variable and $S = S_X$ is its σ -algebra, then X is S-measurable.
- f. (Topic 2900(24), 2 pts.) If X and Y are random variables, then $\delta_{X,Y} = \delta_X \times \delta_Y$.
- g. (Topic 2900(44), 2 pts.) Let G be the grade of a standard normal random variable and let $0 \le a < b \le 1$. Then $\Pr[a < G < b] = b a$.
- h. (Topic 2400(7-8), 2 pts.) For any random variable X, there exists $a \in \mathbb{R}$, such that $\Pr[X < a] = 0$.

THIS PAGE IS FOR TOTALING SCORES PLEASE DO NOT WRITE ON THIS PAGE

| I. a-d. | | |
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| I. e-h. | | |
| II. a-d. | | |
| II. e-h. | | |
| III(1). | | |
| III(2). | | |
| III(3,4). | | |
| III(5). | | |
| III(6). | | |
| III(7). | | |
| III(8). | | |
| III(9). | | |
| III(10). | | |

- III. Computations. Some of your answers may involve Φ , the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)
- 1. (Topic 3200(14-23), 20 pts.) Let X be a random variable whose distribution is χ^2 with two degrees of freedom. Compute $\mathrm{E}[X^2]$.

2. (Topic 2900(47), 20 pts.) Let X_1, X_2, \ldots be iid standard random variables. For all integers $n \geq 1$, let $Z_n := (X_1 + \cdots + X_n)/\sqrt{n}$. Compute $\lim_{n \to \infty} \mathbb{E}[(e^{Z_n} - e)_+]$

3. (Topic 2900(13), 20 pts.) Let Z be a standard normal random variable. Let $\mu:=\delta[Z^2]$ be the distribution of Z^2 . Compute $\int_{-\infty}^{\infty} x^5 \, d\mu(x)$.

4. (Topic 2800(13), 20 pts.) Let $g(x) = x^5$. Let $v(x) = \begin{cases} x^2 + 2, & \text{if } x < 1; \\ x^4, & \text{if } x \ge 1. \end{cases}$ Compute $\int_0^2 g(x) \, dv(x)$. 5. (Topic 3200(2), 15 pts.) Assume that the distribution $\delta[X]$ of the random variable X has probability density function given by $p(x) = \frac{1}{\pi(1+x^2)}$. Let $Y := e^X$. Compute a probability density function f for $\delta[Y]$. Express f(x) explicitly.

6. (Topic 3000(52), 15 pts.) Let C_1, C_2, C_3, \ldots be iid binary random variables such that, for all integers $j \geq 0$, we have $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. Let

$$X := E[(C_1 + \dots + C_{100}) | (C_1 + \dots + C_{50})].$$

Compute $E[X^2]$.

7. (Topic 3400(20), 15 pts.) Let X_1,\ldots,X_{100} be iid normal variables with unknown mean μ and known variance 0.49. Let x_1,\ldots,x_{100} be a sample modeled on X_1,\ldots,X_{100} . Assume that the sample mean $(x_1+\cdots+x_{100})/(100)=5$. Find a 99% confidence interval for μ . (Note: For a standard normal random variable Z, we have $\Pr[|Z|<2.58]=0.99$.)

8. (Topic 3600(21), 15 pts.) Let W_{\bullet} be a Brownian motion. Compute $\mathrm{E}[(W_4)^2(W_{13})^2]$.

9. (Topic 3800(47), 10 pts.) Let W_t be a Brownian motion. Let X_t satisfy

$$dX_t/X_t = 2 dW_t - 3 dt, \qquad X_0 = 1.$$

Compute $E[(X_4)^3]$.

10. (Topic 0026, 10 pts.) Let X_{\bullet} satisfy $dX_t = 4t \, dW_t + \frac{dt}{t^4 + 1}$. Let Y_{\bullet} be defined by $Y_t = t^3 e^{X_t}$. Compute $\mathrm{E}\left[\int_0^4 \frac{dY_t}{e^{X_t}}\right]$.