FM 5011 Fall 2011, Final Exam
Handout date: Thursday 15 December 2011

PRINT NAME: SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. (Topic 1500(28), 3 pts.) Let M be a set. A subset $\mathcal{S}$ of 2^M is called a **σ-algebra** on M if ...

$$[\forall S_1, S_2, \ldots \in \mathcal{S}, \quad S_1 \cup S_2 \cup \cdots \in \mathcal{S}]$$

$$\& \; [\quad \forall S \in \mathcal{S}, \quad M \backslash S \in \mathcal{S}\;]$$

b. (Topic 3000(37), 3 pts.) Let U be a random variable on the probability space $(\Omega, \mathcal{B}, \mu)$ and let $\mathcal{S}$ be a σ-subalgebra of $\mathcal{B}$. A random variable X **represents** $\mathrm{E}[U|\mathcal{S}]$ if ...

[X is $\mathcal{S}$-measurable]

& [$\forall S \in \mathcal{S}$ of positive measure

$E[X|S] = E[U|S]$]

c. (Topic 2900(20), 3 pts.) Let $X : \Omega \to \mathbb{R}$ be a random variable on $(\Omega, \mathcal{B}, \mu)$. The **$\sigma$-algebra** of X is $\mathcal{S}_X = \cdots$

$X^*(\mathcal{B}_{\mathbb{R}})$, where $\mathcal{B}_{\mathbb{R}}$ is the standard σ-algebra on $\mathbb{R}$

d. (Topic 2330(23), 3 pts.) Let μ and ν be probability measures on a Borel space $(\Omega, \mathcal{B})$. We say that μ is **absolutely continuous** with respect to μ, and write $\mu << \nu$, if ...

$$\forall B \in \mathcal{B}, \quad (\nu(B) = 0) \Rightarrow (\mu(B) = 0)$$

e. (Topic 2700(28), 3 pts.) Let μ be a probability measure on $\mathbb{R}$ and let λ denote Lebesgue measure on $\mathbb{R}$. A function $f : \mathbb{R} \to [0, \infty)$ is said to be a **probability density function** for μ if ...

$$\mu = f\lambda$$

f. (Topic 2900(8), 3 pts.) Let $X_\bullet^{(1)}, X_\bullet^{(2)}, \ldots$ be a sequence of processes. Let $X_\bullet$ be a process. We say $X_\bullet^{(n)}$ **converges to** $X_\bullet$ **in finite dimensional marginals**, as $n \to \infty$, if ...

$\forall$ integers $d \geqslant 1$, $\forall t_1, \ldots, t_d \geqslant 0$,

$$\delta\left[X_{t_1}^{(k)}, \ldots, X_{t_d}^{(k)}\right] \xrightarrow[k\to\infty]{} \delta\left[X_{t_1}^{(k)}, \ldots, X_{t_d}^{(k)}\right]$$

g. (Topic 2900(23), 3 pts.) Two random variables X and Y are said to be **independent** if ...

$\forall$ Borel $A, B \subseteq \mathbb{R}$,

$(X \in A)$ is independent of $(Y \in B)$

h. (Topic 2900(35), 3 pts.) Let μ and ν be two probability measures on $\mathbb{R}$. Define $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $A(x, y) = x + y$. Then the **convolution** of μ and ν is given by $\mu * \nu = \cdots$

$$A_*(\mu \times \nu)$$

II. True or False. (No partial credit.)

a. (Topic 2700(47), 2 pts.) Let X and Y be random variables. If their distributions have the same Fourier transforms, then $X = Y$ a.s.

F

b. (Topic 2900(9), 2 pts.) If X and Y are identically distributed random variables, then X^2 and Y^2 are also identically distributed.

T

c. (Topic 3600(21), 2 pts.) If $V_\bullet$ and $W_\bullet$ are Brownian motions, then $V = W$ in finite dimensional marginals.

T

d. (Topic 3000(37), 2 pts.) Let X and Y be random variables. Assume that X and Y are independent. Then $\mathrm{E}[X|Y] = X$.

F

e. (Topic 2900(22), 2 pts.) If X is a random variable and $\mathcal{S} = \mathcal{S}_X$ is its σ-algebra, then X is $\mathcal{S}$-measurable.

T

f. (Topic 2900(24), 2 pts.) If X and Y are random variables, then $\delta_{X,Y} = \delta_X \times \delta_Y$.

F

g. (Topic 2900(44), 2 pts.) Let G be the grade of a standard normal random variable and let $0 \leq a < b \leq 1$. Then $\Pr[a < G < b] = b - a$.

T

h. (Topic 2400(7-8), 2 pts.) For any random variable X, there exists $a \in \mathbb{R}$, such that $\Pr[X < a] = 0$.

F

THIS PAGE IS FOR TOTALING SCORES
PLEASE DO NOT WRITE ON THIS PAGE

I. a-d.

I. e-h.

II. a-d.

II. e-h.

III(1).

III(2).

III(3,4).

III(5).

III(6).

III(7).

III(8).

III(9).

III(10).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. (Topic 3200(14-23), 20 pts.) Let X be a random variable whose distribution is χ^2 with two degrees of freedom. Compute $\mathrm{E}[X^2]$.

Let Z_1, Z_2 be indep. std normal RVs

$$X \overset{\delta}{=} Z_1^2 + Z_2^2$$

$$X^2 \overset{\delta}{=} (Z_1^2 + Z_2^2)^2$$

$$= Z_1^4 + 2Z_1^2 Z_2^2 + Z_2^4$$

Z_1^2, Z_2^2 indep & $\therefore$ uncorrelated

$$E[X^2] = (E[Z_1^4]) + 2(E[Z_1^2])(E[Z_2^2]) + (E[Z_2^4])$$

$$= (3 \cdot 1) + 2(1)(1) + (3 \cdot 1)$$

$$= 3 + 2 + 3 = 8$$

2. (Topic 2900(47), 20 pts.) Let $X_1, X_2, \ldots$ be iid standard random variables. For all integers $n \geq 1$, let $Z_n := (X_1 + \cdots + X_n)/\sqrt{n}$. Compute $\lim_{n\to\infty} \mathrm{E}[(e^{Z_n} - e)_+]$

Z std normal RV

$$= E[(e^{Z} - e)_+]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{x} - e)_+ \, e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} (e^{x} - e) \, e^{-x^2/2} \, dx$$

$$= e^{1/2}\left[\frac{1}{\sqrt{2\pi}} \int_{\cancel{1}\,0}^{\infty} \cancel{e^{x}}\, e^{-x^2/2} \, dx\right] - e\left[\frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} \cancel{e}\cdot e^{-x^2/2} \, dx\right]$$

$$= e^{1/2}[\Phi(0)] - e[\Phi(-1)]$$

3. (Topic 2900(13), 20 pts.) Let Z be a standard normal random variable. Let $\mu := \delta[Z^2]$ be the distribution of Z^2. Compute $\displaystyle\int_{-\infty}^{\infty} x^5 \, d\mu(x)$.

distribution drives expectation

$$= E[(Z^2)^5] = E[Z^{10}] = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$$

4. (Topic 2800(13), 20 pts.) Let $g(x) = x^5$. Let $v(x) = \begin{cases} x^2 + 2, & \text{if } x < 1; \\ x^4, & \text{if } x \geq 1. \end{cases}$

Compute $\displaystyle\int_0^2 g(x) \, dv(x)$.

$$= \left[\int_0^1 x^5 (2x) \, dx\right] + \left[\int_1^1 (x^5)(-2) \, d\delta_1(x)\right] + \left[\int_1^2 x^5 (4x^3) \, dx\right]$$

$$= 2\left[\int_0^1 x^6 \, dx\right] - 2\left[\int_1^1 x^5 \, d\delta_1(x)\right] + 4\left[\int_1^2 x^8 \, dx\right]$$

$$= 2\left[\frac{x^7}{7}\right]_{x:\to 0}^{x:\to 1} - 2\left[x^5\right]_{x:\to 1} + 4\left[\frac{x^9}{9}\right]_{x:\to 1}^{x:\to 2}$$

$$= 2\left[\frac{1}{7}\right] - 2\left[1\right] + 4\left[\frac{2^9 - 1}{9}\right]$$

5. (Topic 3200(2), 15 pts.) Assume that the distribution $\delta[X]$ of the random variable X has probability density function given by $p(x) = \dfrac{1}{\pi(1+x^2)}$. Let $Y := e^X$. Compute a probability density function f for $\delta[Y]$. Express $f(x)$ explicitly.

$P := CDF_{\delta[X]}$ $\quad\quad$ $F := CDF_{\delta[Y]}$

$P' = p$ $\quad\quad$ $F' = f$

$$\begin{aligned} F(x) &= Pr\,[Y \leq x] \\ &= Pr\,[e^X \leq x] \\ &= Pr\,[X \leq \ln x] \\ &= P(\ln x) \end{aligned}$$

$$\begin{aligned} f(x) = F'(x) &= [P'(\ln x)]\left[\frac{1}{x}\right] \\ &= [p(\ln x)]\left[\frac{1}{x}\right] \\ &= \left[\frac{1}{\pi(1+(\ln x))^2}\right]\left[\frac{1}{x}\right] \end{aligned}$$

6. (Topic 3000(52), 15 pts.) Let $C_1, C_2, C_3, \ldots$ be iid binary random variables such that, for all integers $j \geq 0$, we have $\Pr[C_j = 1] = 0.5 = \Pr[C_j = -1]$. Let

$$X \;:=\; \mathrm{E}[\,(C_1 + \cdots + C_{100}) \mid (C_1 + \cdots + C_{50})\,]\,.$$

Compute $\mathrm{E}[X^2]$.

$$X = E[(C_1 + \cdots + C_{50}) \mid (C_1 + \cdots + C_{50})]$$

$$+ E[(C_{51} + \cdots + C_{100}) \mid (C_1 + \cdots + C_{50})]$$

$$= C_1 + \cdots + C_{50}$$

$$+ E[C_1 + \cdots + C_{50}]$$

$$= C_1 + \cdots + C_{50}$$

$$X^2 = \sum_{j=1}^{50} \sum_{k=1}^{50} C_j C_k$$

$$E[X^2] = \sum_{j=1}^{50} \sum_{k=1}^{50} \delta_j^k = \sum_{j=1}^{50} 1 = 50$$

7. (Topic 3400(20), 15 pts.) Let $X_1, \ldots, X_{100}$ be iid normal variables with unknown mean μ and known variance 0.49. Let $x_1, \ldots, x_{100}$ be a sample modeled on $X_1, \ldots, X_{100}$. Assume that the sample mean $(x_1 + \cdots + x_{100})/(100) = 5$. Find a 99% confidence interval for μ. (*Note*: For a standard normal random variable Z, we have $\Pr[|Z| < 2.58] = 0.99$.)

$$\bar{X} := \frac{1}{100}\left[X_1 + \cdots + X_{100}\right] \qquad \bar{x} := \frac{1}{100}\left[x_1 + \cdots + x_{100}\right] = 5$$

$$\sigma := SD[X_1] = \cdots = SD[X_{100}] = \sqrt{0.49} = 0.7$$

$$E[\bar{X}] = \frac{1}{100}\left[\sum_{j=1}^{100} E[X_j]\right] = \frac{1}{100}[100\mu] = \mu$$

$$Var[\bar{X}] = \frac{1}{100^2}\left[\sum_{j=1}^{100} Var[X_j]\right] = \frac{1}{100^2}[100\sigma^2] = \frac{\sigma^2}{100}$$

$\dfrac{\bar{X} - \mu}{\sigma/10}$ is std normal

$$\Pr\left[\left|\frac{\bar{X} - \mu}{\sigma/10}\right| < 2.58\right] = 0.99$$

With 99% confidence, $\left|\dfrac{\bar{x} - \mu}{\sigma/10}\right| < 2.58$,

i.e. $|\mu - \bar{x}| < (2.58)\left(\dfrac{\sigma}{10}\right)$

i.e. $|\mu - 5| < (2.58)(0.07)$

99% confidence interval:

$$\left(5 - (2.58)(0.07),\ 5 + (2.58)(0.07)\right)$$

8. (Topic 3600(21), 15 pts.) Let $W_\bullet$ be a Brownian motion. Compute $\mathrm{E}[(W_4)^2(W_{13})^2]$.

$$Z_1 := \frac{W_4}{2}$$

$$2Z_1 = W_4$$

$$Z_2 := \frac{W_{13} - W_4}{3}$$

$$3Z_2 = W_{13} - W_4$$

$$2Z_1 + 3Z_2 = W_{13}$$

$$= E[(2Z_1)^2(2Z_1 + 3Z_2)^2]$$

$$= E[4Z_1^2(4Z_1^2 + 12Z_1Z_2 + 9Z_2^2)]$$

$$= E[16Z_1^4 + 48Z_1^3Z_2 + 36Z_1^2Z_2^2]$$

Z_1, Z_2 indep std normal

$$= 16(E[Z_1^4]) + 48(E[Z_1^3])(E[Z_2]) + 36(E[Z_1^2])(E[Z_2^2])$$

$$= 16(1\cdot 3) + 48(0)(0) + 36(1)(1)$$

$$= 48 + 0 + 36$$

$$= 84$$

9. (Topic 3800(47), 10 pts.) Let W_t be a Brownian motion. Let X_t satisfy

$$dX_t/X_t = 2\,dW_t - 3\,dt, \qquad X_0 = 1.$$

Compute $E[(X_4)^3]$.

Log-Itô

$$d(\ln X_t) \stackrel{\vee}{=} (2dW_t - 3dt) - \frac{1}{2}(2dW_t - 3dt)^2$$

$$= 2dW_t - 3dt - \frac{1}{2}(4dt)$$

$$= 2dW_t - 5dt$$

$$\ln X_4 = (\ln X_0) + \int_0^4 (2dW_t - 5dt)$$

$$= (\ln 1) + \Big[2W_t - 5t\Big]_{t:\to 0}^{t:\to 4}$$

$$= (0) + [(2W_4 - 5\cdot 4) - 0]$$

$$= 2W_4 - 20$$

Let Z be a std normal RV

$$\stackrel{\delta}{=} 4Z - 20$$

$$X_4 \stackrel{\delta}{=} e^{4Z-20}$$

$$(X_4)^3 \stackrel{\delta}{=} e^{12Z-60}$$

$$E[(X_4)^3] = E[e^{12Z-60}] = (e^{-60})(E[e^{12Z}])$$

$$= (e^{-60})(e^{12^2/2})$$

10. (Topic 0026, 10 pts.) Let $X_\bullet$ satisfy $dX_t = 4t\,dW_t + \dfrac{dt}{t^4+1}$.

Let $Y_\bullet$ be defined by $Y_t = t^3 e^{X_t}$. Compute $\mathrm{E}\left[\int_0^4 \frac{dY_t}{e^{X_t}}\right]$.

$$dY_t = 3t^2 e^{X_t}\,dt + t^3\left(e^{X_t}\,dX_t + \tfrac{1}{2}e^{X_t}(dX_t)^2\right)$$

$$\frac{dY_t}{e^{X_t}} = 3t^2\,dt + t^3\left(\left(4t\,dW_t + \frac{dt}{t^4+1}\right) + \tfrac{1}{2}\left(16t^2\,dt\right)\right)$$

$$= 4t^4\,dW_t + \left(3t^2 + \frac{t^3}{t^4+1} + 8t^5\right)dt$$

$$\mathrm{E}\left[\int_0^4 \frac{dY_t}{e^{X_t}}\right] = \int_0^4 \left(3t^2 + \frac{t^3}{t^4+1} + 8t^5\right)dt$$

$$= \left[t^3 + \tfrac{1}{4}\left(\ln\left(t^4+1\right)\right) + 8\cdot\frac{t^6}{6}\right]_{t:\to 0}^{t:\to 4}$$

$$= 4^3 + \tfrac{1}{4}\left(\ln\left(4^4+1\right)\right) + 8\cdot\frac{4^6}{6}$$