FM 5011 Fall 2011, Midterm #1
Handout date: Thursday 20 October 2011
Time for exam: ONE HOUR

PRINT NAME: SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. [Topic 1200(25), 3 pts.] Let X_n be a sequence of PCRVs. We say $X_n \to \sigma Z + \mu$ in distribution if ...

$$\frac{X_n - \mu}{\sigma} \longrightarrow Z \text{ in distribution}$$

b. [Topic 0100(21), 3 pts.] Let X and Y be non-determinstic PCRVs. The **covariance** of X and Y is defined by $\mathrm{Cov}[X, Y] = \cdots$.

$$\frac{(\mathrm{Var}[X+Y]) - (\mathrm{Var}[X]) - (\mathrm{Var}[Y])}{2}$$

c. [Topic 2330(24), 3 pts.] Two measures μ and ν on a Borel space $(M, \mathcal{B})$ are **equivalent**, written $\mu \approx \nu$, if ...

$\mu \ll \nu$ and $\nu \ll \mu$

d. [Topic 1700(12), 3 pts.] Let $\mathcal{B}$ and $\mathcal{C}$ be σ-algebras on sets M and N, respectively. The **product σ-algebra** on $M \times N$ is defined by $\mathcal{B} \times \mathcal{C} = \cdots$

$$\left\langle \{ B \times C \mid B \in \mathcal{B}\ C \in \mathcal{C} \} \right\rangle_\sigma$$

e. [Topic 2300(6), 3 pts.] Let $(M, \mathcal{B}, \mu)$ be a measure space. A function $s : M \to \mathbb{R}$ is **simple** if ...

(s is measurable)

and ($s(M)$ is finite)

II. True or False. (No partial credit.)

a. [Topic 2300(9), 3 pts.] Let $(M, \mathcal{B}, \mu)$ be a measure space and let $f : M \to [0, \infty]$ be a measurable function. Then $\displaystyle\int_M f\, d\mu$ exists (although it may be equal to ∞).

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b. [Topic 2330(24), 3 pts.] Let $(M, \mathcal{B}, \mu)$ be a measure space and let $f : M \to [0, \infty]$ be a measurable function. Then μ and $f\mu$ are equivalent measures.

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c. [Topic 1100(12), 3 pts.] Let X and Y be independent, identically distributed PCRVs. If $f(t)$ is the Fourier transform of the distribution of X, then the Fourier transform of the distribution of $5(X + Y)$ is $[f(5t)]^2$.

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d. [Topic 0200(30), 3 pts.] Let $(M, \mathcal{B})$, $(N, \mathcal{C})$ and $(P, \mathcal{D})$ be Borel spaces. If $f : M \to N$ and $g : N \to P$ are both Borel maps, then $g \circ f : M \to P$ is also Borel.

T

e. [Topic 1100(18), 3 pts.] If $X_1, X_2, X_3, \ldots$ is a sequence of PCRVs, and if the Fourier transforms of their distributions converge to $e^{-t^2/2}$, then $X_1, X_2, X_3, \ldots \to Z$ in distribution.

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PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1).

III(2ab).

III(3).

III(4,5).

III(6).

III(7).

III(8).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. [Topic 1400(2-8), 10 pts.] Compute $\displaystyle\int_0^\infty (e^{3x} - 2x^6)e^{-x^2/2}\,dx$.

$$= \left[\int_0^\infty e^{3x} e^{-x^2/2}\,dx\right] - 2\left[\int_0^\infty x^6 e^{-x^2/2}\,dx\right]$$

$$= e^{3^2/2}\left[\int_{-3}^\infty e^{-x^2/2}\,dx\right] - \left[\int_{-\infty}^\infty x^6 e^{-x^2/2}\,dx\right]$$

$$= e^{9/2}\left[\Phi(3)\right]\sqrt{2\pi} - \left[1\cdot 3\cdot 5\right]\sqrt{2\pi}$$

2. [Topic 1500(29), 10 pts.] Let $\mathcal{S}$ be the σ-algebra on $[0, 10)$ generated by the sets $[0, 1)$, $[0, 2)$, $[0, 3)$, $\ldots$, $[0, 9)$. How many sets are there in $\mathcal{S}$?

atoms: $[0,1), [1,2), [2,3), \ldots, [9,10)$

$$\#\mathcal{S} = 2^{\#\text{atoms}} = 2^{10}$$

3. [Topic 1200(13-23), 10 pts.] Let $X_1, X_2, \ldots$ be an independent sequence of binary PCRVs such that:

$$\Pr[X_j = -3] = 0.4 \qquad \text{and} \qquad \Pr[X_j = 2] = 0.6.$$

Let $Y_n := (X_1 + \cdots + X_n)/\sqrt{n}$. Let $f_n(t)$ be the Fourier transform of the distribution of Y_n. Compute $\lim\limits_{n\to\infty} (f_n(3))$.

$$\mathcal{F}\delta_{X_j} = \frac{4}{10} e^{3it} + \frac{6}{10} e^{-2it}$$

$$= \frac{4}{10}\left(1 + \cancel{3it} - \frac{1}{2}\cdot 9t^2\right)$$

$$+ \frac{6}{10}\left(1 - \cancel{2it} - \frac{1}{2}\cdot 4t^2\right)$$

$$+ (\varepsilon(t))t^2, \qquad \text{some } \varepsilon(t) \xrightarrow{t\to 0} 0$$

$$\mathcal{F}\delta_{X_j} = 1 - \frac{1}{2}\left(\frac{36}{10} + \frac{24}{10}\right)t^2 + (\varepsilon(t))t^2$$

$$f_n(t) = \left[1 - \frac{1}{2}\left(\frac{60}{10}\right)\frac{t^2}{n} + \left(\varepsilon\left(\frac{t}{\sqrt{n}}\right)\right)\frac{t^2}{n}\right]^n$$

$$f_n(3) = \left[1 - \frac{1}{2}(6)\frac{9}{n} + \frac{\delta_n}{n}\right]^n,$$

$$\text{some } \delta_n \xrightarrow{n\to\infty} 0$$

$$\lim_{n\to\infty} f_n(3) = e^{-3\cdot 9} = e^{-27}$$

4. [Topic 2400(28), 10 pts.] Let λ denote Lebesgue measure on $\mathbb{R}$. Give an example of a sequence of smooth functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

(•) for all $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0$; and

(•) for all integers $n \geq 1$, $\int_{\mathbb{R}} f_n(x)\, d\lambda(x) = 1$.

$$f_n(x) = \Phi'(x-n)$$

$$= \frac{e^{-(x-n)^2/2}}{\sqrt{2\pi}}$$

5. [Topic 1700(35), 5 pts.] Define the PCRV $X : [0,1] \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} 6, & \text{if } 0 \leq \omega \leq 0.2; \\ 7, & \text{if } 0.2 < \omega < 0.9; \\ 8, & \text{if } 0.9 \leq \omega \leq 1. \end{cases}$$

Let λ_1 be the restriction of Lebesgue measure to $[0,1]$. Write the distribution $X_*(\lambda_1)$ of X as a linear combination of point masses.

$$(0.2)\,\delta_6 + (0.7)\,\delta_7 + (0.1)\,\delta_8$$

6. [Topic 1700(19), 10 pts.] Let $S := [1, 1.1] \cup [2, 2.01] \cup [3, 3.001] \cup [4, 4.0001] \cup \cdots$. Let $T := [1, 1.1] \cup [2, 2.02] \cup [3, 3.003] \cup [4, 4.0004] \cup \cdots$. Let λ^2 be Lebesgue measure on $\mathbb{R}^2$. Compute $\lambda^2(S \times T)$. (Express your answer as a quotient of two integers.)

$$\alpha := \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots = \lambda(S)$$

$$10\alpha = \alpha + 1, \quad \text{so} \quad \alpha = \frac{1}{9}$$

$$\beta := \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots = \lambda(T)$$

$$\frac{1}{10}\beta = \qquad \frac{1}{10^2} + \frac{2}{10^3} + \cdots$$

$$\beta - \frac{1}{10}\beta = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots = \alpha = \frac{1}{9}$$

$$\frac{9}{10}\beta = \frac{1}{9} \quad \text{so} \quad \beta = \frac{10}{81}$$

$$\lambda^2(S \times T) = [\lambda(S)][\lambda(T)]$$
$$= \alpha\beta = \frac{1}{9} \cdot \frac{10}{81} = \frac{10}{729}$$

7. [Topic 0200 (25-30), 5 pts.] Let X be a sum of $1,000,000$ independent binary PCRVs, all with the same mean, μ, and all with the same standard deviation, σ. Assume that $\mathrm{SD}[X] = 1,000$ and that $\mathrm{E}[X] = 1,000$. Find μ and σ.

$$1000000\mu = E[X] = 1000$$

so $\mu = \frac{1}{1000}$

$$1000000\sigma^2 = \mathrm{Var}[X]$$
$$= (SD[X])^2$$
$$= (1000)^2$$
$$= 1000000$$

$$\sigma^2 = 1$$

& $\sigma > 0$

so $\sigma = 1$

8. [Topic 1200(2), 10 pts.] For each integer $n \geq 1$, let X_n be a sum of n independent binary PCRVs, all with the same distribution. Assume, for all integers $n \geq 1$, that the uptick and downtick probabilities (of the summands) are between 0.1 and 0.9. Assume, for all integers $n \geq 1$, that X_n is has mean 0 and standard deviation 10. Using the Triangular Central Limit Theorem, compute $\lim_{n\to\infty} \mathrm{E}[(e^{X_n} - e^{30})_+]$.

By ΔCLT, $X_n \longrightarrow 10Z$

$$\mathrm{E}\left[(e^{X_n}-e^{30})_+\right] \xrightarrow{n\to\infty} \mathrm{E}\left[(e^{10Z}-e^{30})_+\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} (e^{10x}-e^{30})_+ \, e^{-x^2/2}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{3}^{\infty} (e^{10x}-e^{30}) \, e^{-x^2/2}\, dx\right]$$

$$= e^{10^2/2}\frac{1}{\sqrt{2\pi}}\left[\int_{3-10}^{\infty} \cancel{e^{10x}}\, e^{-x^2/2}\, dx\right] - \frac{e^{30}}{\sqrt{2\pi}}\left[\int_{3}^{\infty} e^{-x^2/2}\, dx\right]$$

$$= e^{50}\left[\Phi(7)\right] - e^{30}\left[\Phi(-3)\right]$$