## FM 5011 Fall 2011, Midterm #1 Handout date: Thursday 20 October 2011 Time for exam: ONE HOUR



Remember to read to the bottom and to SIGN YOUR NAME BELOW!

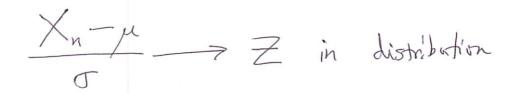
Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

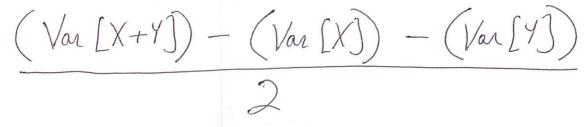
REMEMBER TO SIGN YOUR NAME:

T.	Definitions:	Complete	the :	following	sentences
1.	Dominion.	Compice	ULIC .	DITOWING	politionicos.

a. [Topic 1200(25), 3 pts.] Let  $X_n$  be a sequence of PCRVs. We say  $X_n \to \sigma Z + \mu$  in distribution if . . .



b. [Topic 0100(21), 3 pts.] Let X and Y be non-determinstic PCRVs. The **covariance** of X and Y is defined by  $Cov[X, Y] = \cdots$ .

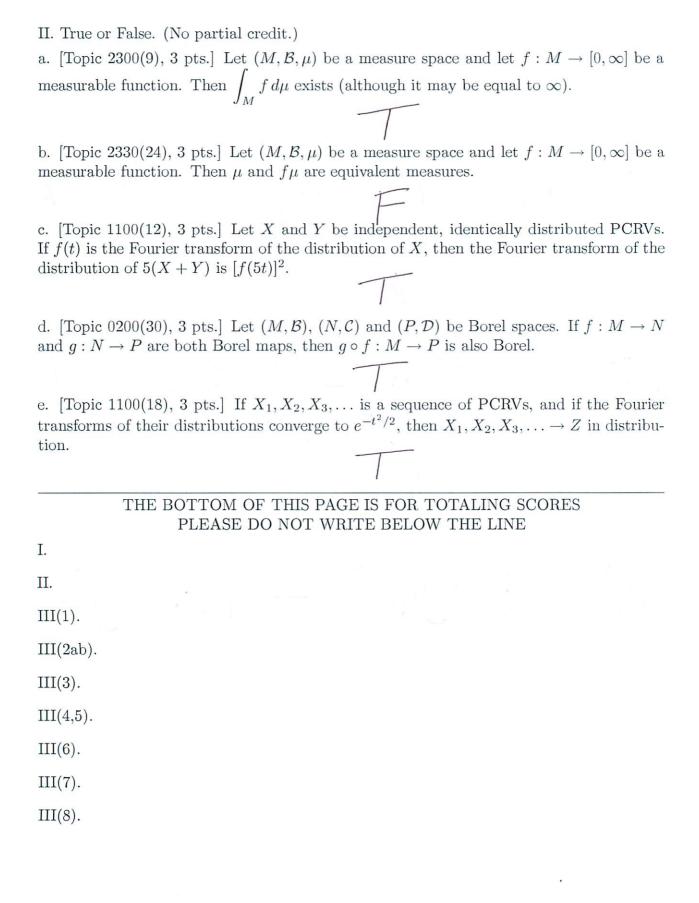


c. [Topic 2330(24), 3 pts.] Two measures  $\mu$  and  $\nu$  on a Borel space  $(M, \mathcal{B})$  are equivalent, written  $\mu \approx \nu$ , if . . .



d. [Topic 1700(12), 3 pts.] Let  $\mathcal{B}$  and  $\mathcal{C}$  be  $\sigma$ -algebras on sets M and N, respectively. The **product**  $\sigma$ -algebra on  $M \times N$  is defined by  $\mathcal{B} \times \mathcal{C} = \cdots$ 

e. [Topic 2300(6), 3 pts.] Let  $(M, \mathcal{B}, \mu)$  be a measure space. A function  $s: M \to \mathbb{R}$  is **simple** if . . .



III. Computations. Some of your answers may involve  $\Phi$ , the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. [Topic 1400(2-8), 10 pts.] Compute 
$$\int_{0}^{\infty} (e^{3x} - 2x^{6})e^{-x^{2}/2} dx$$
.

$$\int_{0}^{\infty} e^{3x} e^{-x^{2}/2} dx - 2 \int_{0}^{\infty} x^{6} e^{-x^{2}/2} dx$$

$$\int_{0}^{\infty} e^{-3x^{2}/2} \int_{-3}^{\infty} e^{-x^{2}/2} dx - \int_{-\infty}^{\infty} x^{6} e^{-x^{2}/2} dx$$

$$= \int_{-3}^{\infty} e^{-x^{2}/2} dx - \int_{-\infty}^{\infty} x^{6} e^{-x^{2}/2} dx$$

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2. [Topic 1500(29), 10 pts.] Let S be the  $\sigma$ -algebra on [0, 10) generated by the sets [0, 1),  $[0, 2), [0, 3), \ldots, [0, 9)$ . How many sets are there in S?

atoms: 
$$[0,1)$$
,  $[1,2)$ ,  $[2,3)$ ...,  $[9,10)$ 

$$# J = 2^{\#atoms} = 2^{10}$$

3. [Topic 1200(13-23), 10 pts.] Let  $X_1, X_2, \ldots$  be an independent sequence of binary PCRVs such that:

$$Pr[X_i = -3] = 0.4$$
 and  $Pr[X_i = 2] = 0.6$ .

Let  $Y_n := (X_1 + \cdots + X_n)/\sqrt{n}$ . Let  $f_n(t)$  be the Fourier transform of the distribution of  $Y_n$ . Compute  $\lim_{n \to \infty} (f_n(3))$ .

$$\begin{aligned}
& \exists S_{X_{3}} = \frac{4}{10} e^{3it} + \frac{6}{10} e^{-2it} \\
& = \frac{4}{10} \left( 1 + 3it - \frac{1}{2} \cdot 9t^{2} \right) \\
& + \frac{6}{10} \left( 1 - 2it - \frac{1}{2} \cdot 4t^{2} \right) \\
& + \left( \varepsilon(t) \right) t^{2}, \quad \text{some } \varepsilon(t) - \frac{t \to 0}{0} \to 0
\end{aligned}$$

$$\begin{aligned}
& \exists S_{X_{3}} = \left[ 1 - \frac{1}{2} \left( \frac{36}{10} + \frac{24}{10} \right) t^{2} + \left( \varepsilon(t) \right) t^{2} \\
& f_{n}(t) = \left[ 1 - \frac{1}{2} \left( \frac{60}{10} \right) \frac{t^{2}}{n} + \left( \varepsilon\left( \frac{t}{10} \right) \right) \frac{t^{2}}{n} \right]^{n}
\end{aligned}$$

$$\begin{aligned}
& f_{n}(3) = \left[ 1 - \frac{1}{2} \left( \frac{6}{10} \right) \frac{t^{2}}{n} + \frac{5n}{n} \right]^{n}, \\
& \text{Some } S_{n} - \frac{n \to \infty}{n} \to 0
\end{aligned}$$

$$\lim_{n\to\infty} f_n(3) = e^{-3.9} = e^{-27}$$

- 4. [Topic 2400(28), 10 pts.] Let  $\lambda$  denote Lebesgue measure on  $\mathbb{R}$ . Give an example of a sequence of smooth functions  $f_n : \mathbb{R} \to \mathbb{R}$  such that
  - (•) for all  $x \in \mathbb{R}$ ,  $\lim_{n \to \infty} f_n(x) = 0$ ; and
  - (•) for all integers  $n \ge 1$ ,  $\int_{\mathbb{R}} f_n(x) d\lambda(x) = 1$ .

$$\int_{n} (x) = \overline{\Phi}'(x-n)$$

$$= \frac{e^{-(x-n)^{2}/2}}{\sqrt{2\pi}}$$

5. [Topic 1700(35), 5 pts.] Define the PCRV  $X:[0,1]\to\mathbb{R}$  by

$$X(\omega) = \begin{cases} 6, & \text{if } 0 \le \omega \le 0.2; \\ 7, & \text{if } 0.2 < \omega < 0.9; \\ 8, & \text{if } 0.9 \le \omega \le 1. \end{cases}$$

Let  $\lambda_1$  be the restriction of Lebesgue measure to [0,1]. Write the distribution  $X_*(\lambda_1)$  of X as a linear combination of point masses.

$$(0.2)$$
  $\delta_6 + (0.7)$   $\delta_7 + (0.1)$   $\delta_8$ 

6. [Topic 1700(19), 10 pts.] Let  $S := [1, 1.1] \cup [2, 2.01] \cup [3, 3.001] \cup [4, 4.0001] \cup \cdots$ . Let  $T := [1, 1.1] \cup [2, 2.02] \cup [3, 3.003] \cup [4, 4.0004] \cup \cdots$ . Let  $\lambda^2$  be Lebesgue measure on  $\mathbb{R}^2$ . Compute  $\lambda^2(S \times T)$ . (Express your answer as a quotient of two integers.)

$$\begin{aligned}
\alpha &:= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots = \lambda(S) \\
10 \alpha &= \alpha + 1, \quad \rho_0 \quad \alpha &= \frac{1}{9}
\end{aligned}$$

$$\beta^2 &= \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \dots = \lambda(T)$$

$$\frac{1}{10} \beta &= \frac{1}{10^2} + \frac{2}{10^3} + \dots = \alpha = \frac{1}{9}$$

$$\beta - \frac{1}{10} \beta &= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots = \alpha = \frac{1}{9}$$

$$\frac{9}{10} \beta &= \frac{1}{9} \qquad \rho_0 \qquad \beta &= \frac{10}{81}$$

7. [Topic 0200 (25-30), 5 pts.] Let X be a sum of 1,000,000 independent binary PCRVs, all with the same mean,  $\mu$ , and all with the same standard deviation,  $\sigma$ . Assume that SD[X] = 1,000 and that E[X] = 1,000. Find  $\mu$  and  $\sigma$ .

$$10000000\mu = E[X] = 1000$$

8. [Topic 1200(2), 10 pts.] For each integer  $n \ge 1$ , let  $X_n$  be a sum of n independent binary PCRVs, all with the same distribution. Assume, for all integers  $n \ge 1$ , that the uptick and downtick probabilities (of the summands) are between 0.1 and 0.9. Assume, for all integers  $n \ge 1$ , that  $X_n$  is has mean 0 and standard deviation 10. Using the Triangular Central Limit Theorem, compute  $\lim_{n\to\infty} \mathbb{E}[(e^{X_n}-e^{30})_+]$ .

By 
$$\Delta CLT$$
  $X_{n} \longrightarrow 102$ 
 $E[(e^{X_{n}} - e^{30})_{+}] \xrightarrow{n \to \infty} E[(e^{102} - e^{30})_{+}]$ 
 $\frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} (e^{10x} - e^{30})_{+} e^{-x^{2}/2} dx \right]$ 
 $\frac{1}{\sqrt{2\pi}} \left[ \int_{3}^{\infty} (e^{10x} - e^{30})_{+} e^{-x^{2}/2} dx \right]$ 
 $e^{10x} \frac{1}{\sqrt{2\pi}} \left[ \int_{3\pi}^{\infty} e^{10x} e^{-x^{2}/2} dx \right] - \frac{e^{30}}{\sqrt{2\pi}} \left[ \int_{3}^{\infty} e^{-x^{2}/2} dx \right]$ 
 $e^{50} \left[ \frac{1}{2} (7) \right] - e^{30} \left[ \frac{1}{2} (-3) \right]$