FM 5011 Fall 2011, Midterm #2
Handout date: Thursday 17 November 2011
Time for exam: ONE HOUR

PRINT NAME:

SOLUTIONS

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind.
Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has **severe consequences**, from a failing grade to expulsion.

REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

a. [Topic 2900(9), 3pts.] If X is a random variable on a probability space $(\Omega, \mathcal{B}, \mu)$, then the **distribution** of X is given by $\delta_X := \cdots$

$$X_*(\mu)$$

b. [Topic 2900(4), 3pts.] A (real-valued) random variable X is L^2 if ...

$$E[X^2] < \infty$$

c. [Topic 2700(13), 3pts.] The CDF of a measure μ on $\mathbb{R}$ is defined by $\mathrm{CDF}_\mu(x) = \cdots$

$$\mu(\ (-\infty, x]\ )$$

d. [Topic 2700(35), 3pts.] Let $\mu_1, \mu_2, \ldots$ be a sequence of probability measures on $\mathbb{R}$. Let μ be another probability measure on $\mathbb{R}$. We say $\mu_1, \mu_2, \ldots \to \mu$ if ...

$$\forall f \in C_B, \quad \int_{\mathbb{R}} f \, d\mu_n \xrightarrow[n \to \infty]{} \int_{\mathbb{R}} f \, d\mu$$

e. [Topic 2900(51), 3pts.] Let $X_1, X_2, \ldots$ be a sequence of random variables on a probability space $(\Omega, \mathcal{B}, \mu)$. Let X be another random variable on $(\Omega, \mathcal{B}, \mu)$. We say $X_1, X_2 \ldots \to X$ **in probability** if ...

$$\forall \varepsilon > 0, \quad \Pr[\,|X_n - X| > \varepsilon\,] \xrightarrow[n \to \infty]{} 0$$

II. True or False. (No partial credit.)

a. [Topic 2450(8), 3 pts.] Let λ be Lebesgue measure on $\mathbb{R}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be measurable. Let $A := \int_{\mathbb{R}} \left[\int_{\mathbb{R}} [f(x,y)](d\lambda)(x) \right] (d\lambda)(y)$ and $B := \int_{\mathbb{R}} \left[\int_{\mathbb{R}} [f(x,y)](d\lambda)(y) \right] (d\lambda)(x)$. Assume both A and B exist and are finite. Then $A = B$.

F

b. [Topic 2500(5), 3 pts.] The Borel spaces $[0,1]$ and $\mathbb{R}^3$, with their standard σ-algebras, are Borel isomorphic.

T

c. [Topic 2700(13), 3 pts.] Let μ and ν be two probability measures on $\mathbb{R}$. Assume that $\mathrm{CDF}_\mu = \mathrm{CDF}_\nu$. Then $\mu = \nu$.

T

d. [Topic 2900(14), 3 pts.] Suppose that A, B, C, D are random variables, all with the same distribution. Then $\mathrm{Var}[A] = \mathrm{Var}[B] = \mathrm{Var}[C] = \mathrm{Var}[D]$.

T

e. [Topic 2900(53), 3 pts.] Let $X_1, X_2, \ldots$ be a sequence of random variables on a probability space $(\Omega, \mathcal{B}, \mu)$. Let X be another random variable on a $(\Omega, \mathcal{B}, \mu)$. If $X_1, X_2, \ldots \to X$ in probability, then $X_1, X_2, \ldots \to X$ almost surely.

F

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PLEASE DO NOT WRITE BELOW THE LINE

I.

II.

III(1).

III(2,3).

III(4).

III(5).

III(6).

III. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. [Topic 2800(14), 15 pts.] Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x, & \text{if } x \leq 0; \\ 1 - x^2, & \text{if } x > 0; \end{cases}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. Compute $\int_{-1}^{1} f\, dg$.

(Hint: An antiderivative of xe^x is $xe^x - e^x$.)

$$\left[\int_{-1}^{0} (e^x)(1)\, dx\right] + \left[\int_{\mathbb{R}} e^x \, d\delta_0(x)\right] + \left[\int_{0}^{1} (e^x)(-2x)\, dx\right]$$

$$\parallel$$

$$\left[e^x\right]_{x:\to -1}^{x:\to 0} + \left[e^x\right]_{x:\to 0} + (-2)\left[xe^x - e^x\right]_{x:\to 0}^{x:\to 1}$$

$$\parallel$$

$$\left[1 - \frac{1}{e}\right] + [1] + (-2)\left[(e-e) - (0-1)\right]$$

$$\parallel$$

$$1 - \frac{1}{e} + 1 - 2[1]$$

$$\parallel$$

$$-\frac{1}{e}$$

2. [Topic 2900(36), 10 pts.] Let Y and Z be independent standard normal random variables. Compute the Fourier transform of the distribution of $Y + Z$.

(*Hint*: $\mathcal{F}\delta_Y = \mathcal{F}\delta_Z = e^{-t^2/2}$.)

$$\mathcal{F}\delta_{Y+Z} = (\mathcal{F}\delta_Y)(\mathcal{F}\delta_Z)$$

$$= (e^{-t^2/2})(e^{-t^2/2}) = e^{-t^2}$$

3. [Topic 2900(8), 10 pts.] Let Y and Z be two independent standard normal random variables. Find constants $a \geq 0$, $b \in \mathbb{R}$ and $c \geq 0$ such that

(•) $\mathrm{Var}[aY + bZ] = 29$;
(•) $\mathrm{Cov}[aY + bZ, cZ] = -6$; and
(•) $\mathrm{Var}[cZ] = 9$;

$c^2 = 9, \quad c \geq 0 \quad \therefore \quad \boxed{c = 3}$

$bc = -6, \quad c = 3 \quad \therefore \quad \boxed{b = -2}$

$a^2 + b^2 = 29, \quad b = -2 \quad \therefore \quad a^2 = 25$

$a^2 = 25, \quad a \geq 0 \quad \therefore \quad \boxed{a = 5}$

4. [Topic 2900(47), 15 pts.] Let $C_1, C_2, \ldots$ be an iid sequence of binary random variables such that, for all integers $j \geq 1$, we have $\Pr[C_j = 1] = 0.5$ and $\Pr[C_j = -1] = 0.5$. For all integers $n \geq 1$, let $X_n = (C_1 + \cdots + C_n)/\sqrt{n}$. Compute $\lim\limits_{n\to\infty} E[e^{4X_n}]$.

Let Z be a std normal RV.

$X_n \xrightarrow[n\to\infty]{} Z$ in distribution against continuous exp-bdd

$$E\left[e^{4X_n}\right] \xrightarrow[n\to\infty]{} E\left[e^{4Z}\right]$$

$$\|$$

$$e^{4^2/2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{e^{4x}}\, e^{-x^2/2}\, dx\right]$$

$$\|$$

$$e^{4^2/2}\,[1]$$

$$\|$$

$$e^8$$

5. [Topic 2900(13), 10 pts.] Let Z be a standard normal random variable. Let $\sigma > 0$ and let $\mu \in \mathbb{R}$. Compute $\mathrm{E}[e^{\sigma Z+\mu}]$.

$$\|$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma x+\mu} e^{-x^2/2}\, dx$$

$$\|$$

$$e^{\sigma^2/2} e^{\mu} \Bigg[\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{e^{\sigma x}} e^{-x^2/2}\, dx}_{1} \Bigg] = e^{\sigma^2/2} e^{\mu}$$

6. [Topic 2900(45), 10 pts.] Let Z be a standard normal random variable. Let G be the grade of Z and let H be the grade of Z^3. Let $\mu := \delta_{G,H}$ be the copula of Z and Z^3. Let $I := [0.2, 0.3]$ and $J := [0.4, 0.5]$. Compute $\mu(I \times J)$.

Hint: $\mathrm{CDF}_{\delta[Z]}(x) = \Phi(x)$ and

$$\mathrm{CDF}_{\delta[Z^3]}(x) = \Pr[Z^3 \leq x] = \Pr[Z \leq \sqrt[3]{x}] = \mathrm{CDF}_{\delta[Z]}(\sqrt[3]{x}) = \Phi(\sqrt[3]{x}).$$

$$G = \Phi(Z)$$

$$H = \Phi\left(\sqrt[3]{Z^3}\right) = \Phi(Z) = G$$

$$\mu(I \times J) = \delta_{G,H}(I \times J)$$

distribution drives probability

$$= \Pr[(G,H) \in I \times J]$$

$$= \Pr[(G \in I) \;\&\; (H \in J)]$$

$G=H$

$$= \Pr[(G \in I) \;\&\; (G \in J)]$$

$I \cap J = \emptyset$

$$= 0$$