FM 5011 Fall 2011, Midterm #2 Handout date: Thursday 17 November 2011 Time for exam: ONE HOUR

PRINT NAME:

Remember to read to the bottom and to SIGN YOUR NAME BELOW!

Closed book, closed notes, no calculators/PDAs; no reference materials of any kind. Show work; a correct answer, by itself, may be insufficient for credit.

I understand the above, and I understand that cheating has severe consequences, from a failing grade to expulsion.

REMEMBER TO SIGN YOUR NAME:

I. Definitions: Complete the following sentences.

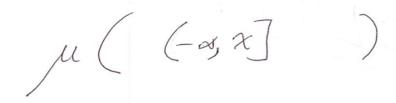
a. [Topic 2900(9), 3pts.] If X is a random variable on a probability space $(\Omega, \mathcal{B}, \mu)$, then the **distribution** of X is given by $\delta_X := \cdots$



b. [Topic 2900(4), 3pts.] A (real-valued) random variable X is L^2 if . . .

$$E[X^2] < \infty$$

c. [Topic 2700(13), 3pts.] The CDF of a measure μ on \mathbb{R} is defined by $\mathrm{CDF}_{\mu}(x) = \cdots$

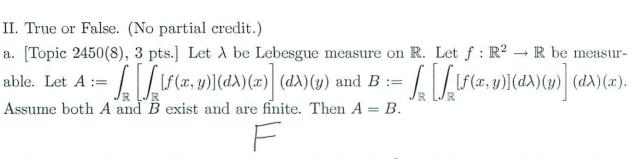


d. [Topic 2700(35), 3pts.] Let μ_1, μ_2, \ldots be a sequence of probability measures on \mathbb{R} . Let μ be another probability measure on \mathbb{R} . We say $\mu_1, \mu_2, \ldots \to \mu$ if ...

$$\forall f \in C_{B_j}$$
 $\int_{\mathbb{R}} f d\mu_n \xrightarrow{n \to \infty} \int_{\mathbb{R}} f d\mu$

e. [Topic 2900(51), 3pts.] Let $X_1, X_2, ...$ be a sequence of random variables on a probability space $(\Omega, \mathcal{B}, \mu)$. Let X be another random variable on $(\Omega, \mathcal{B}, \mu)$. We say $X_1, X_2, ... \to X$ in probability if ...

$$\forall \varepsilon > 0$$
 $P_n \left[|X_n - X| > \varepsilon \right] \xrightarrow{n \to \infty} 0$



b. [Topic 2500(5), 3 pts.] The Borel spaces [0,1] and \mathbb{R}^3 , with their standard σ -algebras, are Borel isomorphic.

c. [Topic 2700(13), 3 pts.] Let μ and ν be two probability measures on \mathbb{R} . Assume that $\mathrm{CDF}_{\mu}=\mathrm{CDF}_{\nu}$. Then $\mu=\nu$.

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d. [Topic 2900(14), 3 pts.] Suppose that A, B, C, D are random variables, all with the same distribution. Then Var[A] = Var[B] = Var[C] = Var[D].

7

e. [Topic 2900(53), 3 pts.] Let X_1, X_2, \ldots be a sequence of random variables on a probability space $(\Omega, \mathcal{B}, \mu)$. Let X be another random variable on a $(\Omega, \mathcal{B}, \mu)$. If $X_1, X_2, \ldots \to X$ in probability, then $X_1, X_2, \ldots \to X$ almost surely.



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I.

II.

III(1).

III(2,3).

III(4).

III(5).

III(6).

III. Computations. Some of your answers may involve Φ , the cumulative distribution function of the standard normal distribution. (Unless otherwise specified, answers must be exactly correct, but can be left in any form easily calculated on a standard calculator.)

1. [Topic 2800(14), 15 pts.] Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x, & \text{if } x \le 0; \\ 1 - x^2, & \text{if } x > 0; \end{cases}$$

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. Compute $\int_{-1}^1 f \, dg$. (Hint: An antiderivative of xe^x is $xe^x - e^x$.)

$$\left[\int_{-1}^{0} (e^{2}\chi(1) dx)\right] + \left[\int_{-1}^{0} e^{2x} dS_{o}(x)\right] + \left[\int_{0}^{1} (e^{2x})(-2x) dx\right]$$

$$\left[e^{x} \right]_{x \mapsto -1}^{x \mapsto 0} + \left[e^{x} \right]_{x \mapsto 0} + \left(-2 \right) \left[x e^{x} - e^{x} \right]_{x \mapsto 0}^{x \mapsto 0}$$

$$[1-\frac{1}{e}] + [1] + (-2)[(e-e)-(o-1)]$$

$$11$$

$$1-\frac{1}{e} + 1 - 2[1]$$

2. [Topic 2900(36), 10 pts.] Let Y and Z be independent standard normal random variables. Compute the Fourier transform of the distribution of Y + Z.

(Hint: $\mathcal{F}\delta_Y = \mathcal{F}\delta_Z = e^{-t^2/2}$.)

- 3. [Topic 2900(8), 10 pts.] Let Y and Z be two independent standard normal random variables. Find constants $a \geq 0$, $b \in \mathbb{R}$ and $c \geq 0$ such that
 - (•) Var[aY + bZ] = 29;
 - (•) $\operatorname{Cov}[aY + bZ, cZ] = -6$; and
 - $(\bullet) \operatorname{Var}[cZ] = 9;$

$$c^{2} = 9$$
, $c > 0$. $c = 3$
 $bc = -6$, $c = 3$. $b = -2$
 $a^{2} + b^{2} = 29$, $b = -2$. $a^{2} = 25$
 $a^{2} - 25$, $a > 0$. $a = 5$

4. [Topic 2900(47), 15 pts.] Let C_1, C_2, \ldots be an iid sequence of binary random variables such that, for all integers $j \geq 1$, we have $\Pr[C_j = 1] = 0.5$ and $\Pr[C_j = -1] = 0.5$. For all integers $n \geq 1$, let $X_n = (C_1 + \cdots + C_n)/\sqrt{n}$. Compute $\lim_{n \to \infty} E[e^{4X_n}]$.

Let Z be a Std normal RV.

Xn n-> a > Z in distribution against continuous exp-feld

 $E[e^{4X_n}] \xrightarrow{n \to \infty} E[e^{4Z}]$

 $e^{4\frac{2}{2}} \int_{-\infty}^{\infty} e^{4x} e^{-x^2/2} dx$

 $e^{4^{2}/2} \begin{bmatrix} 1 \end{bmatrix}$ |1| p^{8}

5. [Topic 2900(13), 10 pts.] Let Z be a standard normal random variable. Let $\sigma > 0$ and let $\mu \in \mathbb{R}$. Compute $\mathrm{E}[e^{\sigma Z + \mu}]$.

$$\int_{\sqrt{2\pi}}^{\infty} \int_{-\alpha}^{\infty} e^{\sigma x + \mu} e^{-x^{2}/2} dx$$

$$\int_{-\alpha}^{\infty} e^{\sigma x} e^{-x^{2}/2} dx = e^{\sigma^{2}/2} e^{\mu}$$

6. [Topic 2900(45), 10 pts.] Let Z be a standard normal random variable. Let G be the grade of Z and let H be the grade of Z^3 . Let $\mu := \delta_{G,H}$ be the copula of Z and Z^3 . Let I := [0.2, 0.3] and J := [0.4, 0.5]. Compute $\mu(I \times J)$.

Hint: $CDF_{\delta[Z]}(x) = \Phi(x)$ and

$$\mathrm{CDF}_{\delta[Z^3]}(x) = \Pr[Z^3 \le x] = \Pr[Z \le \sqrt[3]{x}] = \mathrm{CDF}_{\delta[Z]}(\sqrt[3]{x}) = \Phi(\sqrt[3]{x}).$$

$$G = \overline{\mathcal{I}}(Z)$$

$$H = \overline{\mathcal{I}}(Z) = \overline{\mathcal{I}}(Z) = G$$

$$\mu(\overline{I} \times J) = S_{SH}(\overline{I} \times J)$$

$$\lim_{h \to \infty} \lim_{h \to \infty} \frac{1}{h} = \Pr_{R}[(G, H) \in \overline{I} \times J]$$

$$= \Pr_{R}[(G \in \overline{I}) \& (H \in J)]$$

$$= \Pr_{R}[(G \in \overline{I}) \& (G \in J)]$$

$$= \Pr_{R}[(G \in \overline{I}) \& (G \in J)]$$