Financial Mathematics

Topics in integration

2360-1. Let $\Omega := \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$.

Define prob. measures $\mu, \nu : 2^{\Omega} \to [0, 1]$ by

$$\begin{aligned} \mu(\{\mathrm{A}\}) &= 0.2, & \nu(\{\mathrm{A}\}) &= 0.2, \\ \mu(\{\mathrm{B}\}) &= 0.0, & \nu(\{\mathrm{B}\}) &= 0.0, \\ \mu(\{\mathrm{C}\}) &= 0.3, & \nu(\{\mathrm{C}\}) &= 0.1, \\ \mu(\{\mathrm{D}\}) &= 0.3, & \nu(\{\mathrm{D}\}) &= 0.3, \\ \mu(\{\mathrm{E}\}) &= 0.2, & \nu(\{\mathrm{E}\}) &= 0.4, \end{aligned}$$

Determine whether $\mu << \nu$ and whether $\nu << \mu$.

If $\mu << \nu$, then find $\dfrac{d\mu}{d\nu}$, *i.e.*, find $f : \Omega \to \mathbb{R}$ s.t. $\mu = f\nu$.

If $\nu << \mu$, then find $\dfrac{d\nu}{d\mu}$, *i.e.*, find $g : \Omega \to \mathbb{R}$ s.t. $\nu = g\mu$.

2360-2. Define a PCRV X by

$$X(\omega) = \begin{cases} 3, \text{ if } 0.00 \leq \omega < 0.25 \\ -4, \text{ if } 0.25 \leq \omega \leq 0.50 \\ 1, \text{ if } 0.50 < \omega \leq 0.75 \\ 7, \text{ if } 0.75 < \omega \leq 1.00 \end{cases}$$

Let P denote Lebesgue msr on $\Omega := [0, 1]$.
Let $\mathcal{A}$ denote the set of all changes of measure on (Ω, P), *i.e.*, the set of prob. msrs S on Ω s.t. S is equivalent to P.

$\forall S \in \mathcal{A}$, $\boxed{\mathsf{E}^S[X]} := \int_0^1 X \, dS$.

a. Find $\mathsf{E}^P[X]$.

b. Find $Q \in \mathcal{A}$ s.t. $E^Q[X] = 0$.

c. Is there $R \in \mathcal{A}$ s.t. $\mathsf{E}^R[X] = 6$? Explain.

d. Find $\sup\{E^S[X] \,|\, S \in \mathcal{A}\}$.

2360-3. Let $\mathbb{R}_+ := (0, \infty)$.

Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = x^4$.

Let λ_+ be Lebesgue measure on $\mathbb{R}_+$.

a. Compute $\displaystyle\int_{\mathbb{R}_+} e^{-x^2}\, d(f\lambda_+)(x)$.

b. Compute $\displaystyle\int_{\mathbb{R}_+} e^{-x^{1/2}}\, d(f_*\lambda_+)(x)$.