Financial Mathematics

Basics of random variables

2900-1. For any measure μ on $\mathbb{R}$,
the **center of mass of** μ is $\int_{\mathbb{R}} x\,d\mu(x)$.

a. Find the center of mass of the distribution of the PCRV $Y : [0,1] \to \mathbb{R}$ defined by

$$Y(\omega) = \begin{cases} 3, & \text{if } 0.00 \leq \omega < 0.40 \\ -2, & \text{if } 0.40 \leq \omega < 0.55 \\ -1, & \text{if } 0.55 \leq \omega \leq 0.80 \\ -12, & \text{if } 0.80 < \omega \leq 1.00 \end{cases}$$

b. Compute $(Y_*(\lambda))([-1,3))$, where λ is Lebesgue measure on $[0,1]$.

2900-2. Define a RV Y by

$$Y(\omega) = \begin{cases} 3, & \text{if } 0.00 \le \omega < 0.40 \\ -2, & \text{if } 0.40 \le \omega < 0.55 \\ -1, & \text{if } 0.55 \le \omega \le 0.80 \\ -12, & \text{if } 0.80 < \omega \le 1.00 \end{cases}$$

a. Compute $\Pr[2 \le Y^2 \le 9]$.

b. Let ν be the distribution of Y^2. Compute $\nu([2,9])$.

2900-3. Define a RV Y by

$$Y(\omega) = \begin{cases} 3, & \text{if } 0.00 \leq \omega < 0.40 \\ -2, & \text{if } 0.40 \leq \omega < 0.55 \\ -1, & \text{if } 0.55 \leq \omega \leq 0.80 \\ -12, & \text{if } 0.80 < \omega \leq 1.00 \end{cases}$$

a. Compute $\mathrm{E}[Y]$.

b. Compute $\mathrm{E}[Y^2]$.

c. Let ν be the distribution of Y^2. Compute the center of mass of ν.

2900-4. Let $Z := \Phi^{-1} : (0,1) \to \mathbb{R}$.

Compute the expectation and variance

of $X := Z^2 + e^{3Z+2} : (0,1) \to \mathbb{R}$.

That is, let $\Omega := (0,1)$,

let λ_1 be Leb. msr (restr. to Ω),

and compute

$$\mu := \int_\Omega X \, d\lambda_1, \quad \sigma^2 := \int_\Omega (X - \mu)^2 \, d\lambda_1.$$

Hint: Use Φ to make a change of variables.

2900-5. Let Y be a PCRV.

Suppose $\Pr[Y = 1] = 4/5$

and $\Pr[Y = 9] = 1/5$.

Compute the expectation and variance of Y.

2900-6. Let X be a binary PCRV

s.t. $\Pr[X = 0] = 0.65$

and s.t. $\Pr[X = 4] = 0.35$.

Let $F : \mathbb{R} \to [0, 1]$ be the CDF of δ_X,

defined by $F(x) = \Pr[X \leq x]$.

Let $Y := F(X) = F \circ X$ be the grade of X.

Compute $\Pr[Y < 2/3]$.

2900-7. Let $Z := \Phi^{-1}$ be a std normal variable, $X := 5Z + 2$ normal, with SD $= 5$, mean $= 2$.

Let $F(x) = \text{Pr}[X^3 \leq x]$ be the CDF of δ_{X^3}.

Let $Y := F(X^3) = F \circ X^3$ be the grade of X^3.

a. Compute $\text{Pr}[2/3 < Y]$.

b. Compute $\text{Pr}[Y < 7/8]$.

c. Compute $\text{Pr}[2/3 < Y < 7/8]$.

d. Compute $\text{Pr}[\,(2/3 < Y) \mid (Y < 7/8)\,]$.

e. Compute $\text{Pr}[\,(7/8 < Y) \mid (Y < 2/3)\,]$.

2900-8. Let $\Omega := (0,1)$, with Leb. measure.
Let $Z := \Phi^{-1} : \Omega \to \mathbb{R}$. Let $X := -4Z^5 + 8$.
Let μ be the distribution of $X : \Omega \to \mathbb{R}$.
Define $F : \mathbb{R} \to [0,1]$ by $F(x) = \Pr[X \leq x]$,

so F is the CDF of δ_X.

Let $a := F^{-1}(1/4)$ and $b := F^{-1}(7/8)$.

a. Compute $\mu((-\infty, b])$.

b. Compute $\Pr[X \leq b]$.

c. Compute $\Pr[X < b]$.

d. Compute $\Pr[X \leq a]$.

e. Compute $\Pr[a < X < b]$.

f. Let $Y := F(X) = F \circ X : (0,1) \to [0,1]$,

so Y is the grade of X.

Compute $\Pr[1/4 < Y < 7/8]$.

2900-9. Let Z be a std normal variable,

$$X := 4Z^3 + 8Z + 3.$$

Let $F(x) = \Pr[X \leq x]$ be the CDF of δ_X.

Let $Y := F(X) = F \circ X$ be the grade of X.

a. Compute $\Pr[5/8 < Y]$.

b. Compute $\Pr[Y < 9/8]$.

c. Compute $\Pr[5/8 < Y < 1]$.

d. Compute $\Pr[\,(5/8 < Y) \mid (Y < 7/8)\,]$.

e. Compute $\Pr[\,(5/8 < Y) \mid (Y < 1/8)\,]$.

2900-10. Let $\mu := (0.4)\delta_2 + (0.1)\delta_4 + (0.5)\delta_5$.
Let $\nu := (0.2)\delta_1 + (0.5)\delta_4 + (0.3)\delta_6$.

a. Compute $\mu * \nu$.

b. Compute $\mathcal{F}\mu$ and $\mathcal{F}\nu$.

c. Compute $(\mathcal{F}\mu)(\mathcal{F}\nu)$
and $\mathcal{F}(\mu * \nu)$.

d. Find two independent RVs X and Y
on $\Omega := [0,1] \times [0,1]$ (with Lebesgue msr)
s.t. $\delta_X = \mu$ and $\delta_Y = \nu$.

e. Compute $\mathrm{E}[e^{-itX}]$ and $\mathrm{E}[e^{-itY}]$.

f. Compute $(\mathrm{E}[e^{-itX}])(\mathrm{E}[e^{-itY}])$
and $\mathrm{E}[e^{-it(X+Y)}]$.