Financial Mathematics

Ito’s lemma

3800-1. Let $W_\bullet$ be a Brownian motion.
Let $X_t = e^{3W_t}$ and $Y_t = \cos(2W_t)$.

a. Compute dX_t, using Itô's Lemma.

b. Compute dY_t, using Itô's Lemma.

c. Compute $d(X_tY_t)$, using Itô's Lemma.

d. Compute

$$[(X_t)(dY_t)] + [(dX_t)(Y_t)] + [(dX_t)(dY_t)],$$

using a, b, and the rules of Stochastic Calculus.

3800-2. Let $W_\bullet$ be a Brownian motion.
Let $X_t = e^{3t}$ and $Y_t = \cos(2W_t)$.

a. Compute dX_t, using Itô's Lemma.

b. Compute dY_t, using Itô's Lemma.

c. Compute $d(X_t Y_t)$, using Itô's Lemma.

d. Compute

$$[(X_t)(dY_t)] + [(dX_t)(Y_t)],$$

using a, b, and the rules of Stochastic Calculus.

3800-3. Suppose $dX_t = \sigma_t\, dW_t + \mu_t\, dt$.

Show that $\displaystyle\frac{d(e^{X_t})}{e^{X_t}} = [dX_t] + [1/2][dX_t]^2$,

i.e., show that

$$\frac{d(e^{X_t})}{e^{X_t}} = [\sigma_t\, dW_t + \mu_t\, dt] + [1/2][\sigma_t^2\, dt].$$

3800-4. Let $W_\bullet$ be a BM. Let $X_t := e^{4W_t^2 - t^3}$.

a. Using Itô's Lemma, compute dX_t.

Hint: Let $f(x,t) := e^{4x^2 - t^3}$,

so $X_t = f(W_t, t)$.

b. Let $Y_t := \arctan(2t + X_t)$.

Using Itô's Lemma,

compute dY_t from dX_t above.

Your answer should be in terms of

t, W_t and X_t,

but should not involve dX_t.

Hint: Let $g(x,t) := \arctan(2t + x)$,

so $Y_t = g(X_t, t)$.

3800-5. Suppose $X_\bullet$ solves

$$dX_t = X_t(0.30\,dW_t + 0.01\,dt), \qquad X_0 = 7.$$

Solve the SDE above for $X_\bullet$.

3800-6. Suppose $X_\bullet$ solves

$$dX_t = 0.3\,dW_t - 0.04 X_t\,dt, \qquad X_0 = 3.$$

Let $U_t := e^{(0.04)t} X_t$ and let $V_t := U_t^2$.

a. Compute SDEs for $U_\bullet$ and then $V_\bullet$.

b. Compute $\mathrm{E}[U_9]$ and $\mathrm{E}[V_9]$ from the SDEs.

c. Compute $\mathrm{Var}[X_9]$.

Hint: $\mathrm{Var}[X_9] = (\mathrm{E}[X_9^2]) - (\mathrm{E}[X_9])^2$

$$= (\mathrm{E}[(e^{-0.36}U_9)^2]) - (\mathrm{E}[e^{-0.36}U_9])^2$$