Financial Mathematics

Girsanov's theorem and the martingale representation theorem

3900-1. Let $W_\bullet$ be a BM. Let $X_t := e^{5W_t + 4t^3}$.

a. Find processes $\sigma_\bullet$ and $\gamma_\bullet$

s.t. $dX_t = \sigma_t(dW_t + \gamma_t\, dt)$.

Express σ_t and γ_t in terms of W_t and t, and don't be surprised that $\gamma_\bullet$ is deterministic.

b. $\forall T > 0$, compute the Novikov quantity

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2\, dt\right)\right].$$

c. Let $\tilde{W}_\bullet$ solve $d\tilde{W}_t = dW_t + \gamma_t\, dt$, $\tilde{W}_0 = 0$.

Explain why there's a change of measure Q

s.t. $\tilde{W}$ is a Q-Brownian motion.

d. $\forall T > 0$, show that $\mathrm{E}\left[\int_0^T \sigma_t^2\, dt\right] < \infty$.

e. Explain why $X_\bullet$ is a Q-martingale.

3900-2 a. Compute the expansion of ζ_T below, with two terms (inside exp).

b. Using $d\tilde{W}_t = dW_t + \gamma_t\,dt$, compute the expansion of $\exp(-i\int_0^T \xi_s\,d\tilde{W}_s)$, with two terms (inside exp).

c. Compute the expansion of $\zeta_T \cdot \exp(-i\int_0^T \xi_s\,d\tilde{W}_s)$, with four terms (inside exp).

d. Compute the expansion of ρ_T, with five terms (inside exp).

Proof: Let $\zeta_\bullet := \mathrm{EM}_\bullet[-\gamma]$. two terms $\quad i := \sqrt{-1}$

Let ν have Radon-Nikodỳm derivative ζ_T.

Let $\xi_\bullet : [0,T] \to \mathbb{R}$ be piecewise constant.

Want: $\mathrm{E}[\overset{\text{two}}{\zeta_T} \cdot \overset{\text{two}}{\exp}(-i\int_0^T \xi_s\,d\tilde{W}_s)] = \overset{\text{one}}{\exp}(-\frac{1}{2}\int_0^T \xi_s^2\,ds)$

$\rho_\bullet := \mathrm{EM}_\bullet[-i\xi \overset{\text{five}}{-} \gamma]$ a mart. on $[0,T]$, & $\rho_0 = 1$;

then $\mathrm{E}[\overset{\text{five}}{\rho_T}] = 1$, which expands to this. QED

exercise hint . . .