Financial Mathematics

Models in continuous time – the basics

4000-1. Suppose $X_\bullet$ solves

$$dX_t = X_t(0.3\,dW_t + 0.02\,dt), \qquad X_0 = \ln 7.$$

You may assume, $\forall t > 0$, that $X_t > 0$ surely.

a. Find $\sigma_\bullet$ and $\gamma_\bullet$ s.t. $dX_t = \sigma_t(dW_t + \gamma_t, dt)$.

Express your answers for σ_t and γ_t in terms of X_t and t.

b. Solve the SDE above for $X_\bullet$.

Hint: Remember the trick of setting $Y_t := \ln(X_t)$ and using Itô.

c. $\forall T > 0$, verify that the Novikov quantity

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2\,dt\right)\right] \text{ is finite.}$$

4000-2. Let $dX_t = 0.3\,dW_t + 0.02\,dt$, $X_0 = \ln 7$
be our (real-world) model of the
logarithm of the price of the underlying.
Assume that a risk-free investment of 1 dollar
grows to $e^{(0.01)t}$ dollars at time t.

a. Find the price of a derivative
that pays $(6 - e^{X_8})_+$ dollars at time $t = 8$.

b. Find an expression $F(x,t)$ of x and t
s.t., if the price of the underlying at time t
is equal to x,
then the Δ-hedging strategy in part a above
involves holding $F(x,t)$ of the underlying.