**Notes on Measure Theory**

Definitions and Facts from **Topic 1500**

- For any set $M$, $2^M := \{subsets\ of\ M\}$ is called the **power set** of M.

  *The power set is the "set of all sets".*

- Let $\mathcal{A} \subseteq 2^M$. A function $\mu : \mathcal{A} \to [0, \infty]$ is **finitely additive** if, $\forall integer\ n \geq 1, \forall pw - dj\ A_1, ..., A_n \in \mathcal{A}, \bigsqcup A_j \in \mathcal{A} \Rightarrow \mu(\bigsqcup A_j) = \sum \mu(A_j)$.

  *Finite additivity states that if we have pairwise-disjoint sets in some space, the measure of the union of those disjoint sets is equal to the sum of the measures of the individual sets.*

- Let $\mathcal{A} \subseteq 2^M$. A function $\mu : \mathcal{A} \to [0, \infty]$ is **$\sigma$-additive** if $\mu$ is finitely additive & $\forall pw - dj\ A_1, ..., A_n \in \mathcal{A}, \bigsqcup A_j \in \mathcal{A} \Rightarrow \mu(\bigsqcup A_j) = \sum \mu(A_j)$.

  *$\sigma$-additive differs from finitely additive in that you can add infinitely many things. Specifically, you can add countably many things. $\sigma$-additive improves finitely-additive by making it "infinite".*

- Let $\mathcal{A} \subseteq 2^{\mathbb{R}}$. A function $\mu : \mathcal{A} \to [0, \infty]$ is **translation invariant** if $\forall A \in \mathcal{A}, \forall c \in \mathbb{R}$, we have: $A + c \in \mathcal{A}$ and $\mu(A + c) = \mu(A)$.

  *Translation invariance states that if we measure some interval, we should be able to move it and the measure should not change.*

- $\mathcal{I} := intervals \subseteq 2^{\mathbb{R}}$. $\lambda_0 : \mathcal{I} \to [0, \infty]$ defined by $\lambda_0(I) = [sup\ I] - [inf\ I]$ is called **length**.

  *For intervals based in the reals, the size of the interval is defined as the supremum of the interval minus the infimum.*

- Let $\mathcal{A} \subseteq \mathcal{B} \subseteq 2^M$. Let $\mu : \mathcal{A} \to [0, \infty], \nu : \mathcal{B} \to [0, \infty]$. We say that $\nu$ **extends** $\mu$ if: $\forall A \in \mathcal{A}, \mu(A) = \nu(A)$.

  *If you have two sigma algebras, one larger than the other, and a measure coincides in the sets from the smaller one, then it is said that the measure on the bigger sigma algebra extends the smaller measure.*

- Let $\mathcal{A} \subseteq \mathcal{B} \subseteq 2^M$. A function $\mu : \mathcal{A} \to [0, \infty]$ is **$\sigma$-finite** (on $M$) if $\exists A_1, A_2, ... \in \mathcal{A}$ s.t. $M = \bigcup A_j$ and $\forall j, \mu(A_j) < \infty$.

  *A function is $\sigma$-finite if all of the constituent sets have measure less than infinity.*

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**Fact:** Length is $\sigma$-additive, $\sigma$-finite, and translation invariant.

**Fact:** $\mathcal{I}$ is a near algebra on $\mathbb{R}$, length $\sigma$-additive and $\sigma$-finite.

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- $\mathcal{A}$ is an **algebra** (on $M$) if $M \in \mathcal{A}$ and $\forall A, B \in \mathcal{A}$, we have: $A \backslash B \in \mathcal{A}$.

  *Algebras exist to allow the formation of unions, intersections, and complements. If we have some measure space, we want to be able to partition it into pieces and measure those pieces; algebras allow that to happen.*

- $\mathcal{A}$ is a **$\sigma$-algebra** (on $M$) if $\mathcal{A}$ is an algebra in $M$ and $\bigcup \mathcal{A} \subseteq \mathcal{A}$.

  *A $\sigma$-algebra extends the definition of an algebra by allowing countably many unions and intersections. Like other $\sigma$ definitions, the $\sigma$-algebra allows the notion of "infinite" instead of "finite" operations.*

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**Fact:** $\mathcal{A}$ is an **algebra** (on M) iff $\mathcal{A} \neq null$ and $\mathcal{A}$ is closed under: finite union, finite intersection, and complement in $M$.

**Fact:** $\mathcal{A}$ is a **$\sigma$-algebra** (on M) iff $\mathcal{A} \neq null$ and $\mathcal{A}$ is closed under: countable union, countable intersection, and complement in $M$.

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- The $\sigma$-algebra (on M) **generated** by $\mathcal{S} \subseteq 2^M$ is the intersection of all of the $\sigma$-algebras on $M$ that contain $S$. It is denoted $< \mathcal{S} >_\sigma$.

  *Generation is a process that creates a $\sigma$-algebra with the smallest number of elements that still captures all relevant information about the $\sigma$-algebra.*

- $< \mu_0 >_\sigma := \mu$ is called the **measure generated** by $\mu_0$.

  *The generated measure extends our definition of length to allow for measuring $\sigma$-algebras.*

- A subset of $\mathbb{R}$ is **Borel** if it's an element of $< \mathcal{I} >_\sigma$. The unique extension of length to {Borel sets in $\mathbb{R}$} is called **Lebesgue measure** on $\mathbb{R}$.

  *A subset is Borel if it is an element of the sigma algebra generated by intersecting all intervals. Lebesgue measure can be thought of as the analogue to length on Borel sets. Virtually everything is Borel.*

Definitions and Facts from **Topic 1600**

- $P, Q \subseteq M$ are **essentially equal** (written $P \doteq Q$) if $\exists$null sets $Z, Z' \subseteq M$ s.t. $(P \backslash Z) \bigcup Z' = Q$.

  *Essentially equal sets can be visualized by imagining an interval [0,2]. Imagine sets P:=[0,0.5)(0.5,2] and Q:=[0,1)(1,2]. Although set P does not cover the point 0.5 and set Q does not cover the point 1, since both points have measure zero we say that the two are essentially equal.*

- A subset $C \subseteq M$ is **conull** in $M$ (or $\mu$-**conull**) if $M \backslash C$ is null.

  *If the complement of a subset is null, then the subset is said to be conull.*

- A subset of $P \subseteq M$ is **measurable** (or $\mu$-**measurable**) if $\exists A \in \mathcal{A}$ s.t. $P \doteq A$.

  *If we wish to measure a set P, but do not have a measure to do so, but another essentially equal set A is measurable, then we can measure P also.*

- The **completion** of (w.r.t. $\mu$) is $\bar{\mathcal{A}} := \{\mu - measurable\ sets\}$.

- The **completion** of $\mu$ is the function $\bar{\mu} : \bar{\mathcal{A}} \to [0, \infty]$; defined by: $\bar{\mu}(P) = \mu(A), \forall A \in \mathcal{A}$ s.t. $A \doteq P$.

  *Measurable sets are one step beyond Borel sets because one may need to add things of measure zero. We use the completion of the measure to measure such things, since regular measure cannot be applied. We need the definition of "essentially equal" to make the leap.*

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**Fact:** $A, B \in \mathcal{A}, A \doteq B \Rightarrow \mu(A) = \mu(B)$

**Fact:** $\bar{\mu} : \bar{\mathcal{A}} \to [0, \infty]$ is $\sigma$-additive.

**Fact:** {Borel sets in $\mathbb{R}$} is countably generated.
$< \{(a,b)|a,b \in \mathbb{Q}, a < b\} >_\sigma$

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- A subset of $\mathbb{R}$ is **measurable** if it's an element of the completion of {Borel sets in $\mathbb{R}$} w.r.t. Lebesgue measure.

  *It's hard to make non-measurable sets, or even non-Borel sets. This follows from the earlier definition of a Borel set, which was essentially defined as the sigma algebra generated by intersecting all intervals.*

- A $\sigma$-algebra $\mathcal{A}$ on $M$ is **countably generated** if $\exists$a countable set $\mathcal{C} \subseteq \mathcal{A}$ s.t. $\mathcal{A} =< \mathcal{C} >_\sigma$.

  *If A is a countably generated $\sigma$-algebra on M, then the elements of A are called* ***Borel sets****. If, furthermore, $\mu : \mathcal{A} \to [0, \infty]$ is $\sigma$-finite, then the elements of the completion of A w.r.t. $\mu$ are called measurable sets.*

- A **Borel space** is a set with a countably generated $\sigma$-algebra on it.

  *In an abstract space, if a countably-generated $\sigma$-algebra can be defined, we call it a Borel space.*

- A **measure** on a Borel space $(M, \mathcal{B})$ is a $\sigma$-additive function $\mu : \mathcal{B} \to [0, \infty]$.

- A **measure space** is a Borel space with a $\sigma$-finite measure on it.

- A measure $\mu$ on a Borel space $(M, \mathcal{B})$ is a **probability measure** if $\mu(M) = 1$.

- A measure $\mu$ on $M$ is **finite** if $\mu(M) < \infty$.

- **monotonicity:** $\forall A, B \in \mathcal{A}, A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

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**Fact:** $\mu$ is finite iff, $\forall A \in \mathcal{A}, \mu(A) < \infty$

**Fact:** measures are monotone.

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- For any countable set $M$, **counting measure** on $M$ is the measure $\mu : 2^M \to [0, \infty]$ defined by $\mu(S) = \#S$.

  *A measure on the (countably generated) Borel space* $(M, 2^M)$.

- $\forall$set $M$, $2^M$ is the **discrete** $\sigma$-algebra on $M$ and $\{null, M\}$ is the **indiscrete** $\sigma$-algebra on $M$.

  *If M is a countable set, then the power set of M is the discrete $\sigma$-algebra and the most coarse $\sigma$-algebra is the indiscrete.*

Definitions and Facts from **Topic 1700**

- The $\sigma$-algebra **inherited** (or **restricted**) from $\mathcal{A}$ to $W$ is $\mathcal{A}|W := \{A \bigcap W | A \in \mathcal{A}\}$.

  *TBD*

- A **Borel space** is a set with a countably generated $\sigma$-algebra on it. Sometimes called a **measure space**. A Borel space ($M$,$\mathcal{A}$) is **discrete** is $\mathcal{A} = 2^M$.

  *I always had the impression that a Borel space and a measure space were different things...*

- $\forall z \in \mathbb{C}, \forall \delta > 0, D(z, \delta) := \{w \in \mathbb{C} || w - z| < \delta\}$

- The **standard $\sigma$-algebra** on $\mathbb{C}$ is $\mathcal{B}_{\mathbb{C}} :=< \{D(z, \delta) | z \in \mathbb{C}, \delta > 0\} >_\sigma$.

  *TBD*

- $\forall A \subseteq \mathbb{C}$, the **standard $\sigma$-algebra** on $A$ is $\mathcal{B}_A := \mathcal{B}_{\mathbb{C}}|A$.

  *TBD*

- $\forall$finite $F$, **standard $\sigma$-algebra** on $F$ is $\mathcal{B}_F := 2^F$.

  *TBD*

- $\forall A \subseteq \mathbb{C}$, $\forall$finite $F$, the **standard $\sigma$-algebra** on $A \bigcup F$ is $\mathcal{B}_{A \bigcup F} :=< \mathcal{B}_A \bigcup \mathcal{B}_F >_\sigma$.

  *TBD*

- Let $\mathcal{A}$ and $\mathcal{B}$ be $\sigma$-algebras on $M$ and $N$, respectively. Let $\mathcal{C} := \{A \times B | A \in \mathcal{A}, B \in \mathcal{B}\}$.
  $\mathcal{A} \times \mathcal{B} :=< \mathcal{C} >_\sigma$ is called the **product** of $\mathcal{A}$ and $\mathcal{B}$.

  *TBD*

- $\mu \times \nu :=< \omega >_\sigma$ is called **product** of $\mu$ and $\nu$.

  *TBD*

- Let $\mathcal{B} := \{\textit{Borel sets in } \mathbb{R}\}$. A subset of $\mathbb{R}^2$ is **Borel** iff it's an element of $\mathcal{B} \times \mathcal{B}$. A subset of $\mathbb{R}^3$ is **Borel** iff it's an element of $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$. etc...

  *TBD*

- Let $\lambda$ be Lebesgue measure on $\mathbb{R}$. $\lambda \times \lambda$ is **Lebesgue measure** on $\mathbb{R}^2$. $\lambda \times \lambda \times \lambda$ is **Lebesgue measure** on $\mathbb{R}^3$. etc...

  *TBD*

- A subset of $\mathbb{R}^2$ is **measurable** if it's an element of $\bar{\mathcal{B}} \bar{\times} \bar{\mathcal{B}}$.

  *this principle applies for R n*

- Let $M$ be a set. Let $(N, \mathcal{B})$ be a Borel space. Let $f : M \to N$ be a function.
  $f^*(\mathcal{B}) := \{f^{-1}(B) | B \in \mathcal{B}\}$ is the **pull back** $\sigma$-algebra on M (from $\mathcal{B}$ via $f$).

  *TBD*

- Let $(M, \mathcal{A})$ be a Borel space. Let $(N, \mathcal{B})$ be a Borel space. Let $f : M \to N$ be a function.
  $f$ is $(\mathcal{A}, \mathcal{B})$-**Borel** (or just **Borel**) if $f^*\mathcal{B} \subseteq \mathcal{A}$.

  *The expression* $f^*\mathcal{B} \subseteq \mathcal{A}$ *can be alternatively expressed as* $\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}$

- Let $(M, \mathcal{A}, \mu)$ be a measure space. Let $(N, \mathcal{B})$ be a Borel space. Let $f : M \to N$ be a function.
  $f$ is $(\mu, \mathcal{B})$-**measurable** if $f^*\mathcal{B} \subseteq \bar{A}$ (or $\mu$-**measurable** or just **measurable**).

*The expression $f^*\mathcal{B} \subseteq \mathcal{A}$ can be alternatively expressed as $\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}$.*

- The measure $f_*^{\mathcal{B}} : \mathcal{B} \to [0, \infty]$ defined by $(f_*^{\mathcal{B}}(\mu))(B) = \bar{\mu}(f^{-1}(B))$ is the **push forward** measure on $N$ (on $\mathcal{B}$ from $\mu$ via $f$). (We usually just write $f_*(\mu)$.)

  *TBD*

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**Fact:** Functoriality of push-forward: $(g \circ f)_*(\mu) = (g_*(f_*(\mu))) = (g_* \circ f_*)(\mu)$.

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- Let $M$ be a set. Let $x \in M$.
  The **delta mass** (or **point mass**) at x (in $M$) is the measure $\delta_x : 2^M \to [0, \infty]$ (*or* $\delta_x^M$) defined by $\delta_x(A) = \{1, if\ x \in\ A; 0, if\ x \notin\ A\}$.

  *TBD*

- Let $S \in \bar{\mathcal{B}}$, i.e., let $S$ be measurable. We say $\mu$ is **concentrated off** $S$ if $\bar{\mu}(S) = 0$. We say $\mu$ is **concentrated on** $S$ if $\bar{\mu}(M \backslash S) = 0$.

  *TBD*

- $C \subseteq \mathbb{R}^d$ closed, $\nu$ a measure on $C$. Let $S \subseteq C$ be closed. We say $\nu$ is **supported on** $S$ if $\nu(C \backslash S) = 0$.

  *TBD*

- $C \subseteq \mathbb{R}^d$ closed, $\nu$ a measure on $C$. The **support** of $\nu$ the intersection of all of the closed sets that support $\nu$.

  *TBD*

Definitions and Facts from **Topic 2300**

- We say $f$ is **simple** if both $f : M \to \mathbb{R}$ is measurable and *f(M* is finite, in which case, $\int_M f\ d\mu := \sum_{y \in f(M)} y[\mu(f^{-1}(y))]$

  *TBD*

- For any set $S$, for any $R \subseteq S$, the function $1_R^S : S \to \{0, 1\}$ defined by $1_R^S(s) = \{1\, if\ s \in R; 0\, if\ s \in S \backslash R$ is the **indicator function** of $R$ (in

$S$).

*The indicator function is a simple switching technique, whereby the function equals 1 if s is contained in R, and 0 if s is not contained in R.*

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Definitions and Facts from **Topic 2330**

- We say $f$ is **integrable** or $L^1$ if both $\int_M f_+ d\mu < \infty$ *and* $\int_M f_- d\mu < \infty$.

  *We can say a function is $L^1$ if the integral of both the positive and negative parts of the function are finite.*

- We say $f$ is **integrable on A** or $L^1$ **on A** if $\chi f$ is integrable.

  *In this case, $\chi$ represents an indicator function that returns 1 if A is contained in M and 0 if it is not.*

- Let $(M, \mathcal{B})$ be a Borel space. Let $\mu$ and $\nu$ be two measures on $(M, \mathcal{B})$. We say $\nu$ is **absolutely continuous** w.r.t. $\mu$ if $\forall Z \in \mathcal{B}, \mu(Z) = 0 \Rightarrow \nu(Z) = 0$.

  *One measure is absolutely continuous with respect to another measure if, for some Z contained in the Borel set, Z's measure is the same when measured using both measures. An earlier definition of absolutely continuous used the notation $\nu << \mu$ to express "$\nu$ has at least as many null sets as $\mu$".*

- We say $\mu$ and $\nu$ are **equivalent** if both $\nu << \mu$ and $\mu << \nu$, i.e. if $\{Z \in \mathcal{B} | \mu(Z) = 0\} = \{Z \in \mathcal{B} | \nu(Z) = 0\}$.

  *If both measures of Z yield the same results, we say the measures are equivalent.*

Definitions and Facts from **Topic 2360**

- Measures $\rho$ and $\sigma$ on $(M, \mathcal{B})$ are **mutually singular** if $\exists Z \in \mathcal{B}$ *s.t.* $\rho(Z) = 0$ *and* $\sigma(M \backslash Z) = 0$.

  *TBD*

**Fact:** $h\mu << \mu, \forall measurable\ h : M \to [0, \infty)$

- Let $(M, \mathcal{B}, \mu)$ be a probability space. A **change of measure** (for $(M, \mathcal{B}, \mu)$) is a probability measure $\mu$ on $(M, \mathcal{B})$ s.t. $\mu \approx \nu$.

  *Suppose a measure exists on some Borel set. A change of measure may take place if there is a new measure on the same set which is equivalent (by the above definition of equivalent).*

**Fact:** Sophisticated change of variables formula:
Let $(M, \mathcal{A}, \mu)$ be a measure space.
Let $(N, \mathcal{B})$ be a Borel space.
Let $f : M \to N$ be measurable.
Let $g : N \to \bar{\mathbb{R}}$ be Borel.
$\int_M (g \circ f) d\mu = \int_N g d(f_* \mu)$

**Fact:** How to integrate against h:
$\int_M (fh) d\mu = \int_M f d(h\mu)$
$\int_M [f(x)][h(x)] d\mu(x) = \int_M f(x) d(h\mu)(x)$

**Fact:** $h := \frac{d\nu}{d\mu}$

**Fact:** $\phi_*(\phi' \cdot \lambda) = \lambda$

Definitions and Facts from **Topic 2400**

**Fact:** Monotone Convergence Theorem
Let $(M, \mathcal{A}, \mu)$ be a measure space. Let $f_1, f_2, f_3, ... : M \to [0, \infty]$ be a non-decreasing sequence of measurable functions. Assume $\lim_{n\to\infty} f_n \geq C$ on $M$. Then $\lim_{n\to\infty} \int_M f_n d\mu = \int_M \lim_{n\to\infty} f_n d\mu$.

**Fact:** Fatou's Lemma
Let $(M, \mathcal{A}, \mu)$ be a measure space. Let $g_1, g_2, g_3, ... : M \to [0, \infty]$ be measurable. Then $\int_M \liminf_{n\to\infty} g_n d\mu \leq \liminf_{n\to\infty} \int_M g_n d\mu$.

- A sequence $f_1, f_2, f_3, ... : M \to [-\infty, \infty]$ is **$L^1$-minorized** if $\exists L^1$ function $g : M \to [0, \infty]\ s.t.\ \forall integers\ n \geq 1, -g \leq f_n$.

*TBD*

- A sequence $f_1, f_2, f_3, ... : M \to [-\infty, \infty]$ is $L^1$**-majorized** if $\exists L^1$ function $g : M \to [0, \infty]$ $s.t. \; \forall integers \; n \geq 1, f_n \leq g.$

  *TBD*

- A sequence $f_1, f_2, f_3, ... : M \to [-\infty, \infty]$ is $L^1$**-enveloped** if $\exists L^1$ function $g : M \to [0, \infty]$ $s.t. \; \forall integers \; n \geq 1, -g \leq f_n \leq g.$

  *TBD*

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**Fact:** Bounded Convergence Theorem
Let $(M, \mathcal{A}, \mu)$ be a measure space.
Assume $\mu(M) < \infty$. Let $K > 0$. Let $f_1, f_2, f_3, ... : M \to [-K, K]$ be measurable and ptwise convergent. Then $\int_M \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_M f_n d\mu$.

**Fact:** Dominated Convergence Theorem
Let $(M, \mathcal{A}, \mu)$ be a measure space.
Let $f_1, f_2, f_3, ... : M \to [-\infty, \infty]$ be measurable, $L^1$-enveloped and ptwise convergent. Then $\int_M \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_M f_n d\mu$.

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Definitions and Facts from **Topic 2450**

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**Fact:** Let $I := [0, 1]$. $\forall measurable \; f : M \to I, \exists simple \; s : M \to I, \forall \varepsilon > 0, \; s.t. \; f - \varepsilon \leq s \leq f$ on $M$.

**Fact:** $\forall measurable \; f : M \to [0, \infty], \exists simple \; s_1, s_2, .... : M \to [0, \infty] \; s.t. \; s_1, s_2, ... \leq f \; and \; s.t. \; \lim_{n\to\infty} s_n = f$

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- A Borel space $(M, \mathcal{A})$ is **standard** if $\forall x, y \in M, x \neq y \Rightarrow \exists A \in \mathcal{A} \; s.t. \; x \in A \; and \; s.t. \; y \notin A$

  *Imagine a space $M$ and a $\sigma$-algebra $\mathcal{A}$. One point, $x$, may lie within $\mathcal{A}$. If the Borel space is standard, point $y$ does not lie within $\mathcal{A}$. The $\sigma$-algebra "separates points".*

- A measure space $(M, \mathcal{A}, \mu)$ is **standard** if $(M, \mathcal{A})$ is standard.

Definitions and Facts from **Topic 2500**

- For any set $S$, let $id_S : S \to S$ be the **identity function** on $S$, defined by: $id_S(s) = s$.

  *Definition of an identity function.*

- Two Borel space $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are **isomorphic** (or **Borel isomorphic**) if $\exists f : M \to N$ *Borel* $\exists g : N \to M$ *Borel s.t.* $g \circ f = id_M$ *and* $f \circ g = id_N$.

  *In this case, we say that* $f : M \to N$ *and* $g : N \to M$ *are Borel isomorphisms.*

- Two measure spaces $(M, \mathcal{A}, \mu)$ and $(N, \mathcal{B}, \nu)$ are **isomorphic** (or **measure isomorphic**) if $\exists f : M \to N$ *Borel* $\exists g : N \to M$ *Borel s.t.* $g \circ f = id_M$ *and* $f \circ g = id_N$ *and* $f_*(\mu) = \nu$ *and* $g_*(\nu) = \mu$.

  *In this case, we say that* $f : M \to N$ *and* $g : N \to M$ *are measure isomorphisms.*

- A measure space $(M, \mathcal{A}, \mu)$ is **standard** if $(M, \mathcal{A})$ is standard.

  *See above definition of a Borel space being standard.*

Definitions and Facts from **Topic 2700**

- Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. A **random variable** (or **RV**) on $(\Omega, \mathcal{B}, \mu)$ is a measurable map $\Omega \to \mathbb{R}$.

  *This is the generalized version of a PCRV. Whereas a PCRV is a function from the unit interval to the reals, we now have a map from a measure space to the reals.*

- Let $X : \Omega \to \mathbb{R}$ be a RV on $(\Omega, \mathcal{B}, \mu)$. The **distribution** fof X is $\delta_X := X_*(\mu)$

  *Here X is defined as a random variable (by the definition above). Since X is a random variable, it must have a distribution. That distribution is given by the notation defined here.*

- A measure on $\mathbb{R}$ is **proper** if every bounded interval has finite measure.

  *An example of this is Lebesgue measure on* $\mathbb{R}$.

- For any measure $\mu$ on $\mathbb{R}$, the **cumulative distribution function** (or **CDF**) of $\mu$ is the function $CDF_\mu : \mathbb{R} \to [0, \infty]$ defined by $CDF_\mu(s) = \mu((-\infty, s])$.

  *The cumulative distribution function describes the probability that a variable with a given distribution will be found at a value less than or equal to x. It is cumulative in the sense that as the value x increases, the total value returned by the function increases.*

- $f : \mathbb{R} \to \mathbb{R}$ is **cadlag** if $\forall a \in \mathbb{R}, \lim_{x \to a^+}(f(x)) = f(a)$ *and* $\lim_{x \to a^-}(f(x))$ exists.

  *Cadlag is an acronym for a French phrase that translates to "continuous from right and limit from left," which adequately describes the process taking place.*

- $f : \mathbb{R} \to \mathbb{R}$ is **CDF type** if $f$ is nondecreasing, bounded, cadlag and $f(-\infty) = 0$.

  *TBD*

- $f : \mathbb{R} \to \mathbb{R}$ is a **regular CDF** if $f$ is increasing, continuous, and $f(-\infty) = 0, f(\infty) = 1$.

  *TBD*

- $\mu$ is a **regular distribution** if $\mu$ is a probability measure, $\forall x, \mu(\{x\}) = 0$ *and* $\forall (a < b), \mu((a, b)) > 0$.

  *TBD*

- Let $\mu$ be a measure on $\mathbb{R}$. Then a measurable function $h : \mathbb{R} \to [0, \infty]$ is a **probability density function (PDF)** for $\mu$ if $\mu = h\lambda$.

  *TBD*

- A RV is **standard normal** if $\phi'$ is a PDF of its distribution.

  *TBD*

- $C_B := \{continuous,\ bounded\ functions\ \mathbb{R} \to \mathbb{R}\}$

  *TBD*

- $C_E := \{continuous,\ exponentially-bounded\ functions\ \mathbb{R} \to \mathbb{R}\}$.

  *TBD*

- Let $\mu_1, \mu_2, ...$ and $\nu$ be probability measures on $\mathbb{R}$. Then $\mu_1, \mu_2, ... \to \nu$ means: $\forall f \in C_B, \int_{\mathbb{R}^n} f d\mu_n \to \int_{\mathbb{R}^n} f d\nu$. The same expression holds $\forall f \in C_E$ (continuous, exponentially-bounded).

  *TBD*

- For any probability measure $\mu$ on $\mathbb{R}$, the **Fourier transform** of $\mu$ is the function $\mathcal{F}\mu : \mathbb{R} \to \mathbb{C}$ defined by $\mathcal{F}\mu := \int_{-\infty}^{\infty} e^{-itx} d\mu(x)$.

  *TBD*

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**Fact:** $\mathcal{F}\mu_n \to \mathcal{F}\nu, \forall t \Rightarrow \mu_n \to \nu$.

**Fact:** $\mathcal{F}(\phi'\lambda) = e^{-t^2/2}$

**Corollary:** A RV $X$ is **standard normal** iff $\mathcal{F}\delta_X = e^{-t^2/2}$, i.e. the Fourier transform of the distribution of $X$ is $e^{-t^2/2}$.

**Fact:** $\forall$probability measures $\mu$ and $\nu$ on $\mathbb{R}, \mathcal{F}(\mu * \nu) = [\mathcal{F}\mu][\mathcal{F}\nu]$.

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- The **standard normal distribution** on $\mathbb{R}$ is $\phi'\lambda$.

  *TBD*

- A RV is **standard normal** if $\phi'$ is a PDF on its distribution. i.e.: A RV X is **standard normal** iff $\delta_X = \phi'\lambda$, i.e., the distribution of $X$ is the standard normal distribution.

  *TBD*

- Define $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $A(x, y) = x + y$. For any two measures $\mu$ and $\nu$ on $\mathbb{R}$, the **convolution** of $\mu$ and $\nu$ is the measure $\mu * \nu$ on $\mathbb{R}$ defined by $\mu * \nu := A_*(\mu \times \nu)$.

  *TBD*

- A measure space $(M, \mathcal{B}, \mu)$ is **nonatomic** if $\forall x \in M, \mu(\{x\}) = 0$.

  *A measure space is nonatomic if all the points in M have measure zero.*

Definitions and Facts from **Topic 2800**

- A $\sigma$**-finite signed measure** on $(M, \mathcal{B})$ is a function $\omega : \mathcal{B} \to [-\infty, \infty]$ s.t., for some pair $\mu, \nu$ of $\sigma$-finite measures on $(M, \mathcal{B})$ we have $(\mu(M), \nu(M)) \neq (\infty, \infty)$ and $\omega = \mu - \nu$, i.e., $\forall B \in \mathcal{B}, \omega(B) = (\mu(B)) - (\nu(B))$.

  *TBD*

- $v : [a, b] \to \mathbb{R}$ has **bounded variation** if $\exists$nondecreasing $f, g : [a, b] \to \mathbb{R}$ $s.t.$ $v = f - g$.

  *TBD*

- Let $v : [a, b] \to \mathbb{R}$ have bounded variation, and let $f, g : [a, b] \to \mathbb{R}$ be nondecreasing functions such that $v = f - g$. We define $dv := df - dg$.

  *TBD*

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**Fact:** Let $v : [a, b] \to \mathbb{R}$ be continuous, differentiable on *(a,b)*. Then $v$ has bounded variation. Then $[f(\nu(b))] - [f(\nu(a))] = \int_a^b f'(\nu(t))d\nu(t)$.

---

Definitions and Facts from **Topic 2900**

- An **event** (in $\Omega$ or in $(\Omega, \mathcal{A}, \mu)$) is a measurable subset of $\Omega$.

  *We define an event to be something measurable in a set.*

- The **probability** of an event $E$ is $Pr[E] := \bar{\mu}(E)$. We write $E$ a.s. if Pr[$E$]=1.

  *The probability of an event is defined as the completed measure of the event.*

- Let $(M, \mathcal{B})$ be a standard Borel space. An $(M, \mathcal{B})$**-RV** or $\boldsymbol{M}$**-RV** (on $(\Omega, \mathcal{A}, \mu)$) or on $\Omega$ is a measurable map $\Omega \to M$.

  *Just like a PCRV is a function from* $[0, 1] \to \mathbb{R}$, *an RV works in a similar way, mapping values from* $\Omega$ *to a* $\sigma$*-algebra.*

- Let $X : \Omega \to M$ be an ***M*-RV**. For all Borel $S \subseteq M$, the event $X \in S$ is $\{\omega \in \Omega | X(\omega) \in S\} = X^{-1}(S)$.

  *TBD*

- Referencing the definition above, $X$ is **deterministic** if $\exists p \in M$ s.t. $X{=}p$ a.s.

  *TBD*

- A **RV** (on $(\Omega, \mathcal{A}, \mu)$ or on $\Omega$) is a measurable map $\Omega \to \mathbb{R}$.

  *Like the definition of an **M-RV** from above, the definition of a random variable is a direct analogue to a PCRV mapping from the unit interval to the reals.*

- Let $X : \Omega \to \mathbb{R}$ be a RV on $\Omega$. The **event** $a \geq X$ is the event $X \in (-\infty, a]$; the **event** $a < X \leq b$ is the event $X \in (a, b]$;, etc.

  *If we are trying to measure the probability of an event, we are essentially testing whether the event falls within a certain interval. For example:* $Pr[1 \leq X] = \bar{\mu}(X^{-1}([1, \infty)))$.

- Let $X : \Omega \to \mathbb{C}$ be a $\mathbb{C}$-RV on $\Omega$. The **expectation** or **mean** of $X$ is $E^{\mu}[X] := E[X] := \int_{\Omega} X d\mu$.

  *The definition of expectation for a complex random variable is similar to the definition of expectation for other random variables. We integrate X over the space.*

- $X$ is **integrable** or $L^1$ if $E[|X|]$ is finite; in this case we define: $X^{\circ} := X - (E[X])$, so that $X^{\circ}$ has mean zero.

  *If X is integrable, we define a new* $X^{\circ}$ *such that we normalize the new X to have mean zero.*

- $X$ is **square integrable** or $L^2$ if $E[|X|^2] < \infty$.

  *If the expectation of the square of X is finite, then X is square integrable.*

**Fact:** *$X$ is $L^2 \Rightarrow X$ is $L^1$ and $X^\circ$ is $L^2$*

**Fact:** $X$ is deterministic iff Var[X]=0

**Fact:** $X$ is square integrable if $Var[X] < \infty$

**Fact:** $X$ and $Y$ are uncorrelated if Cov[X,Y]=0

---

- Let $X : \Omega \to \mathbb{R}$ be an $L^2$ RV on $\Omega$. The **variance** of $X$ is $Var[X] := E[(X^\circ)^2]$.

  *TBD*

- Let $X : \Omega \to \mathbb{R}$ be square integrable. $X$ is **standard** if both E[X]=0 and Var[X]=1.

- The **standard deviation** of $X$ is $SD[X] := \sqrt{Var[X]}$.

- Let $X, Y : \Omega \to \mathbb{R}$ be square integrable. The **covariance** of $X, Y$ is Cov[X,Y]:=((Var[X+Y]-Var[X]-Var[Y])/2).

- Let $X, Y : \Omega \to \mathbb{R}$ be square integrable and non-deterministic. The **correlation** of $X, Y$ is Corr[X,Y]:=Cov[X,Y]/(SD[X]SD[Y]).

- For all Borel space $(N, \mathcal{C})$, for all Borel $f : M \to N, f(X) := f \circ X$.

  *Definition of composition. The traditional notation for a function f(X) is defined as f composed with X.*

- The **distribution** of $X$ is $\delta^\mu[X] = \delta[X] = \delta^\mu_X = \delta_X := X_*(\mu)$.

  *Various alternative notations for a distribution.*

---

**Fact:** For all Borel space $(N, \mathcal{C})$, for all Borel $f : M \to N, \delta_{f(X)} = f_*(\delta_X)$

**Fact:** For all Borel $S \subseteq M, Pr[X \in S] = \delta_X(S)$

**Fact:** Say $M = \mathbb{R}$. Then $E[f(X)] = \int_{-\infty}^{\infty} f d\delta_X$

---

- The **joint variable** of $X$ and $Y$ is the $(M \times N) - RV$ $(X, Y) : \Omega \to M \times N$ defined by $(X, Y)(\omega) = (X(\omega), Y(\omega))$.

  *TBD*

- $\delta_{X,Y} := \delta_{(X,Y)}$ is the **joint distribution** of $X$ and $Y$; it's a measure on $M \times N$.

  *TBD*

- Let $(M, \mathcal{B}), (N, \mathcal{C})$ be standard Borel spaces. Define $p : M \times N \to M, q : M \times N \to N$ by $p(x, y) = x$ *and* $q(x, y) = y$. $\forall$measure $\omega$ on $M \times N$, the **marginals** of $\omega$ are $p_*(\omega)$ *and* $q_*(\omega)$; they are measures on $M$ and $N$, respectively.

  *When dealing with joint variables and joint distributions, the individual variables and distributions that make up the joint are known as the marginals of the joint.*

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**Fact:** For all measures $\mu$ on $M$ and $\nu$ on $N$, $p_*(\mu \times \nu) = \mu$ and $q_*(\mu \times \nu) = \nu$. *"The marginals of a product measure are the factor measures."*

**Fact:** For all standard probability space $(\Omega, \mathcal{B}, \mu), \forall M - RV\ X : \Omega \to M, \forall N - RV\ Y : \Omega \to N, p_*(\delta_{X,Y}) = \delta_X$ *and* $q_*(\delta_{X,Y}) = \delta_Y$ *"The marginals of a joint distribution are the individual distributions."*

**Fact:** $E[f(X, Y)] = \int_{\mathbb{R}^2} f(x, y) d\delta_{X,Y}(x, y)$

---

- $\forall E \in \mathcal{A}, \forall F \in \mathcal{A}, E$ and $F$ are **info-equivalent** if $E \Delta F$ is null, where $\mathcal{A}$ is an event.

  *The definition essentially states that if we know whether or not $\omega \in E$, then we also know whether or not $\omega \in F$.*

- For all sets *E,F*, the **symmetric difference** of $E$ and $F$ is $E \Delta F := (E \backslash F) \bigcup (F \backslash E)$.

  *TBD*

- Let $X : \Omega \to M$ be an $M$-RV. The $\sigma$-**algebra** of X is $\mathcal{S}_X^{\mathcal{B}} := \mathcal{S}_X := X^*(\mathcal{B}) := \{X^{-1}(B) | B \in \mathcal{B}\}$.

*TBD*

- For any $\sigma$-subalgebra $\mathcal{S}$ of $\bar{\mathcal{A}}$, we say that $X$ is **$\mathcal{S}$-measurable** if $\bar{\mathcal{S}_X} \subseteq \mathcal{S}$.

  *TBD*

- Two events $E, F \in \bar{\mathcal{A}}$ are **independent** if $\bar{\mu}(E \bigcap F) = [\bar{\mu}(E)][\bar{\mu}(F)]$

  *This definition of independence is analogous to the definition by expectation: E[XY]=E[X]E[Y]. This definition adapts the traditional definition to include events.*

- Two subsets $\mathcal{E}, \mathcal{F} \subseteq \bar{\mathcal{A}}$ are **independent** if, $\forall E \in \mathcal{E}, F \in \mathcal{F}$, $E$ and $F$ are independent.

  *If all the events present in a subset are independent from all the events in another subset, we say the two subsets are independent.*

- An $M$-RV $X$ and a subset $\mathcal{E} \subseteq \bar{\mathcal{A}}$ are **independent** if $\mathcal{S}_X$ and $\subseteq E$ are independent.

  *A random variable on some Borel space M and a subset are independent if the $\sigma$-algebra and the subset are independent. Refer to the definition a few rows above to review the $\sigma$-algebra definition.*

- An $M$-RV $X$ and an $N$-RV $Y$ are **independent** if $\mathcal{S}_X$ and $\mathcal{S}_Y$ are independent.

  *Two random variables on respective Borel spaces are independent if their $\sigma$-algebras are independent.*

---

**Fact:** $X$ and $Y$ are **independent** iff $\delta_{X,Y} = \delta_X \times \delta_Y$.

**Fact:** *X, Y* independent, $(P, \mathcal{D}), (Q, \mathcal{E})$ standard Borel spaces $\Rightarrow \forall$*Borel* $f : M \to P$, $\forall$*Borel* $g : N \to Q$, *f(X)* and *g(Y)* are independent.

**Fact:** Say $M = N = \mathbb{R}$ and $X, Y, L^2$. Then if $X, Y$ are independent then they are uncorrelated. Cov[X,Y]=(E[XY])-(E[X])(E[Y]).

**Fact:** $\mathcal{F}(\mu * \nu) = (\mathcal{F}\mu)(\mathcal{F}\nu)$

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- Define: $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $A(x, y) = x + y$. For all measures $\mu, \nu$ *on* $\mathbb{R}, \mu * \nu := A_*(\mu \times \nu)$ is the **convolution** of $\mu$ and $\nu$.

  *Convolution is the process by which we take the distributions of two random variables and multiply them instead of summing them.*

- The **joint variable** of $X_1, ..., X_n$ is the $M$-RV $(X_1, ..., X_n) : \Omega \to M$ defined by $(X_1, ..., X_n)(\omega) = (X_1(\omega), ..., X_n(\omega))$. For all standard Borel spaces $(\mathcal{P}, \mathcal{D})$, for all Borel $f : M \to \mathcal{P}, f(X_1, ..., X_n) := f((X_1, ..., X_n)) = f \circ (X_1, ..., X_n)$; it's a $P$-RV on $\Omega$.

  *TBD*

- $\delta_{X_1,...,X_n} := \delta_{X_1,...,X_n}$ is the **joint distribution** of $X_1, ..., X_n$; it's a measure on $M$.

  *TBD*

- $X_1, ..., X_n$ are **(jointly) independent** if $\delta_{X_1,...,X_n} = \delta_{X_1} \times ... \times \delta_{X_n}$.

  *TBD*

- For all measure $\tau$ on $M$, the **marginals** of $\tau$ are $(p_1)_*(\tau), ..., (p_n)_*(\tau)$; they are measures on $M_1, ..., M_n$, respectively.

  *TBD*

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**Fact:** For all measures $\mu_1$ on $M_1, ..., \mu_n$ on $M_n, \forall k, (p_k)_*(\mu_1 \times ... \times \mu_n) = \mu_k$.

**Fact:** For all standard probability space $(\Omega, \mathcal{A}, \mu), \forall M_1 - RV\ X_1, ..., \forall M_n - RV\ X_n,\ all\ on\ \Omega, \forall k, (p_k)_*(\delta_{X_1,...,X_n}) = \delta_{X_k}$

**Fact:** For all Borel $S_1 \subseteq M1, ..., S_n \subseteq M_n, Pr[(X_1 \in S_1) \& ... \& (X_n \in S_n)] = (Pr[X_1 \in S_1]) ... (Pr[X_n \in S_n])$

**Fact:** For all standard Borel spaces $(P_1, \mathcal{D}_1), ..., (P_n, \mathcal{D}_n), \forall f_1 : M_1 \to P_1, ..., \forall f_n : M_n \to P_n$, all Borel, $f_1(X_1), ..., f_n(X_n)$ are jointly independent.

**Fact:** $\delta_{X_1+...+X_n} = \delta_{X_1} * ... * \delta_{X_n}$
**Corollary:** $\mathcal{F}\delta_{X_1+...+X_n} = (\mathcal{F}\delta_{X_1})...(\mathcal{F}\delta_{X_n})$

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- Let $X$ be a random variable. Let $F : \mathbb{R} \to [0,1]$ be the CDF of (the distribution of) $X$. The **grade** of $X$ is gr$[X]$:=$F(X)$.

  *TBD*

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**Fact:** If $X$ has no values of positive probability, i.e., if, $\forall c \in \mathbb{R}, Pr[X = c] = 0$, *then* $\delta[gr[X]]$ is Lebesgue measure on [0,1].

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- The **joint distribution** of $X_1, ..., X_n$ is $\delta[X_1, ..., X_n] := (X_1, ...X_n)_*(\mu)$, a probability measure on $\mathbb{R}^n$.

  *TBD*

- The **copula** of $X_1, ..., X_n$ is $cop[X_1, ..., X_n] := \delta[gr[X_1], ..., gr[X_n]]$.

  *TBD*

- $X_1, X_2, ...$ $M$-RVs,... (all on $\Omega$) $X_1, X_2, ...$ are **iid** means both $X_1, X_2, ...$ are (jointly) independent and for all integers $j, k, \geq 1, \delta_{X_j} = \delta_{X_k}$

  *TBD*

Definitions and Facts from **Topic 3000**

- Let $(\Omega, \mathcal{A}, \mu)$ be a standard probability space. Let $E$ be an event, i.e., a measurable subset of $\Omega$, so $E \in \bar{\mathcal{A}}$. The **probability** of $E$ is defined by $Pr[E] := \bar{\mu}(E)$.

  *TBD*

- Let $E$ and $F$ be events. The **conditional probability** of $E$ given $F$ is defined by $Pr[E|F] := \frac{\bar{\mu}(E \bigcap F)}{\bar{\mu}(F)}$.

  *TBD*

- Let $E$ be an event and let $X$ be an $L^1$ RV. The **conditional expectation** of $X$ given $E$ is $E[X|E] := \frac{1}{\mu(E)} \int_E X d\mu$.

  *TBD*

- The **conditional expectation** of $V$ given $\mathcal{P}$ is the RV $E[V|\mathcal{P}] : \Omega \to \mathbb{R}$ defined by $(E[V|\mathcal{P}])(\omega) = E[V|P_\omega]$. Here, $\mathcal{P}$ is a finite partition of $\Omega$.

  *TBD*

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**Fact:** Let $U := E[V|\mathcal{P}]$. Then $U$ is $< \mathcal{P}_\sigma >$-measurable, and, $\forall P \in < \mathcal{P} >_\sigma$ of positive measure, $E[U|P] = E[V|P]$.

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- TBD

Definitions and Facts from **Topic 3100**

- Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $\Omega$. We say that $\mathcal{P}$ is **finer** than $\mathcal{Q}$ if: $\forall P \in \mathcal{P}, \exists Q \in \mathcal{Q}\ s.t.\ P \subseteq Q$.

  *If the question "Which set in $\mathcal{P}$ contained $\omega$?" gives enough info to answer "Which set in $\mathcal{Q}$ contained $\omega$?", then we say that $\mathcal{P}$ is finer than $\mathcal{Q}$.*

---

**Fact:** $\mathcal{P}$ finer than $\mathcal{Q} \Rightarrow \forall Q \in \mathcal{Q}, \exists P_1, ..., P_k \in \mathcal{P}\ s.t.\ Q = P_1 \sqcup ... \sqcup P_k$.

**Fact:** $\mathcal{P}$ finer than $\mathcal{Q}$ implies any $\mathcal{Q}$-measurable RV is $\mathcal{P}$-measurable.

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- **The Tower Law:** Let $V$ be a $L^1$ RV. Let $\mathcal{P}$ and $\mathcal{Q}$ be finite, positive measure partitions of $\Omega$. Assume that $\mathcal{P}$ is finer than $\mathcal{Q}$. Then $E[E[V|\mathcal{P}]|\mathcal{Q}] = E[V|\mathcal{Q}]$.

  *Forcing $\mathcal{P}$-measurability is weaker than forcing $\mathcal{Q}$-measurability, so doing both is redundant.*

- Let $\mathcal{S}$ and $\mathcal{T}$ be $\sigma$-subalgebras on $\Omega$. We say that $\mathcal{S}$ than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{S}$.

  *TBD*

- **The Power Tower Law:** Let $V$ be an $L^1$ RV. Let $\mathcal{S}$ be a $\sigma$-algebra on $\Omega$. Then $E[E[V|\mathcal{S}]] = E[V]$.

  *TBD*

Definitions and Facts from **Topic 3200**

- For all functions $f, g : \mathbb{R} \to \mathbb{R}$, the **convolution** of $f$ and $g$ is the function $f * g$ defined by $(f * g)(x) = \int_{-\infty}^{\infty} [f(t)][g(x - t)]dt$.

  *This definition was used on the final exam from last year in one of the first computation problems*

- $\Gamma(s) := [\int_{-\infty}^{\infty} z^x e^{-e^x} dx]_{z:\to e^s}$

  *This is the definition of the gamma function, which is used to calculate the PDF for the chi-squared distribution. There is a simpler way to define the gamma function in a practical way, as given in the two definitions below.*

- For integer values of $n$, the result of the gamma function is given by $(n - 1)!$

- For non-integer values of $n$, the result of the gamma function is given by $\frac{(2n)!}{4^n n!}\sqrt{\pi}$. Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

- $(*^n \psi)(x) = \frac{x^{(n/2)-1} e^{-x/2}}{2^{n/2}\Gamma(n/2)}$

  *This function yields the PDF for a chi-squared distribution with n degrees of freedom. Note that it relies on the above definition of the gamma function.*

---

**Fact:** Let $(M, \mathcal{B}), (N, \mathcal{C}), (P, \mathcal{D})$ be Borel spaces. Let $F : M \times N \to P$ be Borel. Let $x \in M$ and let $\nu$ be a measure on $N$. Then $F_*(\delta_x \times \nu) = (F(x, \bullet))_*(\nu)$.

**Fact:** Let $(M, \mathcal{B}), (N, \mathcal{C})$ be Borel spaces. Let $F : M \to N$ be a Borel isomorphism. Let $\mu$ be a measure on $M$. Let $g : M \to [0, \infty]$ be measurable. Then $F_*(g\mu) = [g \circ F^{-1}][F_*(\mu)]$.

**Fact:** If $f$ is a PDF for $\mu$ and if $g$ is a PDF for $\nu$ then $f * g$ is a PDF for $\mu * \nu$.

Definitions and Facts from **Topic 3300**

- TBD

Definitions and Facts from **Topic 3400**

- TBD

Definitions and Facts from **Topic 3500**

- The **exponential distribution** describes the time between events that occur continuously and independently at a constant average rate.
- The CDF for the exponential distribution is as follows: $CDF_{\delta_X}(x) := \{1 - e^{-\alpha x}, if\ x \geq 0; 0, if\ x \leq 0$
- The PDF for the exponential distribution is as follows: $PDF_{\delta_X}(x) := \{\alpha e^{-\alpha x}, if\ x > 0; 0, if\ x < 0$

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**Fact:** $X$, $Y$ independent $\Rightarrow \delta_{X+Y} = \delta_X * \delta_Y$

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- TODO: Gamma, Poisson, and empirical distributions

Definitions and Facts from **Topic 3600**

- Fix a probability space $(\Omega, \mathcal{B}, \mu)$. For all Borel spaces $(\mathbb{T}, \mathcal{A})$, a $\mathbb{T}$-process is a function $X : \mathbb{T} \to \{RVs\ on\ \Omega\}\ s.t.\ (\omega, t) \mapsto (X(t))(\omega) : \Omega \times \mathbb{T} \to \mathbb{R}$ is measurable.

  *A process is a series of random variables together in a sequence that describe the evolution of a path of some kind.*

- A **process** is a $[0, \infty)$-process.

  *By default, a process is defined on the positive real numbers.*

- A **spacetime-process** is an $(\mathbb{R} \times [0, \infty))$-process.

  *A process may be defined in terms of both space and time. For instance, a Brownian motion has parameters that describe both the position of a particle and a related time index.*

- A process $X_\bullet$ ($X_t$ *or* $X(t)$) is **continuous** if $\forall\omega \in \Omega, t \mapsto X_t(\omega) : [0,\infty) \to \mathbb{R}$ is continuous.

  *Some processes are continuous, like Brownian motion, and others are not, like a Levy process.*

- For any set $\mathbb{T} \subseteq \mathbb{R}$, a **$\mathbb{T}$-filtration** is a function $\mathcal{F}_\bullet : \mathbb{T} \to \{\sigma - subalgebras\}$ s.t. $t, u \in \mathbb{T}, t \leq u \Rightarrow \mathcal{F}_t \subseteq \mathcal{F}_u$.

  *A filtration can be thought of a $\sigma$-algebra that becomes increasingly finer as time passes. The $\sigma$-algebra allows for the measurement of the process. If one thinks of data appearing on a screen and that data is changing, a filtration is a collection of those data.*

- A RV $X : \Omega \to \mathbb{R}$ is **$\mathcal{S}$-measurable** if for all Borel $B \subseteq \mathbb{R}, X^{-1}(B) \in \mathcal{S}$.

  *TBD*

- $X_\bullet$ is **$\mathcal{F}_\bullet$-adapted** means: $\forall t \in \mathbb{T}, X_t$ is $\mathcal{F}_t$-measurable.

  *TBD*

- The **filtration** of $X_\bullet$ is the filtration $\mathcal{F}_\bullet^X$ defined by $\mathcal{F}_t^X := < \bigcup_{s \leq t} \mathcal{S}_{X_s} >_\sigma$.

  *As mentioned in the definition of filtration above, the filtration is an increasingly finer $\sigma$-algebra.*

- The $(t_1, ..., t_d)$**-marginal** of $X_\bullet$ is the joint distribution $\delta[X_{t_1}, ..., X_{t_d}]$.

  *TBD*

- $X_\bullet = Y_\bullet$ **in finite dimensional (f.d.) marginals**, written $X_\bullet \stackrel{\delta}{=} Y_\bullet$ means: for all integers $d \geq 1, \forall t_1, ...t_d \in [0,\infty), \delta[X_{t_1}, ..., X_{t_d}] = \delta[Y_{t_1}, ..., Y_{t_d}]$.

  *TBD*

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**Fact:** *Any* process is adapted to its filtration.

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**Various Useful Facts You Should Know**

- $E[c\mathcal{A}] = c(E[\mathcal{A}])$

- $Var[c\mathcal{A}] = c^2 Var[\mathcal{A}]$
- $SD[c\mathcal{A}] = |c| SD[\mathcal{A}]$

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- $E[c + \mathcal{A}] = c + E[\mathcal{A}]$
- $Var[c + \mathcal{A}] = Var[\mathcal{A}]$
- $SD[c + \mathcal{A}] = SD[\mathcal{A}]$

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- $E[\sum^n \mathcal{A}] = n \times E[\mathcal{A}]$
- $Var[\sum^n \mathcal{A}] = n \times Var[\mathcal{A}]$
- $SD[\sum^n \mathcal{A}] = \sqrt{n} \times SD[\mathcal{A}]$

*Finite is to algebra as countable is to sigma algebra*
*sigma-additive is a more "robust" form of additivity than finitely-additive??*
*sigma is another way to say infinite*