Pricing and hedging in incomplete markets

Chapter 10

From Chapter 9:

- Pricing Rules:
  Market complete+nonarbitrage$\Longrightarrow$ Asset prices

- The idea is based on perfect hedge:

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0$$

- With completeness, any contingent claim can be perfectly hedged.

- With nonarbitrage, V_0 could pin down.

Also From Chapter 9:

- Market completeness breaks down when there are even small jumps
- So without perfect hedges, the risk to do hedging can't be completely ruled out, we have to find ways out.

In this chapter:

- Merton's approach(10.1): ignore the extra risks$\Longrightarrow$pin down pricing and hedging
- Superhedging (10.2): leads to a bound for prices(preference-free, but the bound is too wide)
- Expected utility max(10.3): choosing hedge by min some measure of hedging errors$\Longrightarrow$utility indifference price
- Special case of the above where the loss function is quadratic (10.4)

Merton's Approach:

- In Merton:

$$S_t = S_0 \exp\left[\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right]$$

  W_t: SBM; N_t:Poisson process with $\lambda; Y_i \sim N(m, \delta^2)$

- He assigns a choice from many risk-neutral measures:

$$Q_M : S_t = S_0 \exp\left[\mu^M t + \sigma W_t^M + \sum_{i=1}^{N_t} Y_i\right]$$

Merton's Approach:

- Q_M just shift the drift of the BM, and left the jumps unchanged

- Rationale: jump risks are diversifiable, so no risk premium/no change of measure upon it.

- Application: Euro option with $H(S_T)$ has price process:

$$\Pi_t^M = e^{-r(T-t)} E^{\mathbb{Q}_M}[H(S_T) | \mathcal{F}_t]$$

Merton's Approach:

- Furthermore, since S_t is a Markov process(under $\mathbb{Q}_M$), so $\mathcal{F}_t$ contains as much info as S_t, thus:

$$\Pi_t^M \ = \ e^{-r(T-t)} E^{\mathbb{Q}_M}[(S_T - K)^+ | S_t = S]$$

- Then by conditioning on the # of jumps N_t, we can express Π_t^M as a weighted sum of B-S prices, finally, we get(set $\tau = T - t$):

$$\Pi(\tau, S; \sigma) = e^{-r\tau} E[H(Se^{(r-\sigma^2/2)\tau + \sigma W_\tau})]$$

Merton's Approach:

- For call and put options ,apply Ito to $e^{-rt}C(t, S_t)$.

$$\hat{\Pi}_t^M = e^{-rt}\Pi_t^M = E^{\mathbb{Q}_M}[e^{-rT}(S_T - K)^+ | \mathcal{F}_t]$$

- the discounted value is a martingale under $\mathbb{Q}_M$, so

$$\hat{\Pi}_T^M - \hat{\Pi}_0^M = \hat{H}(S_T) - E^{\mathbb{Q}_M}[H(S_T)]$$

- Merton gives the hedging portfolio (ϕ_t^0, ϕ_t): $\phi_t = \frac{\partial \Pi^M}{\partial S}(t, S_{t-})$ and $\phi_t^0 = \phi_t S_t - \int_0^t \phi dS$

Merton's Approach:

- From this self-financing strategy, the risk from the diffusion part is hedged, but the discounted hedging error is:

$$\hat{H} - e^{-rT} V_T(\phi) = \hat{\Pi}_T^M - \hat{\Pi}_0^M - \int_0^t \frac{\partial \Pi^M}{\partial S}(u, S_{u-}) d\hat{S}_u$$

- Go back to Merton's rational, how could we hedge jump risk: he assumes the jumps across the stocks are indenp, so in a large market a diversified portfolios such as market index would not have jumps, 'coz they cancel out each other.

Superhedging:

- A conservative approach to hedge: $\mathbb{P}(V_T(\phi) = V_0 + \int_0^t \phi dS \geq H) = 1$ Here ϕ is said to superhedge against the claim H.

- Defn:The cost of superhedging: the cheapest superhedging strategy,

$$\Pi^{sup}(H) = inf\{V_0, \exists \phi \in S, \mathbb{P}(V_0 + \int_0^T \phi dS \geq H) = 1$$

Superhedging:

- Intuition: When some option writer/seller is willing to take the risk at some certain price, it means he can at least partially hedge the option with a cheaper cost, thus the this price represents an upper bound for the option.
- Similarly, the cost of superhedging a short position in H, given by $-\Pi^{sup}(-H)$ gives a lower bound on the price.
- Henceforth, we pin down an interval:

$$[-\Pi^{sup}(-H), \Pi^{sup}(H)]$$

Superhedging:

Prop10.1 Cost of superhedging:

- Consider a European option with a positive payoff H on an underlying asset described by a semimartingale $(S_t)_{t\in[0,T]}$ and assume that

$$\sup_{\mathbb{Q}\in M(S)} E^{\mathbb{Q}}[H] < \infty$$

Then the following duality relation holds:

$$\inf_{\phi\in S}\{\hat{V}_t(\phi), \mathbb{P}(V_T(\phi) \geq H) = 1\} = esssup E^{\mathbb{Q}}[\hat{H}|\mathcal{F}_t]$$

Superhedging:

Prop10.1 Cost of superhedging(con'd):

- In particular, the cost of the cheapest superhedging strategy for H is given by

$$\Pi^{sup}(H) = esssup_{\mathbb{Q}\in M_a(S)} E^{\mathbb{Q}}[\hat{H}]$$

where $M_a(S)$ is the set of martingale measure absolutely continuous wrt to $\mathbb{P}$

Superhedging:

Prop10.1 Cost of superhedging(comments):

- preference-free method: no subjective risk aversion parameter nor ad hoc choice of a martingale measure

- in terms of equivalent martingale measures, superhedging cost corresponds to the value of the option under the least favorable martingale measure

Superhedging:

Application of Prop 10.1: Superhedging in exponential-Levy processes: Prop10.2

- So we have $S_t = S_0 exp X_t$ where (X_t) is a Levy process, if X has infinite variation, no Brownian component, negative jumps of arbitrary size and Levy measure ν : $\int_0^1 \nu(dy) = +\infty$ and $\int_{-1}^0 \nu(dy) = +\infty$ then the range of prices is:

$$\left[\inf_{\mathbb{Q}\in M(S)} E^{\mathbb{Q}}[(S_T - K)^+], \sup_{\mathbb{Q}\in M(S)} E^{\mathbb{Q}}[(S_T - K)^+ \right]$$

Superhedging:

Application of Prop 10.1: Superhedging in exponential-Levy processes: Prop10.2

- If X is a jump-diffusion process with diffusion coefficient σ and compound Poisson jumps then the price range for a call option is:

$$\left[C^{BS}(0, S_0; T, K; \sigma), S_0\right]$$

Superhedging: Comments

- From the above, the superhedging cost is too high. Consider $S_t = S_0 exp(\sigma W_t + aN_t)$, apply prop10.1, we find that the superhedging cost is given by S_0, so however small the jump is, the cheapest superhedging strategy for a call option is a complete hedge.

Utility Maximization

- "As if" method: the agent is picking some strategy to max utility level:

$$\max_Z E^{\mathbb{P}}[U(Z)]$$

usually, $U : \mathbb{R} \longrightarrow \mathbb{R}$ is concave, increasing, and $\mathbb{P}$ could be seen either as a prob distribution objectively or subjectively describe future events.

- The concavity of U is related to risk aversion of the agent. say $U(x) = ln(x)$, $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$

Utility Maximization: Certainty equivalent

- Another way to measure risk aversion: $c(x, H)$
- $U(x + c(x, H)) = E[U(x + H)] \Longrightarrow c(x, H) = U^{-1}(E[U(x + H)]) - x$
- Intuition: at the same level x, faced with the same H, the higher compensation you require, the more risk averse you are
- Notice: c is not linear in H, c depends on x

Utility Maximization: Utility indifference price

- The agent wants to max his final wealth: $V_T = x + \int_0^T \phi_t dS_t$:

$$u(x,0) = \sup_{\phi \in S} E^{\mathbb{P}}[U(x + \int_0^T \phi_t dS_t)]$$

- Suppose now it buys an option, with terminal payoff H, at price p, then

$$u(x-p,H) = \sup_{\phi \in S} E^{\mathbb{P}}[U(x-p+H+\int_0^T \phi_t dS_t)]$$

Utility Maximization: Utility indifference price

- The utility indifference price is defined as price $\pi_U(x, H)$:

$$u(x, 0) = u(x - \pi_U(x, H), H)$$

- Notice:
  1.π_U is not linear in H
  2.π_U depends on initial wealth, except for special utility like: $U(x) = 1 - e^{-\alpha x}$
  3.To same U, same x, same H, buying and selling derives different price:
  $u(x, 0) = u(x + p, -H)$

Utility Maximization: More comments

- The "As if" method: from vNM, Savage
- Hard to identify U and $\mathbb{P}$, and there is homogeneity among agents
- Attack to nonlinearity: remedies–quadratic hedging(where the utility is : $U(x) = -x^2$

Utility Maximization: Quadratic hedging

- As if the agent is choosing so to min the hedging error in a mean square sense.

- Different criterion to be min in a least squares sense can be:
  1.hedging error at maturity $\Longrightarrow$ "Mean-variance hedging";
  2.hedging error measure locally in time $\Longrightarrow$ local risk min.

- The two approaches are equivalent if the discounted price is a martingale measure.

Going Further: "Optimal" martingale measures

- By fund theorem , choosing an arbitrage-free pricing is choosing a martingale measure $\mathbb{Q} \sim \mathbb{P}$

- More general, we're choosing prob measures according to:

$$J_f(\mathbb{Q}) = E^{\mathbb{P}}\left[f(\frac{d\mathbb{Q}}{d\mathbb{P}})\right]$$

  where $f : [0, \infty) \longrightarrow \mathbb{R}$ is str convex, J_f a measure of deviation from the prior $\mathbb{P}$

Going Further: “Optimal” martingale measures

- Some example: relative entropy:

$$H(\mathbb{Q},\mathbb{P}) = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

- quadratic distance:

$$E\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right]$$

Going Further: "Optimal" martingale measures

- More on relative entropy: here $f = x \ln x$

$$H(\mathbb{Q}, \mathbb{P}) = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = E^{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

- So given (S_t) the minimal entropy martingale model is defined as a martingale (S_t^*) such that the $\mathbb{Q}^*$ of S^* minimizes the relative entropy wrt $\mathbb{P}$ among all martingale process:

$$\inf_{\mathbb{Q} \in M^a(S)} H(\mathbb{Q}, \mathbb{P})$$

Going Further: “Optimal” martingale measures

- Interpretation for min entropy martingale model: minimizing relative entropy corresponds to choosing a martingale measure by adding the least amount of info to the prior model.
- Existence: ? But for exp-Levy, nice result(analytic computable) in Prop10.7