Chapter 4

Mathematical Background

Stopping Times

- Stopping time:

$$\{\tau \leq t\} \in \mathcal{F}_t$$

- Indicator process:

$$N_\tau(t) := 1_{\{\tau \leq t\}}$$

- Predictable stopping time: it has an announcing sequence.
- Totally inaccessible stopping time: No predictable stopping time can give any information.

$$P(\tau = \tau' < \infty) = 0$$

for any τ' predictable.

Hazard Rate

- Let τ be a stopping time and $F(T)$ its distribution function. Its hazard rate is defined as.

$$h(t,T) = \frac{f(t,T)}{1 - F(t,T)}$$

where $F(t,T) = P(\tau \leq T | F_t)$

- Interpretation:

$$h(t,T) = lim_{\Delta t -> 0} \frac{1}{\Delta t} P(\tau \leq T + \Delta t | \tau > t)$$

- Or, by looking at:

$$F(t,T) = 1 - e^{-\int_t^T h(t,s) ds}$$

we see that, again, it is like forward rates.

Point Processes

$$\{\tau_i, i \in \mathbb{N}\} = \{\tau_1, \tau_2, ...\}$$

Counting Process

$$N(t) := \sum_i 1_{\{\tau_i \le t\}}$$

Predictable Compensator Process

$$M(t) = N(t) - A(t) \text{ is a martingale}$$

If A is differentiable we define the intensity as:

$$A(t) = \int_0^t \lambda(s) ds$$

Point Processes II

- Assume that A is differentiable.
- These type of models are called intensity models (chapter 7).
- All the models in chapter 9 don't satisfy this.

Hazard/Intensity

Hazard rates and intensity are related, under some conditions:

$$\lambda(t) = h(t, t)$$

There are two ways of viewing a counting process:

- As a stochastic process (predictable compensator, intensities, etc.)
- By looking at the distribution of the next jump time (using hazard rates)

Hazard/Intensity II

- If we know $P(t, T)$ and it is differentiable wrt T (at $T = t$) then (under conditions of theorem 4.1) :

$$\frac{dA(t)}{dt} = -\frac{\partial}{\partial T}|_{T=t} P(t, T) = h(t, t)$$

- Converse is not true.
- Starting from the intensity does not always give easy access to the survival probability.

Marked Point Processes and the Jump Measure

- A marked point process is a point process in which the jumps are stochastic:

$$\{(\tau_i, Y_i), i \in \mathbb{N}\} = \{\tau_1, \tau_2, ...\}$$

- One way to generalize the counting process is:

$$X(t) := \sum_i Y_i 1_{\{\tau_i \leq t\}}$$

- However, sometimes Y could take values that are not numbers (the name of the defaulting company, jumps in the rating classes etc.)
- Because of this we use a different approach: the jump measure.

Marked Point Processes and the Jump Measure II

- We first define the concept of random measure: $\nu : \Omega x \mathcal{E} x \mathcal{B}(\mathbb{R}_+) - > \mathbb{R}_+$ is a random measure if for every $\omega \in \Omega, \nu(\omega, \cdot, \cdot)$ is a measure on $((Ex\mathbb{R}_+), \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+))$ and $\nu(\omega, E, 0) = 0$ identically.
- We can use random measures to construct stochastic processes by integrating.
- The jump measure of a marked point process is a random measure:

$$\mu(\omega, E', [0,t]) = \int_0^t \int_{E'} \mu(\omega, de, ds) :=$$

$$\sum_{i=1}^{\infty} 1_{\{\tau_i(\omega) \leq t\}} 1_{\{Y_i(\omega) \in E'\}}$$

- By integrating against the jump measure we can represent functionals of the marked point process.

The Compensator Measure

- The idea here is that, given a random measure, there exists a predictable random measure so that for every predictable stochastic function $f(\omega, e, t)$ the process defined by:

$$M(\omega, t) := \int_0^t \int_E f(\omega, e, s)\mu(\omega, de, ds) -$$

$$\int_0^t \int_E f(\omega, e, s)\nu(\omega, de, ds)$$

  is a local martingale.
- Many times we can separate the probability that an event occurs from the conditional distribution of the marker given that an event has occurred.

$$\nu(de, dt) = K(t, de)dA(t) \text{ with } \int_E K(t, de) = 1$$

The Compensator Measure II

In discrete time:

- Suppose

$$X(\omega, t) = \int +0^t \int_E f(s,e)\mu(de, ds)$$

- In discrete time:

$$X(t_n) - X(t_{n-1}) = \int_E f(t_n, e)\mu_n(de)$$

- f has to be adapted (for X to be).
- We will ask it to be predictable: at time t_{n-1} we will know what f will be at time t_n conditioned on Y.
- Define $\nu_n(de) = P(Y \in de \text{ and } \tau = t_n | \mathcal{F}_n)$

The Compensator Measure III

- So:

$$E((X(t_n) - X(t_{n-1}))|\mathcal{F}_{n-1}) = \int_E f(t_n, e)\nu_n(de)$$

- We can now construct the compensator:

$$A(t_n) - A(t_{n-1}) = \int_E f(t_n, e)\nu_n(de)$$

- Then A is predictable and $X - A$ is a martingale.

The Compensator Measure IV

Examples

- Poisson Process $N(t)$ with intensity λ (constant)
  - Compensator measure $\nu(de, dt) = \delta_{Y=1}(de)\lambda dt$
  - Conditional distribution $dA(t) = \lambda dt, K(de) = \delta_{Y=1}(de)$
- Poisson Process $N(t)$ with intensity $\lambda(t)$ (stochastic)
  - Compensator measure $\nu(de, dt) = \delta_{Y=1}(de)\lambda(t)dt$
  - Conditional distribution $dA(t) = \lambda(t)dt, K(de) = \delta_{Y=1}(de)$
- Marked inhomogeneous Poisson Process I
  - Marker: $Y \sim N(0, 1)$.
  - Compensator measure $\nu(de, dt) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2} \lambda(t) dedt$
  - Conditional distribution
    $dA(t) = \lambda(t)dt, K(de) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2} de$

The Compensator Measure V

More Examples

- Marked Poisson process II
  - Marker Y is the value of a geometric brownian motion at time t (the time of the jump).
  - Compensator measure $\nu(de, dt) = \delta_{Y=S(t-)}(de)\lambda(t)dt$
  - Conditional distribution $dA(t) = \lambda(t)dt, K(de) = \delta_{Y=S(t-)}(de)$
- Lognormal Jump Diffusion
  - Jump times triggered by a Poisson process with parameter λ.
  - Marker Y (log of the jump size) is $N(0, 1)$.
  - Compensator measure $\nu(de, dt) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2}\lambda(t)dedt$

The Compensator Measure VI

More Examples

- First hitting time process
  - Arrival time is the first time that a geometric brownian motion $S(t)$ hits a barrier.
  - No marker
  - Compensator measure $\nu(dt) = dA(t)$ where

$$dA(x) = \begin{cases} 1 & \text{if the barrier is hit } S(t) = \bar{K} \\ 0 & \text{otherwise} \end{cases}$$

- Since the default arrival is predictable its compensator is the process itself.

The Compensator Measure VII

More Examples

- A (maybe not so) unusual process
  - Compensator measure $\nu(de, dt) = \frac{1}{|e|} de dt$ for $0 \notin de$ where
  - This process has an infinite number of very small jumps and a few larger ones.
  - If $[a, b]$ is an interval away from zero then jumps of a size in $[a, b]$ occur with an intensity of

$$\lambda_{[a,b]} = \int_a^b \frac{1}{|e|} de$$

  - So, the process can be viewed as a collection of Poisson processes, one Poisson process per interval in $\mathbb{R}$. The intensity converges to infinity the closer we get to zero.
  - In the book he assumes that the processes have a finite number of jumps in any finite interval. So, processes like this are excluded.

The Compensator Measure VIII

More Examples

- A very simple process
  - Jumps occur at $\tau_1 = 2, \tau_2 = 4, \tau_3 = 8, \ldots$
  - This is known at the beginning.
  - This is known at the beginning so it is predictable and then its compensator is the jump measure itself:

$$\nu(de, dt) = \delta_{t=\tau_i}(dt)$$

Itô's Lemma For Jump Processes

- The processes considered have RCLL paths.
- Notation $\Delta X(t) := X(t) - X_{-}(t)$, $X^d(t) := \sum_{s \leq t} \Delta X(s)$, $X^c(t) := X(t) - X^d(t)$.
- Let $X = (X^1, ..., X^n)$ be an $n-$dimensional semi-martingale with a finite number of jumps and f a teice differentiable function on $\mathbb{R}^d$. Then $f(X)$ is also a semi-martingale and:

$$f(X(t)) - f(X(0)) = \sum_{i=1}^{n} \int_0^t \frac{\partial f(X_{-}(s))}{\partial x_i} dX^{c,i}(s) +$$

$$\frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f(X_{-}(s))}{\partial x_i \partial x_j} d < X^{c,i}, X^{c,j} > (s) +$$

$$\sum_{s \leq t} \Delta f(X(s))$$

- The jump times τ_i and the jump sizes $\Delta X(\tau_i)$ define a marked point process.
- This marked point process has a jump measure μ_x (which puts mass 1 on the jump times and sizes of the jumps). and a compensator measure ν_x.
- The process X can be rewritten:

$$dX(t) = dX^c(t) + \int_{\mathbb{R}^\ltimes} x\mu_x(dx, dt)$$

Itô's Lemma For Jump Processes III

- Using the jump measure:

$$f(X(t)) - f(X(0)) = \sum_{i=1}^{n} \int_0^t \frac{\partial f(X_-(s))}{\partial x_i} dX^{c,i}(s) +$$

$$\frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f(X_-(s))}{\partial x_i \partial x_j} d < X^{c,i}, X^{c,j} > (s) +$$

$$\int_0^t \int_{\mathbb{R}^n} f(X_-(s) + x) - f(X_-(s)) \mu_x(dx, ds)$$

- In a lot of applications X can be written as a jump diffusion process

$$dX^i = \alpha_i dt + \sum_{k=1}^{K} \sigma_{ik} dW_k + \int_{\mathbb{R}^n} h_i(x) \mu_X(dx, dt)$$

- And the compensator measure ν can be decomposed as

$$\nu_X(dx, dt) = K(t, dx) dA(t)$$

Applications of Itô's Lemma II

- Can do Itô to find $f(X)$ and its compensator.
- In this case the predictable compensator is the sum of the usual drift and:

$$\int_0^t \left(\int_{\mathbb{R}^n} f(X_-(s) + x) K(s, dx) - f(X_-(s)) \right) dA(s)$$

which compensates for the influence of the jumps.

$$\int_{\mathbb{R}^n} f(X_-(s) + x) K(s, dx)$$

represents the expected value of f after a jump at time s.

Itô product and quotient rule.

- Let Y and Z be

$$\frac{dY}{Y_-} = \alpha^y + \sum_{k=1}^{K} \sigma_k^y dW_k(s) + \int_{\mathbb{R}^n} h^y(x)\mu_X(dx, dt)$$

$$\frac{dZ}{Z_-} = \alpha^z + \sum_{k=1}^{K} \sigma_k^z dW_k(s) + \int_{\mathbb{R}^n} h^z(x)\mu_X(dx, dt)$$

- so, the jumps of both processes are driven by the jumps of a third process X.
- So doing Itô can find the process $g(Y, Z) = YZ$.

Applications of Itô's Lemma IV

The stochastic exponential

- Let X be a stochastic process with $\Delta X \geq -1$. Then $Y(t)$ is called the stochastic exponential of X iff Y solves:

$$dY(t) = Y_{-}(t)dX(t)$$

- If X has finitely many jumps:

$$Y(t) = e^{X^c(t) - X^c(0) - \frac{1}{2} < X^c > (t)} \prod_{s \leq t} (1 + \Delta X(s))$$

Martingale Measure

- Let Q be a probability measure If for every dividend-free traded asset with price process $p(t)$ the discounted process $\frac{p(t)}{b(t)}$ is a martingale under Q then Q is called a martingale measure.
- This is important because its existence is equivalent to absence of arbitrage.

Change of numeraire

- Radon-Nikodym: Given two measures Q and P so that $P << Q$ $(Q(A) = 0 => P(A) = 0)$ there exists a density L so that $E^P(X) = E^Q(LX)$ for all measurable X.
- In a dynamic model we define $L(t) = E^Q(L|\mathcal{F}_t)$ then, if X is $\mathcal{F}_T$-measurable:

$$E^P(X|\mathcal{F}_t) = E^Q(LX|\mathcal{F}_t) = E^Q(E^Q(LX|\mathcal{F}_T)|\mathcal{F}_t) =$$

$$= E^Q(E^Q(L|\mathcal{F}_T)X|\mathcal{F}_t) = E^Q(L(T)X|\mathcal{F}_t) =$$

$$L(t)E^Q(\frac{L(T)}{L(t)}X|\mathcal{F}_t)$$

Girsanov Theorem

- It tells us how probabilistic properties of processes change when we change measures.
- A brownian motion under a measure Q does not need to be a brownian motion under P.
- Jump measures don't change (since path are unchanged) but compensator measures will change (since compensators determine probabilities.

Girsanov Theorem II

- Assume a probability space with a brownian motion $(W_Q(t))$ and a marked point process $\mu(de, dt)$ with its compensator $\nu_Q(de, dt) = K_Q(de)\lambda_Q(t)dt$.
- Define a process L as:

$$\frac{dL(t)}{L(t-)} = \varphi(t)dW_Q(t) + \int_E (\Phi(e,t) - 1)(\mu(de, dt) - \nu_Q(de, dt))$$

- Then:

$$dW_P(t) = dW_Q(t) - \varphi(t)dt \text{ is a } P\text{-brownian motion}$$

Girsanov Theorem III

- The compensator under P is:

$$\nu_P(de, dt) = \Phi(t, e)\nu_Q(de, dt)$$

- If $\psi(t) = \int_E \Phi(e, t)K_Q(t, de)$ and $L_E(e, t) = \Phi(e, t)/\psi(t)$ for $\psi(t) > 0$ and $L_E(e, t) = 1$ otherwise. Then the intensity under P becomes:

$$\lambda_P(t) = \psi(t)\lambda_Q(t)$$

- The conditional distribution of the marker is

$$K_P(t, de) = L_E(e, t)K_Q(de)$$

Change of measure/Change of numeraire technique

- Let $p(t)$ be a price (under the money market numeraire, so discounted..).
- Then:

$$\frac{p(t)}{b(t)} = E^Q(\frac{X_T}{b(T)}|\mathcal{F}_t)$$

- Consider a different numeraire $A(t)$, and consider also the process $\frac{dP}{dQ}|_t = \frac{A(t)}{A(0)b(t)}$
- Then X/A is a P-martingale iff $\frac{X_t}{A(t)}\frac{A(t)}{A(0)b(t)}$ is a Q-martingale.