Numerical Methods for American Options

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Given a contingent claim, $C = C(t, S)$,we are concerned with operators of the form

$$\begin{aligned} L^x f(x) &= a_1(x)\frac{\partial f}{\partial x} + a_2(x)\frac{\partial^2 f}{\partial x^2} \\ &+ \int f(x + a_0(x,y)) - f(x) - a_0(x,y)\frac{\partial f}{\partial x}(x)\nu(dy), \end{aligned}$$

and solutions satisfying

$$\frac{\partial C}{\partial t}(t,S) + L^S C(t,S) - rC(t,S) = 0$$

a_0, a_1, and a_2, depend on how we model the dynamics of the underlying.

For example, if an underlying is modeled in the original Black-Scholes framework,

$$\frac{dS}{S_t} = \mu dt + \sigma_t dW,$$

we have the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 C}{\partial S^2} + r\frac{\partial C}{\partial S}S - rC = 0.$$

Now if we assume the risk-neutral dynamics of some asset, S, are given by

$$S_t = S_0 \exp(rt + X_t), \tag{1}$$

where X is a Levy process with characteristic triplet (σ^2, ν, γ) under the (not necessarily unique!) measure $\mathbb{Q}$,

L is given by

$$L^x f(x) = rxf'(x) + \frac{\sigma^2 x^2}{2} f''(x) + \int_{\mathbb{R}} f(xe^y) - f(x) - x(e^y - 1) f'(x) \nu(dy)$$

We also must impose boundary conditions.

For a vanilla option, we need only state what the payoff function, $H = H(S_T)$, is at expiration.

In this case we have the problem

$$\frac{\partial C}{\partial t}(t, S) + L^S C(t, S) - rC(t, S) = 0$$

on $[0, T) \times (0, \infty)$, with boundary condition

$$C(T, S) = H(S), \;\; \forall S > 0.$$

For an American put option, P, with strike, K, we may construct a free boundary problem

$$\begin{aligned}
P(\cdot,S):[0,T]\to[0,\infty) \text{ nonincreasing, convex} & \qquad 0\le b(t)\le K \\
\frac{\partial P}{\partial t}(t,S)+L^S P(t,S)-rP(t,S)=0 & \qquad \text{while } S>b(t) \\
\lim_{S\downarrow b(t)} P(t,S)=K-b(t) & \qquad \forall t\in[0,T) \\
\lim_{S\downarrow b(t)} \frac{\partial P}{\partial S}(t,S)=-1 & \qquad \forall t\in[0,T) \\
P(T,S)=(K-S)^+ & \qquad \forall S\in[0,\infty) \\
P(t,S)>(K-S)^+ & \qquad \text{if } S>b(t) \\
P(t,S)=(K-S)^+ & \qquad \text{if } S\le b(t)
\end{aligned}$$

There is a result that says for a jump diffusion model with finite intensity, an American put option is the unique pair (P, b) satisfying the above conditions.

But, there are few results (and fewer in English alone) about free boundary problems for PIDE's.

So if we are modeling stocks using jump processes, we have to be more clever.

Luckily, the jump diffusion ($\sigma > 0$)case leads to a linear complementarity problem exactly as in the Black-Scholes case.

Specifically, an American put must satisfy

$$\frac{\partial P}{\partial t}(t,S) + L^S P(t,S) - rP(t,S) \leq 0$$

$$P(t,S) - (K-S)^+ > 0$$

$$\left(\frac{\partial P}{\partial t}(t,S) + L^S P(t,S) - rP(t,S)\right)\left(P(t,S) - (K-S)^+\right) = 0$$

$$P(T,S) = (K-S)^+$$

To solve the above LCP, we use the penalty method. (I am not *exactly* sure how this works, though...any takers?)

The idea goes like this: replace the above equations with the PIDE

$$\frac{\partial P}{\partial t}(t, S) + L^S P(t, S) - rP(t, S) = \rho \max(H - P, 0),$$

where in the limit, as $\rho \to \infty$ the solution satisfies $P \geq H = (K - S)^+$.

With this modified PIDE, we impose boundary conditions

$$
\begin{aligned}
P(S,t) &= 0 \qquad && S \to \infty \\
\frac{\partial P}{\partial t} + L^S P - rP &= \frac{\partial P}{\partial t} - rP \qquad && S \to 0
\end{aligned}
$$

We are now in a position to discretize everything in sight and apply an implicit scheme.

Before doing this, we make the change of variables $\tau = T - t$.

In this case, we solve

$$\frac{\partial P}{\partial \tau}(\tau, S) = L^S P(\tau, S) - rP(\tau, S) + \rho \max(H - P, 0),$$

with obvious modifications for the boundary conditions.

As before, we need to truncate the space dimension and large jumps.

Further, we approximate the integral term and derivatives in the obvious way (last week's stuff).

The only new ingredient is dealing with $\rho \max(H - P, 0)$.

Using the same notation as before, we construct a diagonal matrix R by

$$\begin{aligned}(R(P^{n+1}))_{ii} &= M \text{ if } P_i^{n+1} < H_i \\ &= 0 \text{ otherwise}\end{aligned}$$

where M is chosen sufficiently large.

We are therefore looking at

$$\frac{P^{n+1} - P^n}{\Delta t} = DP^n + JP^n - rP^n + R(P^{n+1})H$$

where D is the (appropriate) differential operator approximation matrix, and J is the integral operator matrix.

So what kind of results do we have?

For the uniform grid case, if there exists a constant, C, such that

$$\frac{\Delta t}{\delta S} < C$$

as Δt, $\Delta S \to 0$, (and one other condition) then the discrete solution solves

$$\begin{aligned} \hat{L}P_i^{n+1} &\geq 0 \\ P^{n+1} - H &\geq -\frac{C'}{M} \\ (\hat{L}P^{n+1} = 0) &\vee \left(-\frac{C'}{M} \leq P^{n+1} - H \leq \frac{C'}{M}\right) \end{aligned}$$

where C' is independent of M, Δt, and ΔS.

Of course we can look at explicit, implicit, and mixed schemes (like Crank-Nicholson) in the above.

I didn't see it written up anywhere, but it would seem that a Galerkin method could also be used here with slight modifications from the original PIDE.