Chapter 10

Models for Default Correlation

July 30, 2007

Linear Default Correlation

- Consider two obligors A and B, fixed tie horizon T.
- $P(''A \text{ defaults before } T'') = p_A$ and similar for p_B.
- To know the correlation we need also the joint distribution or the conditional probabilities.

$$\rho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1 - p_A)p_B(1 - p_B)}}$$

- If we have two obligors the pairwise correlation is enough. If we have more combinations.
- So, for larger portfolios it is impossible to describe all the possible events with pairwise correlations.

Size of the impact of default correlation

- If p_A and p_B are not too big and the correlation is not negligible then the joint probabilities of default and the conditional probabilities are dominated by ρ_{AB}.

Price bounds for FtD Swaps

- Clearly, the CDS spread on an FtD has to be bigger than the spread of the worst credit in the portfolio.
- Also, it has to be lower than the sum of the credit spreads of all the credits in the basket.
- The exact value depends on the correlation.
- In a CDO the holder of the equity piece is long correlation and holder of the senior piece is short correlation.

The need for theoretical models of default correlation

- It is hard to get data on joint defaults.
- The alternative is to use credit spreads.
- However, no clear justification for the size and strength of the link between credit spread correlation and default correlation.
- Main problem is that to describe the full joint default probbilities is too complex. For N obligors there are 2^N joint default events. Compare this to the case of describing the dynamics of N continuous variables in which we can assume that they follow a normal (or lognormal) distribution and then the correlation structure is given by $N(N-1)/2$ parameters.

Extreme cases

Independent Defaults

- If we have N obligors, exposure, recovery and probability of default between times 0 and T are the same L, π, p.
- If X is the number of defaults the the loss is $X(1-\pi)L$.
- If defaults are independent then the loss distribution is binomial (N, p).

Perfectly dependent defaults

- If, under the same assumptions on L, π, p, the default are perfectly correlated the distribution is similar (up to scale) to the loss distribution for 1 obligor. So, it is also a binomial distribution but with 1 draw.

Binomial Expansion Method (BET)

- Used by Moody's to assess the default risk in bond and loan portfolios.
- Not based on a formal portfolio default risk model, can be inaccurate and generally unsuitable for pricing but has become a market standard for risk assessment.
- It is based on the extreme cases. Since both are binomial, define a diversity score D as the number of independent obligors (and we aggregate on those the dependent ones so L becomes LN/D).

Factor Models

- Assume that the default is triggered by by the change in the value of the assets of the company. The value of the assets are normal ($V_n(T)$). Obligor n defaults if $V_n(T) < K_n$.
- The probability of default is given by $K_n = \Phi^{-1}(p_n)$.
- Still need to specify the correlation matrix. One trick to reduce dimensionality is to use factors.
- One factor model $V_n(T) = \sqrt{\rho} Y + \sqrt{1-\rho}\epsilon_n$, where Y is a common factor and ϵ_n is idiosyncratic.
- So, we have reduced the number of correlation parameters to 1.
- Another way of saying this is: conditional on the systematic factor Y, the firm's values and the defaults are independent.

Factor Models II

The distribution of the defaults

- Assume that default happens if $V_i(T)$ reaches K.
- So, conditional on y the probability of default is $p(y) = \Phi(\frac{K-\sqrt{\rho}y}{\sqrt{1-\rho}})$.
- Using the fact that conditionally on the value of Y the default probabilities are independent, can compute the distribution of the discrete variable which counts how many defaults happened between 0 and T:

$$P(X \leq m) = \sum_{n=0}^{m} \binom{N}{n} \int_{-\infty}^{\infty} p(y)^n (1-p(y))^{N-n} \phi(y) dy$$

- This can be used to give a density for the random variable giving the loss in a portfolio containing many assets.

Correlated defaults in intensity models

Four approaches:

- Correlate the intensities: can't reproduce realistic levels of dependence (low).
- Joint default events: has an unrealistic distribution of defaults over time and it is difficult to implement and calibrate (defines three Cox processes, two driving defaults in isolation and one driving joint defaults).
- Infectious defaults: good intuition but hard to calibrate and lacks tractability.
- Incorporate a copula function.

Correlated defaults in firm value models

- Firms A and B default only at time T if $V_A < K_A$ or $V_B < K_B$, where V_A, V_B are the logs of the value of the assets.
- V_A and V_B follow brownian motions.
- $DW_A dW_B = \rho dt$.
- Since the defaults happen only at time T we can assume $V_A(T) \sim N(0,1)$ and the same thing for B.
- So the default probabilities are $p_A = \Phi(K_A)$, $p_B = \Phi(K_B)$ and the joint def prob $p_{AB} = \Phi_\rho(K_A, K_B)$.

Correlated defaults in firm value models II

- Now, if share prices are functions of the firm value then (by Itô) local correlation between share prices and firm value processes should coincide.
- This is is not true in reality: calibration of firm value (one-factor) models give lower local correlation than share prices.
- Possibly due to liquidity?
- Another problem: timing. Some people proposed models to solve this in this context.

Copula Functions

- So far all the approaches considered had problems:
  - Firm-value: timing.
  - Barrier based firm value: hard to calibrate and implement.
  - Intensity-based: large number of parameters.
- The idea of copulas is to separate the individual term structure of default risk from the dependency model.
- Remember: is X is a continuous random variable then $U = F_X(X)$ is uniform. Also, if U is uniform then $Y = F^{-1}(U)$ has distribution F (F continuous).

Copula Functions II

The most common definition is:

A copula is a function $C : [0,1]^I \rightarrow [0,1]$ so that
a) There are random variables $U_1, ..., U_I$ taking values in $[0,1]$ such that C is their distr function.
b) C has uniform marginal distributions $C(1,..,1,u_i,1,...,1) = u_i$.

However that definition does not say what should C satisfy (important when building copulas by other methods than just taking a given distribution)

A copula is a function $C : [0,1]^I \rightarrow [0,1]$ so that
a) $C(u) = 0$ if at least one of the u_is is zero.
b) $C(1,...,1,u_i,1...,1) = u_i$.
c) The volume of every hypercube is nonnegative.

Copula Functions III

Sklar's Theorem:

Let $X_1, ..., X_I$ be random variables with marginals $F_1, ..., F_I$ and joint distr F. Then there exist a copula C so that $F(x) = C(F_1(x_1), ..., F_I(x_I))$. So C is the distribution function of the random vector of uniform variables $(F_1(x_1), ..., F_I(x_I))$. Moreover, if all the marginals are continuous then C is unique.

Copula Functions IV

Neat Property:

$$f_X(x_1, \ldots, x_n) = f_U(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)) \prod_{i=1}^{n} f_{X_i}(x_i)$$

Copula Functions IV

Examples

- Product Copula: $\Pi^I(v_1, ..., v_I) = v_1.v_2...v_I$. Gives independence.
- Gaussian: $C_\Sigma(u) = \Phi_\Sigma(\phi_1^{-1}(u_1), ..., \phi_I^{-1}(u_I))$. Notice that the u_is can now be taken as $F(y_i)$ for random variables Y_i with a non-normal distribution. So the result would be a joint distribution with chosen marginals but Gaussian dependence structure.
- Similarly, can define a t-copula.

Copula Functions

Tail Dependence

For a bivariate copula C:

- C as upper tail dependence parameter λ_U if:

$$\lim_{u \to 1} \frac{1 + C(u,u) - 2u}{1 - u} = \lambda_U > 0$$

- C as lower tail dependence parameter λ_L if:

$$\lim_{u \to 0} \frac{C(u,u)}{u} = \lambda_L > 0$$

They give a measure of how many data points are concentrated in the upper and lower squares.
For Gaussian copulas there is no tail dependence, so extreme events happen almost independently, the same is not true for t-copulas (page 333).

Copula Functions V

Dependence Concepts

If X is a random vector with copula $C(u)$ and f_i are strictly increasing functions, then C is also the copula of the vector formed by the $f_i(X_i)$.

The problems with correlation

- It's a linear measure: variables that are strongly (but not linearly) related can have low correlation.
- Take two normal variables with correlation ρ, form lognormal distributions from them. These have correlation different from ρ.
- Depends on the marginals, not just the dependence structure (copula).

Copula Functions VI

Concordance

Let (x, y) and $(\bar{x}, \bar{y})$ be two observations from a vector of continuous random variables.
Then $(x, y))$ and $(\bar{x}, \bar{y})$ are said to be concordant if $(x, y) - (\bar{x}, \bar{y}) > 0$ and discordant if it is < 0.
Main result:

- If (X, Y) and $(\bar{X}, \bar{Y})$ are independent vectors of continuous random variables with joint distr H and $\bar{H}$, common marginals F (for X) and G (for Y) and copulas C and $\bar{C}$ (so $H(x, y) = C(F(x), G(y)))$. Then the difference between the probability of concordance and the probability of discordance is given by:

$$Q = Q(C, \bar{C}) = 4 \int\int_{[0,1]^2} \bar{C}(u, v) dC(u, v) - 1$$

Also, Q is symmetric.

Copula Functions VII

Definition of a measure of concordance

κ is a measure of concordance between two variables X and Y with copula C if:

- $-1 \leq \kappa_{X,Y} \leq -1$ and $\kappa_{X,X} = 1, \kappa_{X,-X} = -1$.
- It's symmetric.
- If they are indep κ is 0.
- $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$
- Pointwise limit is preserved.
- κ preserves the order.

Kendall's tau and Spearman's rho

- Kendall's τ is defined as

$$\tau(X,Y) = P((X-\bar{X})(Y-\bar{Y}) > 0) - P((X-\bar{X})(Y-\bar{Y}) < 0)$$

- Can be proved that $\tau(X,Y) = 4E(C(U,V)) - 1$, where C is the copula of (X,Y) and U, V are uniform.
- Spearman ρ is defined as the correlation of the grades:

$$\rho_S(X,Y) = \rho(F(X), G(Y))$$