Portfolio Analysis: Optimization

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Managing a dynamic portfolio, Π, requires:

- The ability to forecast future states.
- A balance between risk and return.

These are different mathematical problems, but both must be addressed for a portfolio manager.

Dual-time Dynamics (Joseph Breeden) provides tools to forecast. We discussed, but will not post here:

- Heuristics.
- Original iterative method.
- A new iterative method.
- Convergence properties.
- Future directions.

We derive an optimal balance between risk and return via a single factor credit risk model. We will discuss:

- Modeling returns of obligors.
- Model estimates from historical data.
- A Markowitz type optimization problem (QP).
- Lack of robustness.
- A robust optimization problem.
- Future directions.

To begin, let's make an optimal portfolio of loans.

We begin with a Merton style model of default, modeling the return rate of the ith obligor in the portfolio by

$$r_i = \sqrt{\rho}\,Y + \sqrt{1-\rho}\,Z_i, \tag{1}$$

with $Y \sim N(0,1)$ and $Z_i \sim N(0,1)$

We say that the ith obligor defaults if r_i falls below some threshold value c_i.

We define c_i using the historical default DP_i:

$$\mathbb{P}(r_i < c_i) = DP_i,$$

or, equivalently,

$$c_i = \Phi^{-1}(DP_i).$$

The probability of default, conditioned on Y, is

$$\begin{aligned} p_i(y) &= \mathbb{P}\left(Z_i < \frac{c_i - \sqrt{1-\rho}\, Y}{\sqrt{\rho}} \mid Y = y \right) \\ &= \Phi\left(\frac{c_i - \sqrt{1-\rho}\, y}{\sqrt{\rho}} \right), \end{aligned} \tag{2}$$

We also assign a loss statistic, $L_i(Y) = L_i$ to each obligor, with

$$L_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$L_i \sim B(1, p_i(Y))$

η_i will denote the loss given default for the ith obligor.

The weight of the ith obligor in the portfolio is given by w_i.

The vector $\mathbf{w} = (w_1, \ldots, w_N)'$ will be the decision variable in all of our optimization problems, and

The loss of the portfolio, $L_\Pi(Y) = L_\Pi$ is

$$L_\Pi = \sum_{j=1}^{N} w_j \eta_j L_j, \tag{4}$$

and

$$\begin{aligned} E(L_\Pi) &= \sum_{j=1}^{N} w_j \eta_j E(L_j) \\ &= \sum_{j=1}^{N} w_j \eta_j DP_j. \end{aligned}$$

The loss variance of the portfolio is given by

$$
\begin{aligned}
V(L_\Pi) &= E(L_\Pi^2) - E(L_\Pi)^2 \\
&= \sum_{1 \le i,j \le N} w_i w_j \eta_i \eta_j E(L_i L_j) - \left(\sum_i w_i \eta_i E(L_i) \right)^2 .
\end{aligned}
$$

The joint default probability, $E(L_i L_j)$, is calculated by

$$\begin{aligned} E(L_i L_j) &= \mathbb{P}(L_1 = 1,\, L_j = 1) \cdot 1 + \mathbb{P}((L_1, L_j) \neq (1,1)) \cdot 0 \\ &= \mathbb{P}(r_i < c_i, r_j < c_j). \end{aligned}$$

Estimating from data, we say

$$JDP_{ij} = E(L_i L_j) = \Phi_2(\Phi^{-1}(DP_i), \Phi^{-1}(DP_j); \rho). \tag{5}$$

So that we have

$$V(L_\Pi) = \sum_{1 \le i,j \le N} w_i w_j \eta_i \eta_j JDP_{ij} - \left(\sum_i w_i \eta_i DP_i \right)^2$$

We may write this all as

$$E(L_\Pi) = \mu_0'\mathbf{w} \tag{6}$$

$$V(L_\Pi) = \mathbf{w}'\boldsymbol{\Sigma}_0\mathbf{w}, \tag{7}$$

where

$$\begin{aligned}
\boldsymbol{\Sigma}_0 &= \mathbf{EJE} - \mathbf{M}, \\
\mathbf{E} &= diag(\eta_i), \\
\mathbf{J} &= (JDP_{ij}), \\
\mu_0 &= (\eta_1 DP_1, \ldots, \eta_N DP_N)', \text{ and} \\
\mathbf{M} &= \mu\,\mu'.
\end{aligned}$$

This puts us in a position to phrase a Markowitz type optimization problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{w}'\boldsymbol{\Sigma}_0\mathbf{w} \\ \text{subject to} \quad & \mu_0'\mathbf{w} \leq \alpha \\ & \mathbf{1}'\mathbf{w} = 1 \end{aligned} \tag{8}$$

The solutions to this optimization problem are sensitive to to parameter estimates.

We would therefore like to rephrase the problem as

$$\begin{aligned} \text{minimize} \quad & \max_{\Sigma \in \mathcal{Q}} (\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}) \\ \text{subject to} \quad & \max_{\mu \in \mathcal{M}} \mu' \mathbf{w} \leq \alpha \\ & \mathbf{1}' \mathbf{w} = 1, \end{aligned} \tag{9}$$

Let

$$[\underline{DP}_i, \overline{DP}_i]_\alpha$$

denote the $(1-\alpha) \cdot 100\%$ confidence interval for the probability of default in class i.

We may use these bounds to construct the uncertainty sets we require.

For a given confidence level, we define $\mathcal{M}$ by

$$\mathcal{M} = \mathcal{M}_\alpha \quad = \quad \{\mu \mid \mu^L \leq \mu \leq \mu^U\}, \tag{10}$$

where $v \leq w$ means $v_i \leq w_i$ for $i = 1, \ldots, N$, and

$$\begin{aligned} \mu^L &= (\eta_1 \underline{DP}_1, \ldots, \eta_N \underline{DP}_N)' \\ \mu^U &= (\eta_1 \overline{DP}_1, \ldots, \eta_N \overline{DP}_N)' \end{aligned}$$

We next define matrices, Σ^U and Σ^L by

$$\begin{aligned}\Sigma^U &= \mathbf{E}J^U\mathbf{E} - \mu^L(\mu^L)' \\ \Sigma^L &= \mathbf{E}J^L\mathbf{E} - \mu^U(\mu^U)'\end{aligned}$$

where J^U and J^L are given by

$$\begin{aligned}(J^U)_{lm} &= \overline{JDP}_{lm} \\ (J^L)_{lm} &= \underline{JDP}_{lm}.\end{aligned}$$

These estimates of $\overline{JDP}_{lm}$ and $\underline{JDP}_{lm}$ are derived from the uncertainty in DP_i.

We define $\mathcal{Q}$ by

$$\mathcal{Q} = \mathcal{Q}_\alpha = \{\Sigma \mid \Sigma^L \leq \Sigma \leq \Sigma^U,\, \Sigma \in \mathcal{S}_+\} \tag{11}$$

since interpretation of the model requires positive semidefiniteness.

Using uncertainty sets of this form yields a saddle point problem that can be solved using interior point methods.

Future work will include:

- Analyzing the qualitative and quantitative difference between the robust and nonrobust solutions using real data from GMAC.
- Building a multifactor model and a resulting robust optimization problem. This will likely be a SOCP.
- Building a model for prepayment.
- Making the models multiperiod.
- Using Dual-time Dynamics to produce scenario based forecasts for optimization into the future.