Put-Call Parity

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May 22, 2006

Recall that a replicating portfolio of a contingent claim determines the claim's price.

This was justified by the "no arbitrage" principle.

Using this idea, we obtain a relationship between a European call and a European put option.

Consider the following portfolios:

Portfolio 1: A European call option, and cash at time t equal to Ke^{-rT}

Portfolio 2: A European put option, and one share of the underlying, S

Here, K is the strike price for both the call and the put, and T is the time to expiration for both options.

At maturity, Portfolio 1 is worth

$$K + \text{max}(S_T - K, 0) = \text{max}(S_T, K),$$

and Portfolio 2 is worth

$$S_T + \text{max}(K - S_T, 0) = \text{max}(K, S_T).$$

Since the options are European, the prices today of these portfolios must be the same.

If the prices were different, there would be an arbitrage opportunity (just like the gold example).

We have now that

$$c_0 + Ke^{-rT} = p_0 + S_0 \qquad (1)$$

This relationship is *put-call parity*, and holds for European options.

Notice that this relationship is not model dependent-we only relied on the no arbitrage principle.

We got a hint that puts and calls are related when we saw the implied volatility skew for crude oil:

Put & Call Volatility Smiles: March 29, 2006 for Exercise May 29,2006
(Call is dashed/Put is solid)
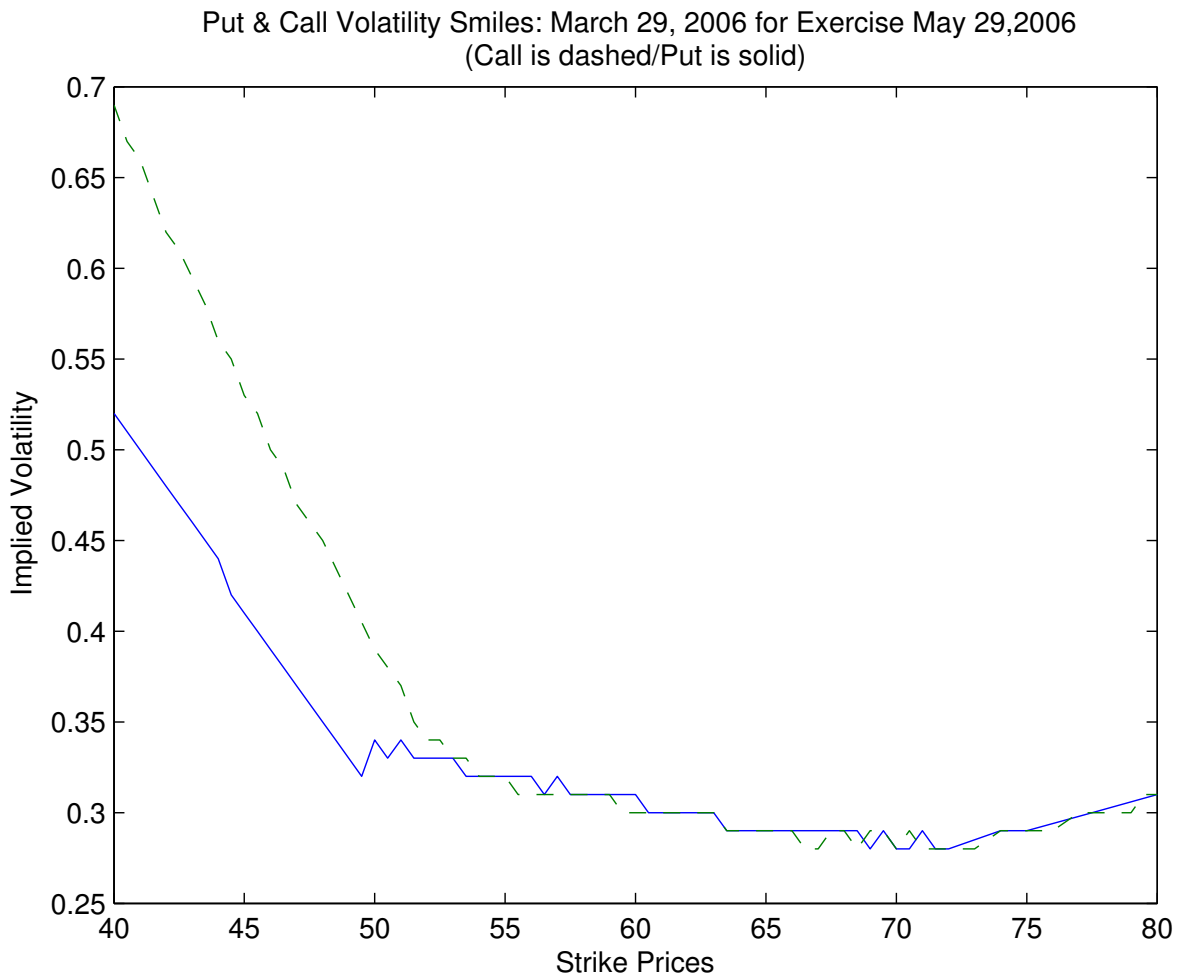
Recall that the only unobservable quantity in the Black-Scholes price of a European call or put was the volatility.

Suppose that for a certain volatility, σ, we obtain prices p_B and c_B for a put and call (resp.) from the Black-Scholes model.

Since we assume the BS model is arbitrage free, we must have

$$c_{B,0} + Ke^{-rT} = p_{B,0} + S_0.$$

Further, if the market is arbitrage free, we must have a similar relationship for market prices, p_M and c_M:

$$c_{M,0} + Ke^{-rT} = p_{M,0} + S_0.$$

This yields

$$c_{B,0} - c_{M,0} = p_{B,0} - p_{M,0}$$

Now suppose the implied volatility for a call is 19%. When we use this volatilty,

$$c_{B,0} - c_{M,0} = 0$$

Hence, from the relation above, the market price of the put must yield the same implied volatility.

So we have:

The implied volatility for a European call option is the same as the implied volatility for a European put option.

What about American options?

Recall an American option is the same as a European option except that exercise may occur any time up to and including the expiration of the option.

We can't use the same reasoning to obtain a put-call parity result here since actions may be taken at any time before maturity.

We will denote an American calls and puts by C and P respectively.

Clearly, we must have

$$C \geq c$$

since an American option has all of the properties of a European option and more.

Now consider the two portfolios:

Portfolio A' : one European call option and cash equal to Ke^{-rT}

Portfolio B' : one share

Assuming the cash is invested at the risk free interest rate, Portfolio A' is worth $\max(S_T, K)$ at maturity.

Portfolio B' is worth S_T.

So, we must have $c \geq S_0 - Ke^{-rT}$

Because options have no downside risk, we have

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$

And hence

$$C \geq \max(S_0 - Ke^{-rT}, 0) > S_0 - K.$$

But this implies that the option will never be exercised early.

So an American call option (on a nondividend paying stock) is the same as a European call option.

American puts are a different story.

First, for European puts, by considering the portfolios

Portfolio A'': one European put option and one share

Portfolio B'': cash equal to Ke^{-rT}

we can show that

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

Since early exercise is always possible for an American option, and since $P \geq p$, we have

$$P \geq K - S_0$$

Determining when to exercise an American put is determined by a threshold amount S_f.

We also observe that

$$P \geq p = c + Ke^{-rT} - S_0$$

Hence we have an inequality for American options (since $C = c$) where we once had equality.

For American options on nondividend paying stocks, we obtain the following result:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

The second inequality has already been discussed. The first inequality is obtained by (you guessed it) considering the appropriate portfolios.

We next derive a put-call parity equation for an asset value model developed by Merton.

For a given firm, the model is given as follows:

- A_t is the asset value process of the firm.

- E_t is the equity process of the firm.

- D_t is the process describing the firm's debt obligation.

- We assume $A_t = E_t + D_t$.

- Further, $(A_t)_{t \geq 0}$, $(E_t)_{t \geq 0}$ both follow geometric Brownian motion (recall Black-Scholes).

- The debt obligation of the firm has the cash profile of a zero coupon bond with maturity T and face value F.

Because of the last bullet, the cash profile of debt is simple:

- At time $t = 0$, debt holders (think banks) pay to the firm an amount of capital equal to D_0

- At time $t = T$, these debt holders receive an amount of capital equal to F.

The debt holders have some risk if the value of the firm's assets at time T can be below F. That is, riskiness is involved if

$$\mathbb{P}[A_T < F] > 0,$$

in which case we must have $D_0 < Fe^{-rT}$.

If $A_T < F$, the firm will default on its debt, and the debt holders will only receive a fraction of F.

The debt holders may want to neutralize this risk. One way to do this is to go long a put option on the asset, A, with strike price F, and time to maturity T.

It is easy to show that the debt holder is now gauranteed to receive F at time T

The debt holders are now completely hedged.

The hedged portfolio for the debt holders consists of a loan and a put option, and has value at time $t = 0$ of

$$D_0 + P_A,$$

where P_A is the put as described above.

Because the hedged portfolio is worth F at time T surely, we must have

$$D_0 + P_A = Fe^{-rT},$$

or

$$D_0 = Fe^{-rT} - P_A. \qquad (2)$$

We may also view equity in this option theoretic light. Suppose that at time T, the shareholders decide to liquidate the firm. That is, pay off all debts and receive capital for all assets. We will consider two cases.

- If $A_T > F$, the shareholders receive an amount of capital equal to $A_T - F$.

- If $A_T < F$, the value of the assets of the firm is not great enough to pay off the debt. Hence there is nothing left over for shareholders and their payoff is 0.

This sounds like a call option. In fact, we see the payoff for equity holders is $\max(A_T - F, 0)$. By no arbitrage arguments, we must have

$$E_0 = C_A, \qquad (3)$$

where C_A is a call option with underlying A, strike price F, and time to maturity T.

Putting this all together, we have

$$A_0 = E_0 + D_0 = C_A + Fe^{-rT} - P_A,$$

rearranging,

$$A_0 + P_A = C_A + Fe^{-rT},$$

which is just put-call parity with underlying A.

Before concluding, we note two results of the model.

- Since $E_0 = C_A$, equity holders prefer assets with more volatility.

- Because $D_0 = Fe^{-rT} - P_A$, debt holders (think banks) are naturally short a put and therefore they prefer lower volatility.

There are some issues with the above model. The biggest is that the asset value process is not observable.

We may approximate the firm's equity (shares of stock) and its debt.

Abstracting the asset value process requires a little more work, and can be tackled another day.