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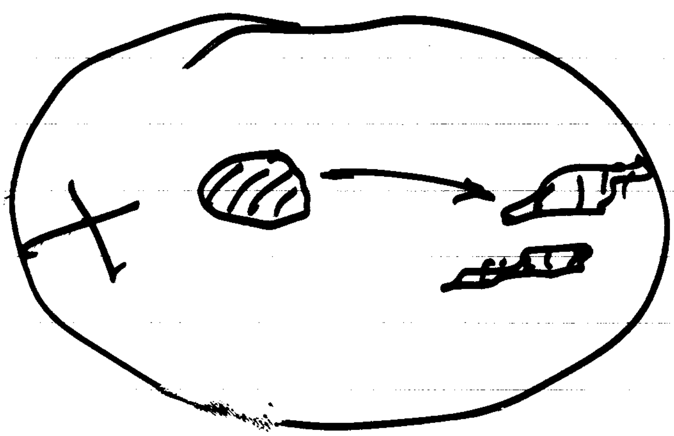
PATTERNS IN THE STARS, RECURRENCE in DYNAMICAL SYSTEMS & THE COMBINATORIAL BACKGROUND for Non-Conventional Ergodic Theorems



$T: X \to X$

Conventional Ergodic Averages:

$$(*) \quad \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) = \frac{1}{N}\{f(x)+f(Tx)+f(T^2x)+\cdots+f(T^{N-1}x)\}$$

where

$x \in X$, a space endowed with "measure" μ with $\mu(X) < \infty$

and $T: X \to X$ is <u>measure preserving</u>

<u>Ergodic Theorem</u>: The limit of $(*)$ as $N \to \infty$ exists (i) almost everywhere (ii) in L^p, $p \geq 1$ for $f \in L^p$

We'll have something to say about what that limit is.

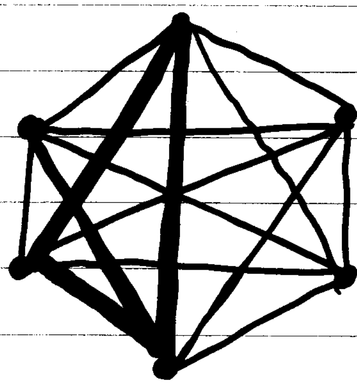
Non-Conventional Ergodic Averages:

$$(**)\begin{cases} \frac{1}{N}\sum_{0}^{N-1} f(T^n x)\, g(T^{2n}x)\, h(T^{3n}x) \\ \frac{1}{N}\sum_{0}^{N} f(T^n x)\, g(T^{n^2}x)\, h(T^{n^3}x)\, k(T^{n^3+17n}x) \end{cases}$$

THEOREM (B. HOST & B. KRA): The limits of averages as in $(**)$ exist in L^2 when all functions are bounded and measurable

The interest in expressions as in (**)

comes from RAMSEY THEORY



For any 2-coloring of the edges of the complete graph with 6 vertices, there will be at least one monochromatic triangle

The RAMSEY phenomenon:

Inside certain "rich" structures, certain patterns will be inevitably found

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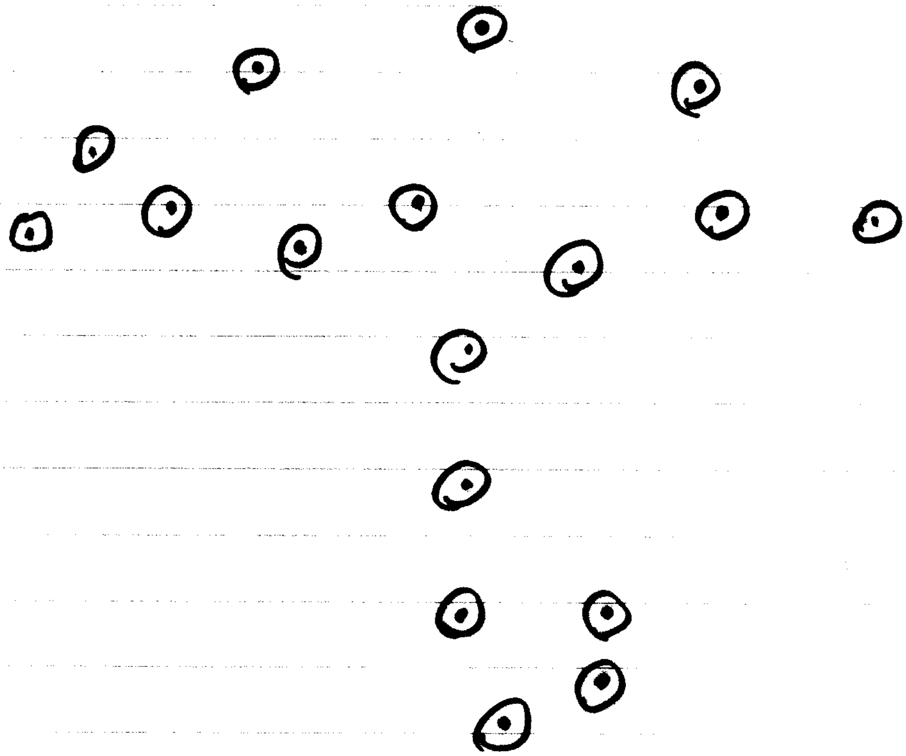

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CONSTELLATION THEOREM (KATZNELSON & F.)

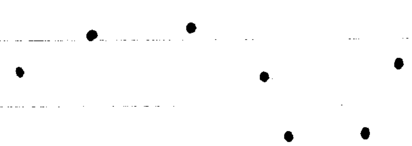
Let $S \subset \mathbb{R}^m$ be a set of positive (upper) density; let $F \subset \mathbb{R}^m$ be <u>any</u> finite set. Then $\exists\, m \in \mathbb{N},\ \vec{u} \in \mathbb{R}^m$ so that

$$mF + \vec{u} \subset S$$

($S \subset \mathbb{R}^m$ has positive upper density if $\exists$ balls B_r with arbitrarily large radius r and a $\delta > 0$ with $\dfrac{vol(S \cap B_r)}{vol(B_r)} > \delta$ for each ball

(Only) Proof of this uses ergodic theory

<u>Example</u>

$F =$ 

$S =$ stars (as tiny splotches in visual plane or in space)

What is analogous statement for $\mathbb{Z}$?

If $S \subset \mathbb{Z}$ has positive upper density (e.g., if $\frac{1}{N} \#(S \cap \{1, 2, \ldots, N\}) \to \delta > 0$) and F is finite, $\exists\, m, u$ with

$$mF + u \subset S$$

$F \subset \{a, a+1, a+2, \ldots, a+\ell-1\}$; take $F = \{a, \ldots, a+\ell-1\}$

$mF+u$ is arithmetic progression with ℓ terms

THEOREM (Szemerédi) A subset of $\mathbb{Z}$ of positive upper density contains arbitrarily long arithmetic progressions.

$\mathbb{Z}$: $\cdot \bullet \times \bullet \cdot \cdot \cdot \times \bullet \cdot \cdot \cdot \times \cdot \bullet \cdot \bullet \times \cdot \cdot \cdot \times \bullet \cdot \bullet \cdot \bullet \cdot \bullet \cdot \cdot$

THEOREM (van der Waerden) In <u>any</u> partition

$$\mathbb{Z} = C_1 \cup C_2 \cup \cdots \cup C_r$$

one of the C_j contains arbitrarily long arithmetic progressions.
(Conjectured by Erdös-Turán in 30's)

We think of $\mathbb{Z}$ as measure space, with DENSITY as measure:

$$\text{DENS}(\text{evens}) = 1/2, \quad \text{DENS}(7\mathbb{Z}+3) = \frac{1}{7}$$

$$\underline{\text{DENS}}\ (\text{primes}) = 0, \quad \text{DENS}(A \cup B) = \text{DENS}(A) + \text{DENS}(B) - \text{DENS}(A \cap B)$$

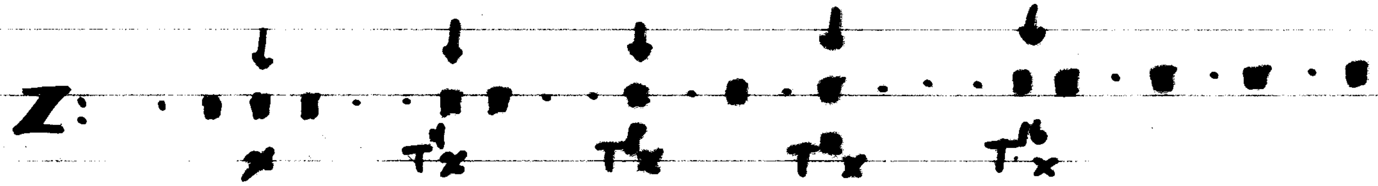
$$\prod \left(1 - \frac{1}{p}\right) = 0$$

(But DENS is <u>not</u> <u>σ-additive</u>:

$$\mathbb{Z} = \bigcup \{n\}, \quad \text{DENS}\{n\} = 0 \ )$$

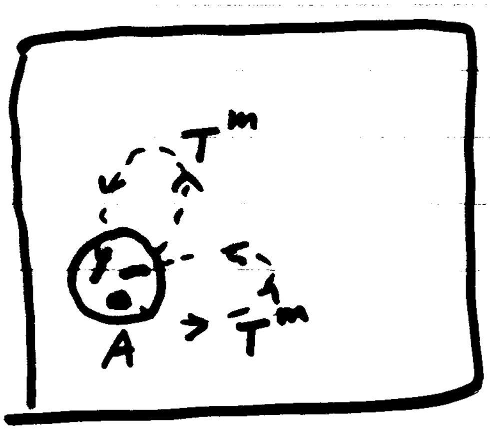
Define $T : \mathbb{Z} \to \mathbb{Z}$ by $Tx = x+1$

T preserves density!



Szemerédi's Theorem $\Longleftrightarrow$

(MRT) <u>Multiple Recurrence Theorem in Ergodic Theory</u>



$\mu(A) > 0 \Longrightarrow$

for any k $\exists$ m

$$\mu(A \cap T^{-m}A \cap T^{-2m}A \cap \cdots \cap T^{-km}A)$$

What about $k=1$? $\quad \mu(A \cap T^{-m}A) > 0$

Must every set return to itself?

this is <u>Poincaré's Recurrence theorem</u>

THEOREM: If $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, $T: X \to X$ a measure preserving transformation, $A \in \mathcal{B}$ with $\mu(A) > 0$, then for some $m > 0$

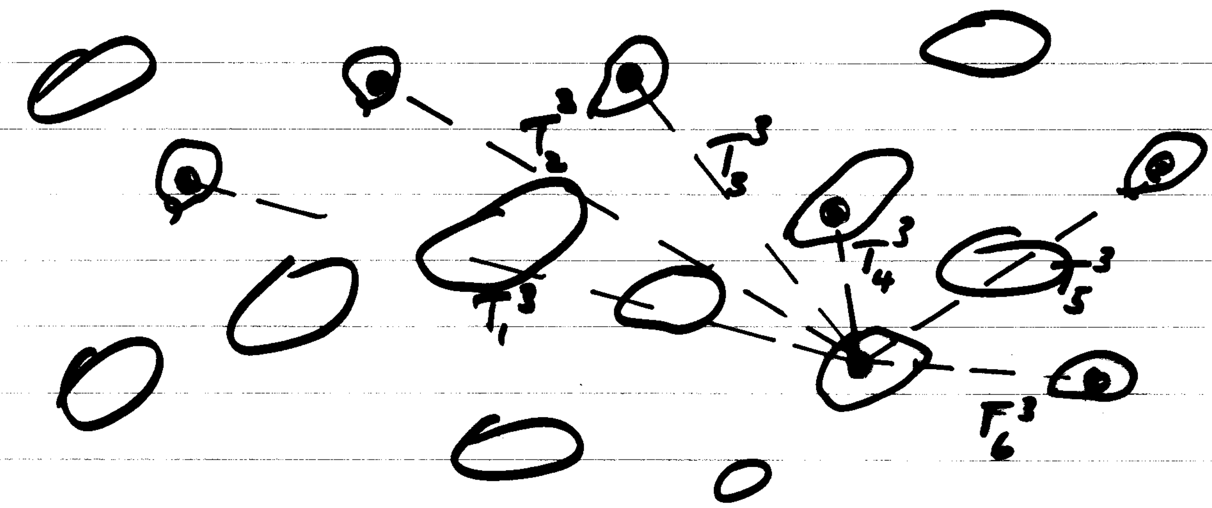
$$\mu(A \cap T^{-m}A) > 0.$$

MRT is special case of <u>Commuting Mult. Rec. Th.</u>

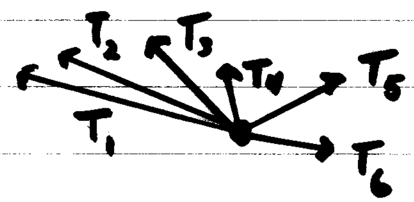
THEOREM: If $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, $T_1: X \to X$, $T_2: X \to X$, ... $T_k: X \to X$ k commuting measure preserving transformations, $A \in \mathcal{B}$ with $\mu(A) > 0$, then for some $m > 0$

$$\mu(A \cap T_1^{-m}A \cap T_2^{-m}A \cap \cdots \cap T_k^{-m}A) > 0$$

Comm. Mult. Rec. Thm $\Rightarrow$ Constellation thm



T_i is translation by corresponding vector:



Proof of Poincaré Recurrence via Ergodic Theorem

We'll prove that if $\mu(A) > 0$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n} A) > 0$$

Principle: Behavior <u>on the average</u> is easier to control

Lemma: If T is measure preserving, then for f integrable

$$\int f(Tx)\,d\mu(x) = \int f(x)\,d\mu(x).$$

Pf. True for $f = 1_A$ where $f(Tx) = 1_{T^{-1}A}(x)$ and

LHS $= \mu(T^{-1}A)$ & RHS $= \mu(A)$ & these are equal.

Linear combinations of these are dense in L^1. $\square$

By Ergodic thm, if $f = 1_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

$$\bar{f}(x) = \lim \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \quad \text{exists (in } L^2)$$

Note that $\bar{f}(x) = \bar{f}(Tx)$ and $\int \bar{f}\,d\mu = \int f\,d\mu = \mu(A)$

In particular, $\bar{f}$ is not a.e. 0.

Claim $\int \bar{f}(x) f(x)\,d\mu(x) > 0$.

Suppose $= 0$. Replace x by $T^n x$:

$$\int \bar{f}(x)\, f(T^n x)\,d\mu(x) = 0$$

Average over n. This $\Rightarrow$

$$\int \bar{f}(x) \cdot \bar{f}(x)\,d\mu(x) = 0$$

i.e. $\bar{f} = 0$ a.e. But $\int \bar{f}\,d\mu = \mu(A) \neq 0$. So

$$\int \bar{f}(x) f(x)\,d\mu(x) = \lim \frac{1}{N} \sum \int f(T^n x) f(x)\,d\mu(x).$$

$$\int 1_A(T^n x)\, 1_A(x)\,d\mu(x) = \mu(A \cap T^{-n}A)$$

Similarly one can prove:

$$\liminf \frac{1}{N}\sum_{n=0}^{N-1} \mu(A\cap T_1^{-n}A \cap T_2^{-n}A \cap \dots T_k^{-n}A) > 0.$$

This is

$$\liminf \int 1_A(x)\left(\frac{1}{N}\sum_{n=0}^{N-1} 1_A(T_1^n x)\, 1_A(T_2^n x) \dots 1_A(T_k^n x)\right) d\mu(x).$$

Does limit exist?

? ? ? ? ? ?

By Host-Kra limit exists for

$T_1 = T,\ T_2 = T^2, \dots T_k = T^k$

and we can identify limit!!

In all cases, multiple recurrence is established by analyzing (non-conventional) ergodic averages.

In addition to commuting mult. recurrence

(Katznelson, F) $\liminf \frac{1}{N}\sum \mu(A\cap T_1^{-n}A \cap \dots \cap T_k^{-n}A) > 0$

we have for $\mu(A)>0$:

(Bergelson-Leibman) $\lim \frac{1}{N}\sum \mu(A\cap T^{-p_1(n)}A\cap T^{-p_2(n)}A\cap\dots\cap T^{-p_k(n)}A)>0$

where $p_1(n),\dots p_k(n)$ are <u>polynomials</u> taking integer values on $\mathbb{Z}$ and with $p_i(0)=0$

For example:

$$\lim \frac{1}{N}\sum \mu(A\cap T^{-n}A\cap T^{-n^2}A\cap T^{-n^3}A)>0$$

$\Rightarrow$ Polynomial Multiple Recurrence:

e.g.: $\exists\, n: \mu(A\cap T^{-n}A\cap T^{-n^2}\cap T^{-n^3}A)>0$

General Principle: A <u>recurrence pattern</u> in dynamics is also a <u>recurrence</u> <u>pattern</u> for sets of positive (upper) density in $\mathbb{Z}$

$\Rightarrow$ If $S\subset\mathbb{Z}$ has positive (upper) density then it contains arbitrary "polynomial progressions"

$$a,\ a+n,\ a+n^2,\ a+n^3$$

&

$$b,\ b+m^2,\ b+2m^2,\ b+3m^2,\ b+4m^2$$

etc.

(But not necessarily $c,\ c+2n^2+1$)

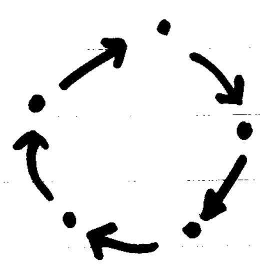
$$x-y=m^2 \quad (\text{Sarkozy})$$

Underlying idea: In presence of sufficient <u>randomness</u>/<u>mixing</u> (ergodicity,

$$\frac{1}{N}\sum f_1(T^n x)\, f_2(T^{2n} x)\cdots f_k(T^{kn} x)$$

$$\xrightarrow{L^2} \int f_1(y)\,d\mu(y) \int f_2(y)\,d\mu(y) \cdots \int f_k(y)\,d\mu(y)$$

In case of (pure) <u>periodicity</u>:



$X=\mathbb{Z}/5\mathbb{Z}$

$$\lim_{N\to\infty} \frac{1}{N}\sum_0^{N-1} f_1(T^n x)\, f_2(T^{2n} x)\cdots f_k(T^{kn} x) =$$

$$\frac{1}{5}\sum_{j=0}^{4} f_1(x+j)\, f_2(x+2j)\cdots f_k(x+kj)$$

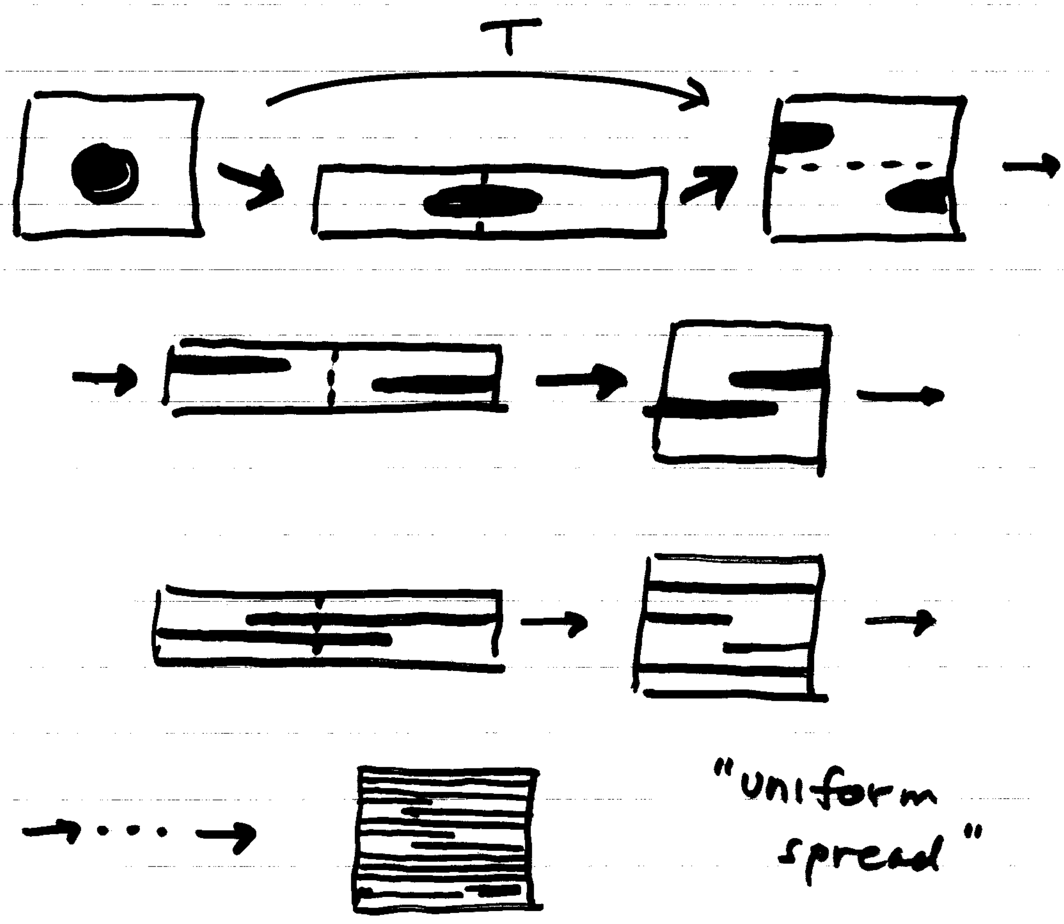
Idea: Integrate out the random effects

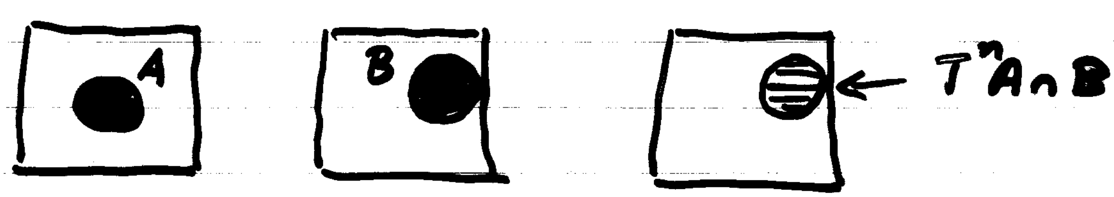
We assume heretofore that $\mu(X) = 1$ or $(X, \mathcal{B}, \mu)$ is probability space

<u>**Independence**</u>: $\mu(A \cap B) = \mu(A)\mu(B)$

(Strong) <u>**Mixing**</u>: $\mu(A \cap T^{-n}B) \longrightarrow \mu(A)\mu(B)$

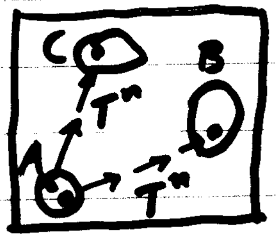
E.g. Baker's Transformation (kneading)





$$\mu(A \cap T^{-n}B) = \mu(T^n A \cap B)$$

Weak MIXING: (chaotic behavior)



If $A, B, C \in \mathcal{B}$ and $\mu(A) > 0$, $\mu(B) > 0$, $\mu(C) > 0$ then $\exists n$ so that

$$\mu(A \cap T^{-n} B) > 0 \ \& \ \mu(A \cap T^{-n} C) > 0$$

Note that (strong) mixing $\Rightarrow$ weak mixing

Theorem: If $(X, \mathcal{B}, \mu, T)$ is weak mixing, $f_1, f_2, \dots f_k$ bounded meas. functions, then

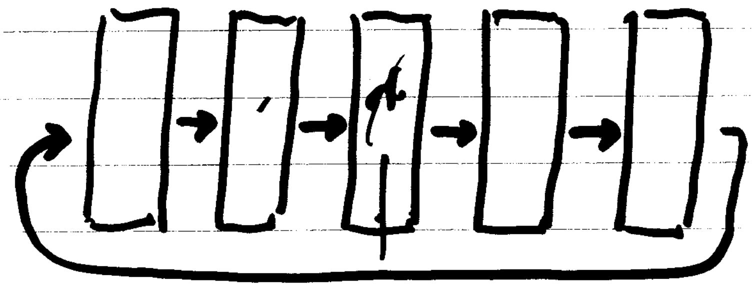
$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_k(T^{kn} x) \xrightarrow{L^2}$$

$$\int f_1 \, d\mu \int f_2 \, d\mu \cdots \int f_k \, d\mu$$

In particular: If $f_1 = f_2 = \dots = f_k = 1_A$, limit is $\mu(A)^k$ and

$$\frac{1}{N} \sum \mu(A \cap T^{-n} A \cap \dots \cap T^{-kn} A) = \frac{1}{N} \sum \int 1_A(x) 1_A(T^n x) \cdots 1_A(T^{kn} x) \, d\mu$$

$$\longrightarrow \mu(A)^{k+1} > 0$$



$T: X_i \to X_{i+1}$

$X = X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4$

Suppose each step is (weak) mixing

Note: If T is weak mixing, so is T^5

Write $S = T^5$

Want $\lim_{N\to\infty} \frac{1}{N} \sum f(T^n x)\, g(T^{2n} x)\, h(T^{3n} x)$

$N = 5M \qquad n = 5m + r$

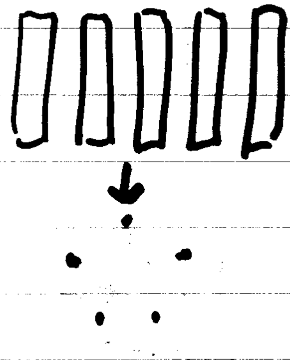
$$\frac{1}{5}\Big\{ \frac{1}{M}\sum f(S^m x)\, g(S^{2m} x)\, h(S^{3m} x) +$$
$$\frac{1}{M}\sum f(TS^m x)\, g(T^2 S^{2m} x)\, h(T^3 S^{3m} x) +$$
$$\cdots + \frac{1}{M}\sum f(T^4 S^m x)\, g(T^8 S^{2m} x)\, h(T^{12} S^{3m} x) \Big\} \to$$

$$\boxed{\frac{1}{5}\left\{ \alpha_i \beta_i \gamma_i + \alpha_{i+1}\beta_{i+2}\gamma_{i+3} + \alpha_{i+2}\beta_{i+4}\gamma_{i+1} + \cdots + \alpha_{i+4}\beta_{i+3}\gamma_{i+2} \right\}}$$

where $\alpha_j = 5\int_{X_j} f\, d\mu \quad \beta_j = 5\int_{X_j} g\, d\mu \quad \gamma_j = 5\int_{X_j} h\, d\mu$

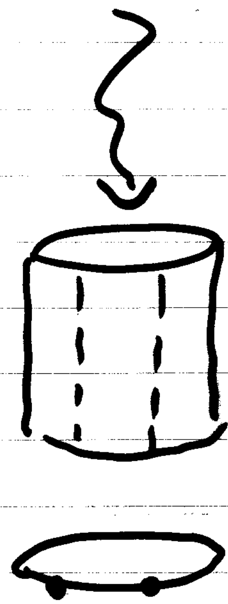
and $x \in X_i$

X



This will be model for general (ergodic) case

$\mathbb{Z}/5\mathbb{Z}$



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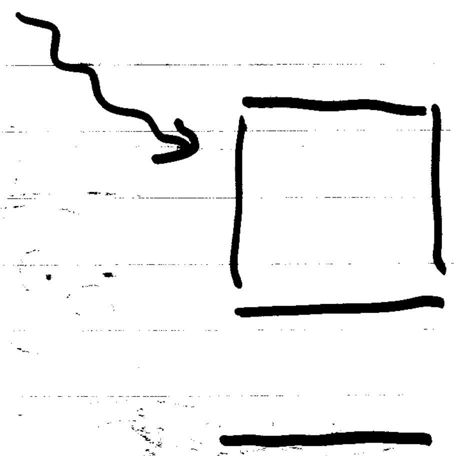
compact abelian group

$\rightarrow$ rotation in gp



(X, μ)

$(G/\Gamma, m)$

G will be nilpotent gp, Γ subgroup

m = haar measure